Lecture 9

$$\boxed{k[x]/(x^2-1) \quad \boxed{\text{Lee's Alg}}}$$

$x^2 = 1$

$\Delta(1) = 1\otimes x + x\otimes 1$

$\Delta(x) = x\otimes x + 1\otimes 1$

$\varepsilon(x) = 1, \ \varepsilon(1) = 0$

$r = \frac{1+x}{2}$ | $\varepsilon(r) = \frac{1}{2}$

$g = \frac{1-x}{2}$ | $\varepsilon(g) = -1/2$

$r^2 = r, \ g^2 = g, \ r + g = 1$

$rg = gr = 0$

$\Delta(r) = 2r\otimes r \quad , \varepsilon(1) = 0$

$\Delta(g) = -2g\otimes g$

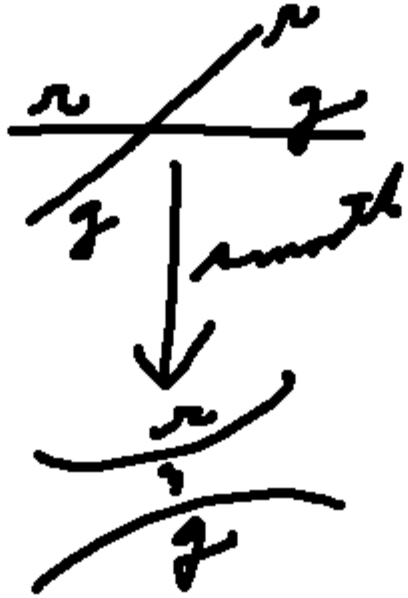
$\boxed{\bullet = x}$

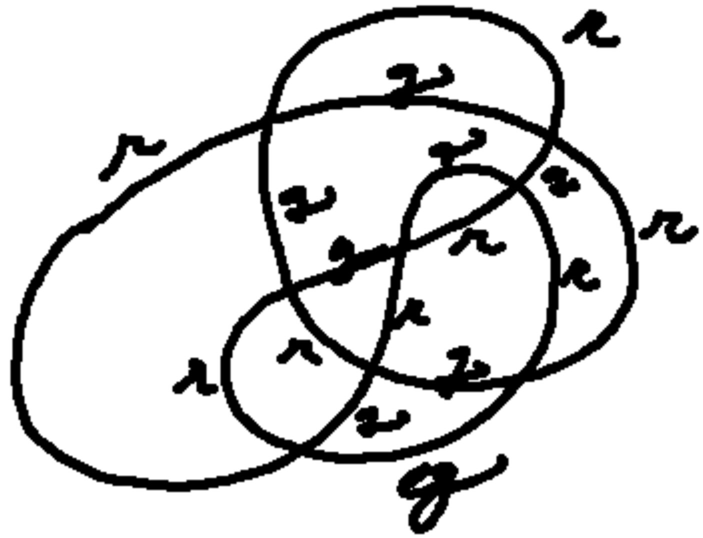
$r - g = x$

Tubing Relation

$$\boxed{\alpha = \varepsilon(\alpha(r-g))1 + \varepsilon(\alpha)(r-g)}$$

$$\begin{aligned} g &= \varepsilon(g(r-g))1 + \varepsilon(g)(r-g) \\ &= \varepsilon(-g)1 + \varepsilon(g)(r-g) \\ &= \frac{1}{2} + \frac{g-r}{2} = \frac{r+g+g-r}{2} = g \ \checkmark \end{aligned}$$





2-coloring + smoothing yields Lee cycles on classical diagrams

$rg = \phi$

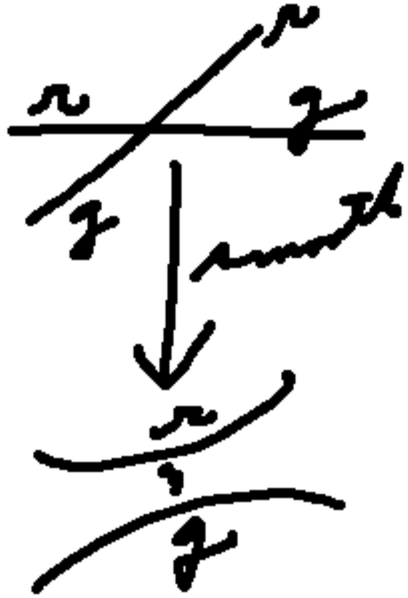
$\partial S = \phi$

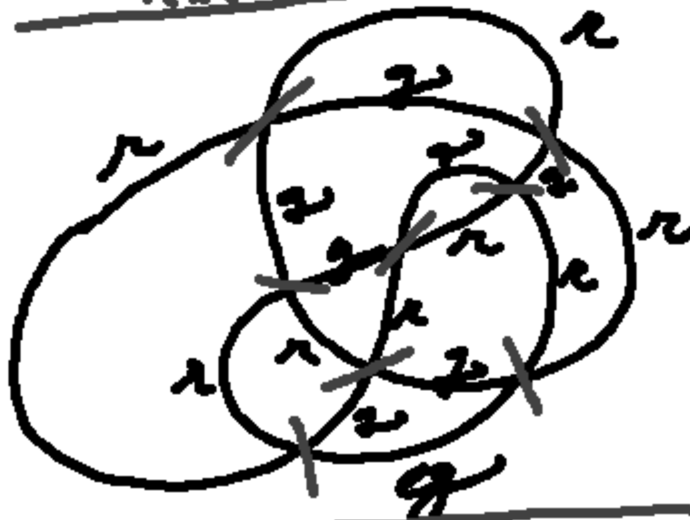
Ans: No!

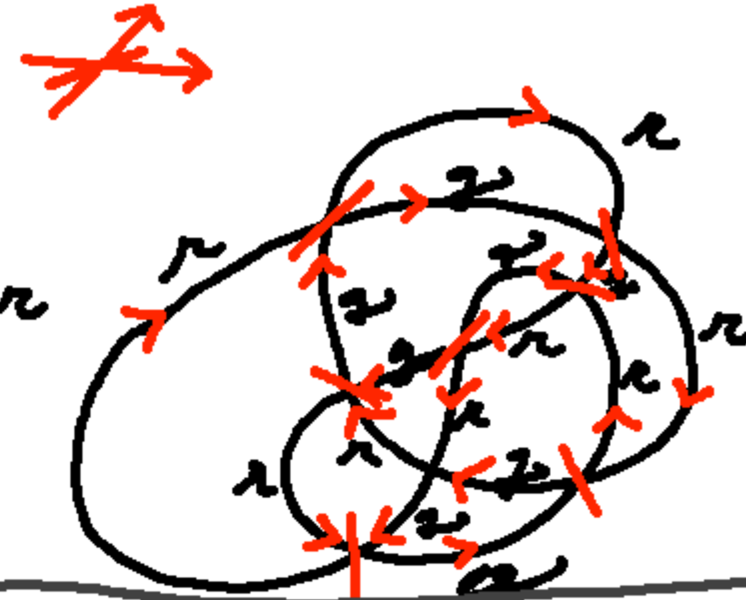
$\Delta(g) = -2g \otimes g$

Does this ? happen.

Classical Diag

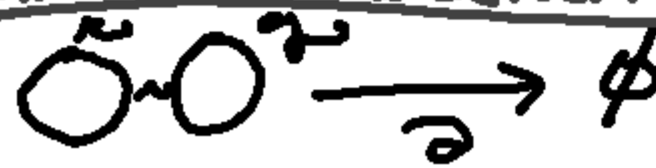






Lem

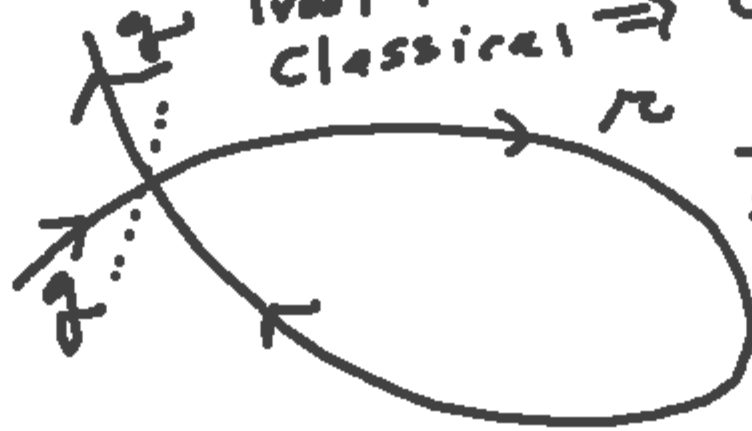
Color Smoothings = Seifert Smoothings



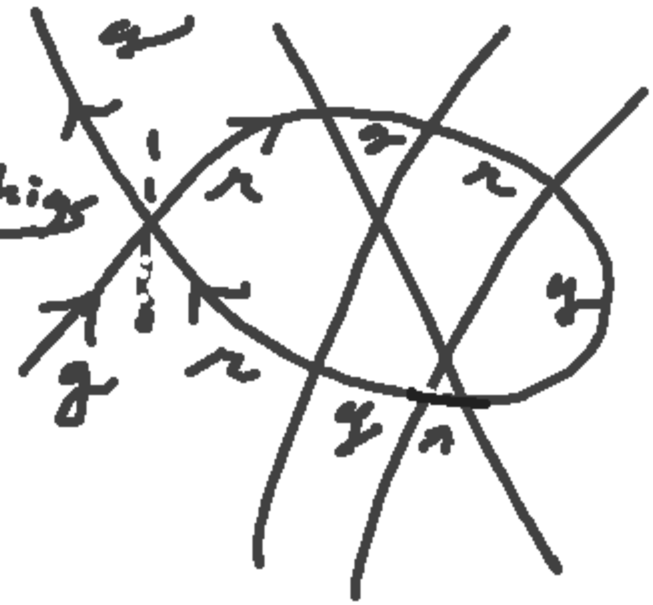
And we can see the Seifert smoothings do not have self-touching sites.

∴ These smoothings are Lee cycles.

Proof that classical ⇒ Color Smoothings

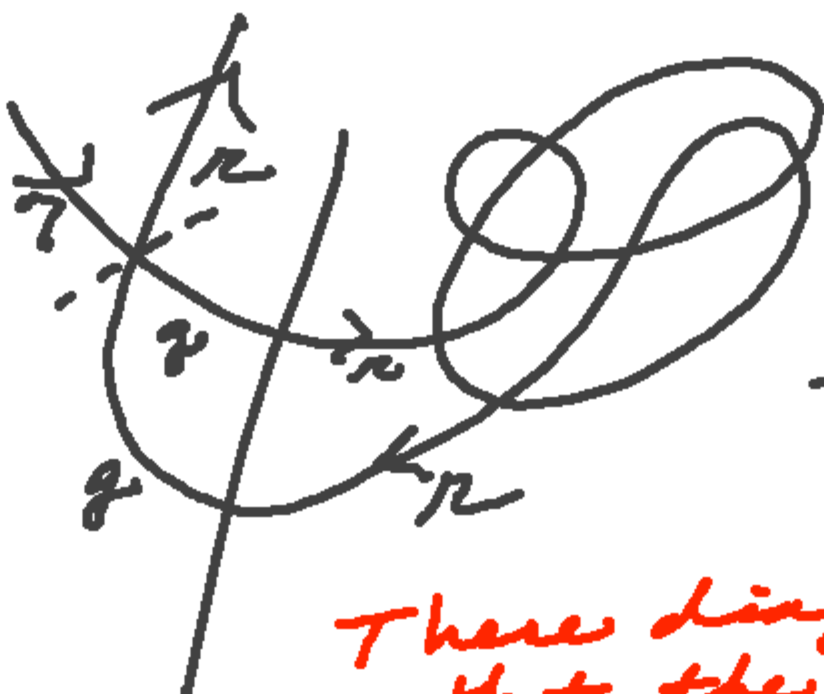


= Seifert Smoothing



even # crossing

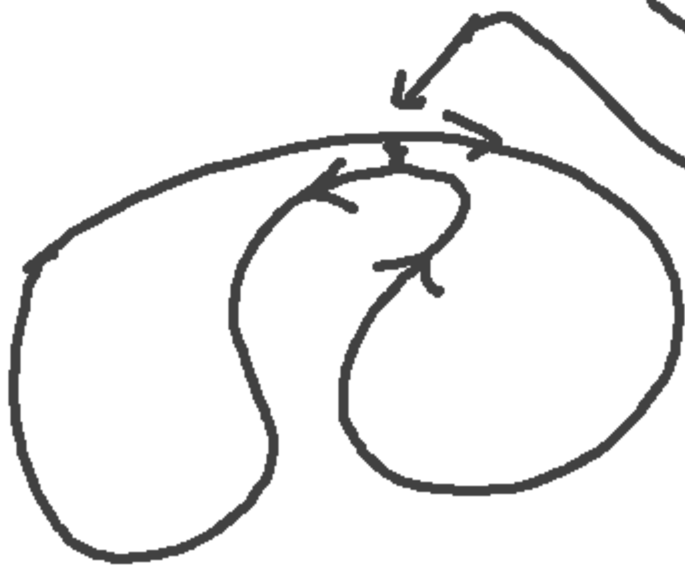
true by Jordan Curve Theorem



These diagrams prove that the color smoothings on classical diagrams are Seifert smoothings.

Seifert smoothings are not self-touching.

Seifert Circles



not a Seifert smoothing.

All S.S. have form

∴ every Seif smoothing gives a cycle in Kh Homology.

∴ no Seif Circle touches itself.

Thm: $\mathcal{L}^*(K) = $ Lie Homology (K)

$\Longrightarrow \dim \mathcal{L}^*(K) = 2^{\#\mathrm{Comp}(K)}$

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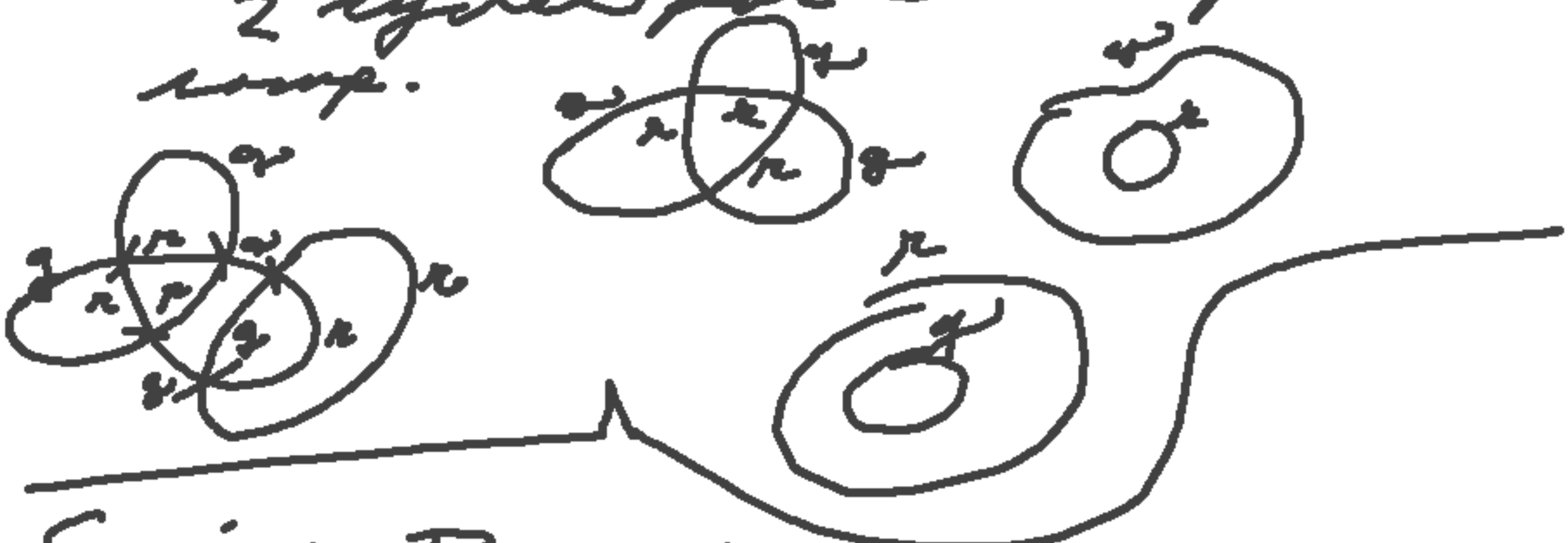
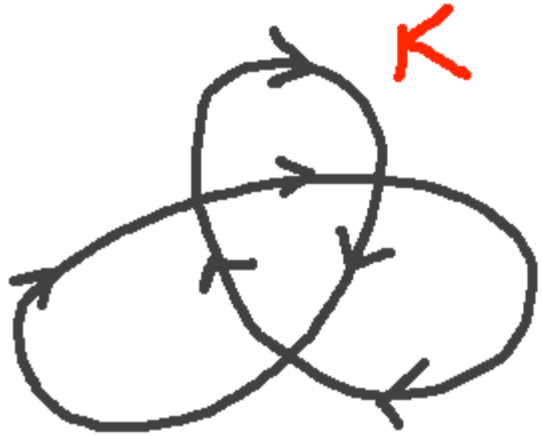
2 cycles for each single comp.



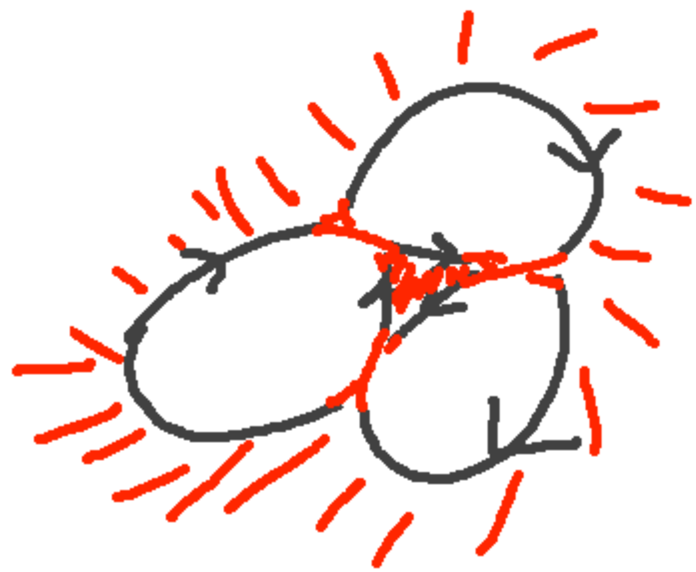
Exercise: Prove Thm.

K



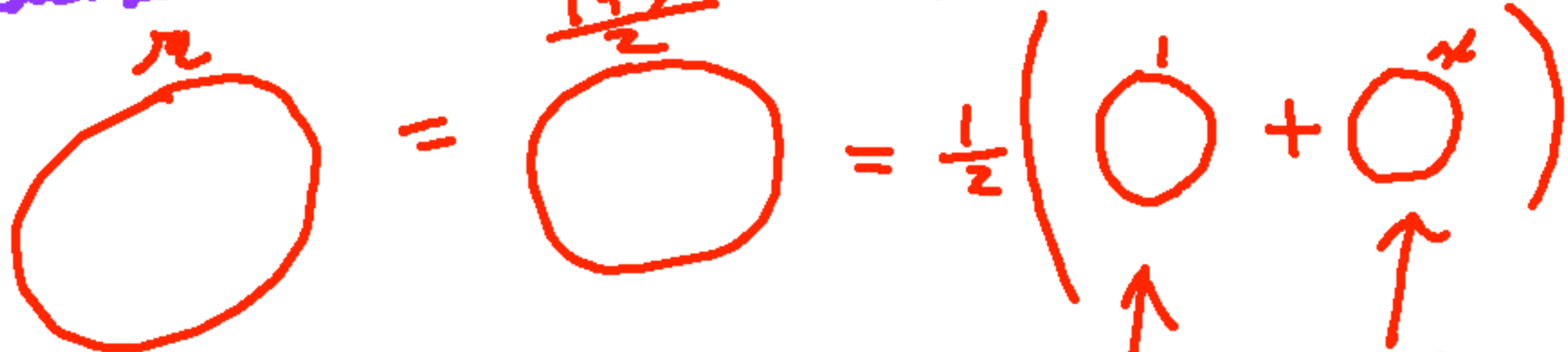
classical Seifert surface

$\partial F = K$



disks bounding Seif circuits

j grading in Lee complex is not preserved by ∂.

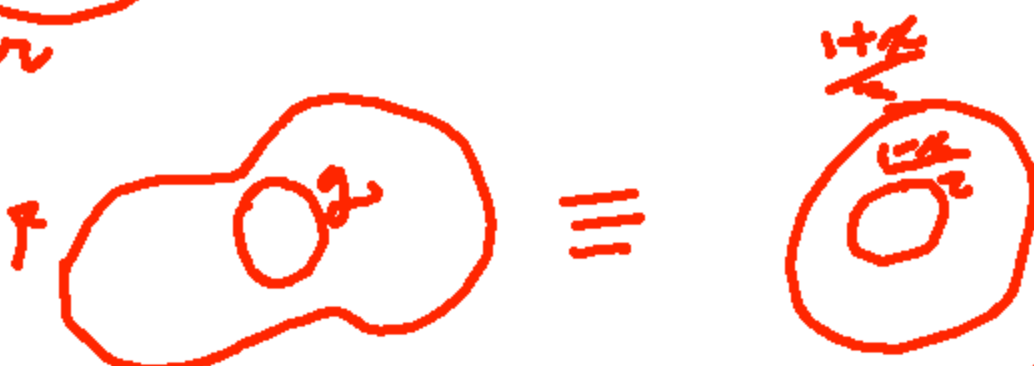


$\#B = 0$

$$\boxed{j = \#B + \#(1) - \#(x)}$$

contrib to $j = -1$

contrib to $j = 0$



= sum of various loop labellings

In Khovanov Complex

$$j(\partial\alpha) = j(\alpha).$$

$\#B=0$

$j = 1_{\text{one}}$ $\qquad$ $j = 1_B + 1_{\text{one}} = 2$

In Lee j not preserved.

$$J_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} [K],$$

$[K] = \langle \times \rangle \to \langle \asymp \rangle - q) ($ $\qquad$ $O = q + q^{-1}$

$[K] \Leftrightarrow \sum q^i \chi(H^{*i})$

$4v = 2e$

$\boxed{e = 2v}$

$n = \#$ crossings
$r = \#$ Seif circles.

Genus of Seifert Surface

$$\boxed{g = \frac{-r+n+1}{2}}$$

Prove this via Euler's Formula

$v - e + f = 2 - 2g$

$n = 3$
$r = 2$

$$g = \frac{-2+3+1}{2} = 1$$

$v = n$
$e = 2n$
$f = r+1$

$n - 2n + r + 1$
$= 2 - 2g$

$-n + r - 1 = -2g$

$$g = \frac{n-r+1}{2}$$

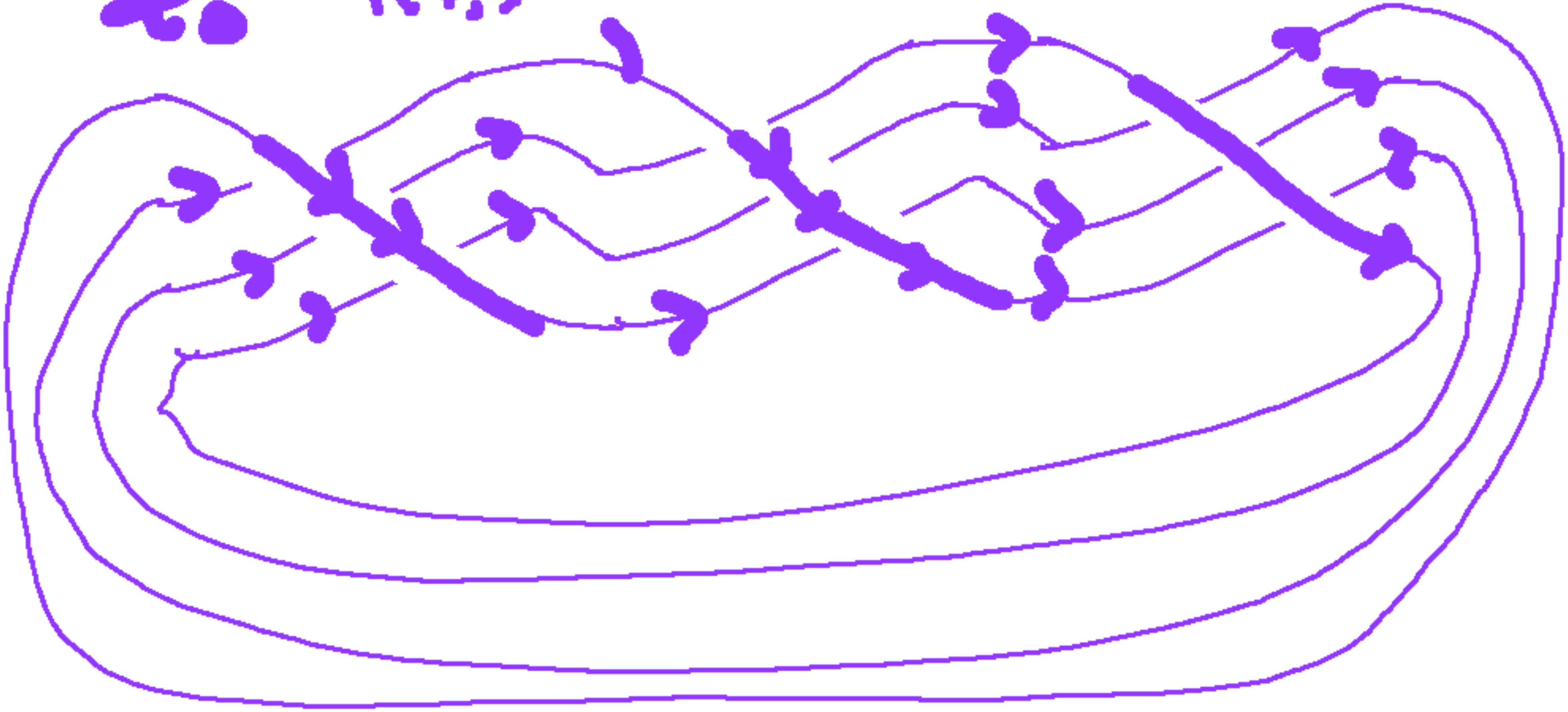
✓

$\cong$

$\cong$

$\cong$ punc(Torus)

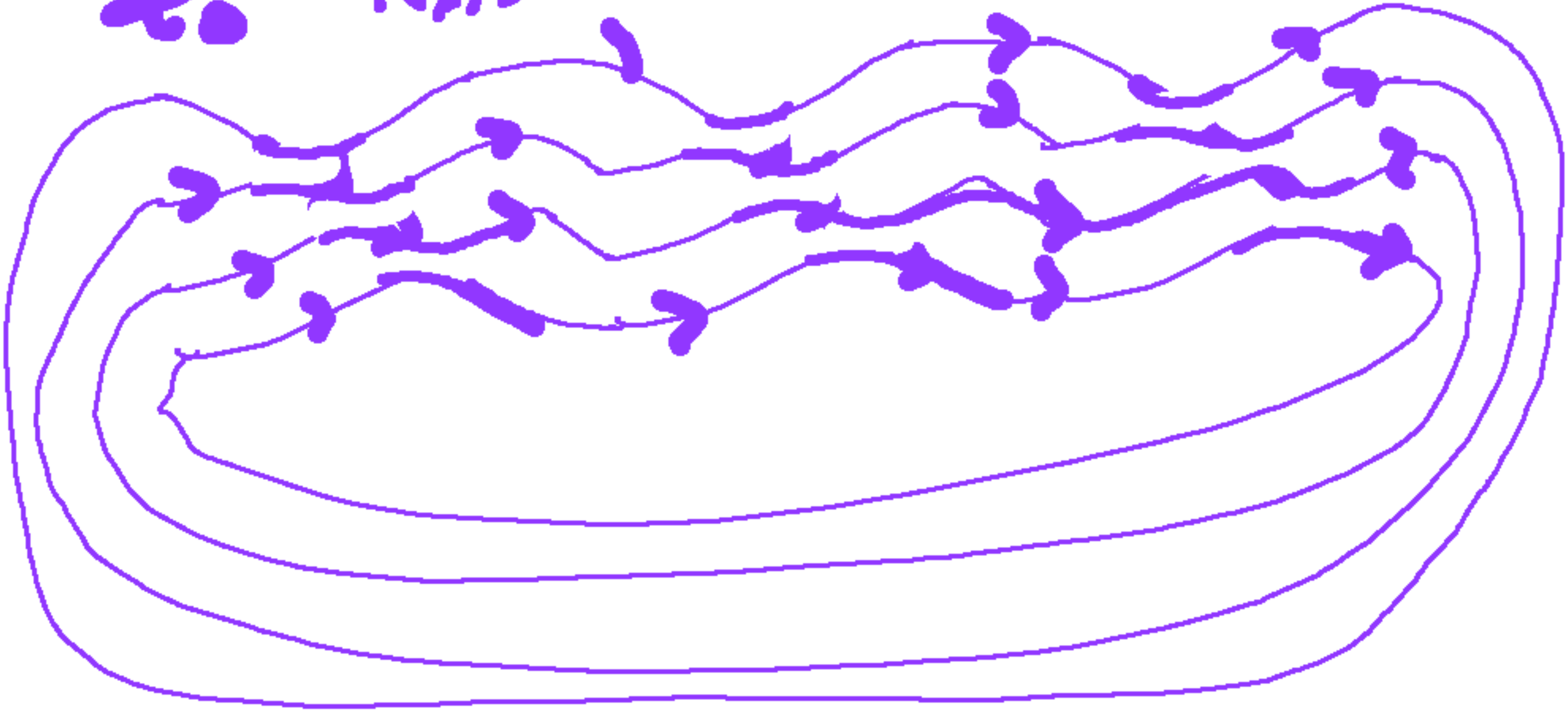
eg. $K_{4,3}$



$K_{p,q}$ $(p-1)q$ crossings

p Seif $\odot$'s.

K#3



Lee's Algebra

$$\left\{\begin{array}{l} k[x]/(x^2-1) = \mathcal{A} \\ x^2 = 1 \\ \Delta(1) = 1\otimes x + x\otimes 1 \\ \Delta(x) = x\otimes x + 1\otimes 1 \\ \varepsilon(x) = 1,\ \varepsilon(1) = 0 \end{array}\right\}$$

This also gives a link homology theory. Now the second grading j is not preserved. But

$$j(\partial\alpha) \geq j(\alpha)$$

for each chain α. This means that one can use j to filter the chain complex for Lee homology.

The result is a spectral sequence that starts from Khovanov homology and converges to Lee homology.

Lee homology is simple:

$$\dim \mathcal{L}ee^{*}(K) = 2^{\#\mathrm{comp}(L)}$$

and behaves well under link concordance.

Rasmussen uses this relation to define invariants of links that give lowerbounds for the 4-ball genus + determine it for torus links.

Rasmussen Invariant (Uses spectral sequence from Khovanov to Lee.)

We have the j-grading on $C_*(K)$ for a diagram K and the fact that for Lee's algebra $j(\partial s) \geq j(s)$. Rasmussen uses a normalized version of this grading denoted by $q(s)$ (adjusted for invariance of the normalized Jones polynomial.)

Then one makes a filtration

$$F^k C_*(K) = \{ v \in C_*(K) \mid q(v) \geq k \}$$

and given $\alpha \in Lee_*(K) = \mathcal{L}_*(K)$ define

$$S(\alpha) \underset{df}{=} \max\{ q(v) \mid [v] = \alpha \}$$

$$s_{min}(K) \underset{df}{=} \min\{ S(\alpha) \mid \alpha \in \mathcal{L}_0(K), \alpha \neq 0 \}$$

$$s_{max}(K) \underset{df}{=} \max\{ S(\alpha) \mid \alpha \in \mathcal{L}_0(K), \alpha \neq 0 \}$$

$$\boxed{s(K) = \frac{s_{min}(K) + s_{max}(K)}{2}}$$

Facts: 0) $s_{max}(K) = s_{min}(K) + 2$ so $s(K) \in \mathbb{Z}$.

1) $s(K)$ is a concordance invar of K.

2) $s(K)$ is additive under connected sum.

3) $s(K^*) = -s(K)$

4) If K is a positive knot diagram (all crossings) then $s(K) = -r + n + 1$ where $r = \#$ of loops in canonical smoothing, $n = \#$ crossings.

5) $s(K_{pr}) = (p-1)(r-1)$ for K_{pr} a (p,r) torus knot.

6) $|s(K)| \leq 2g^*(K)$ where $g^*(K)$ is the least genus spanning surface for K in the four ball.

7) $g^*(K_{pr}) = \frac{(p-1)(r-1)}{2}$ (Milnor's Conjecture).

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$\begin{cases} n_+ = \# \text{ of } + \text{ crossings in } K. \\ n_- = \# \text{ of } - \text{ crossings in } K. \end{cases}$

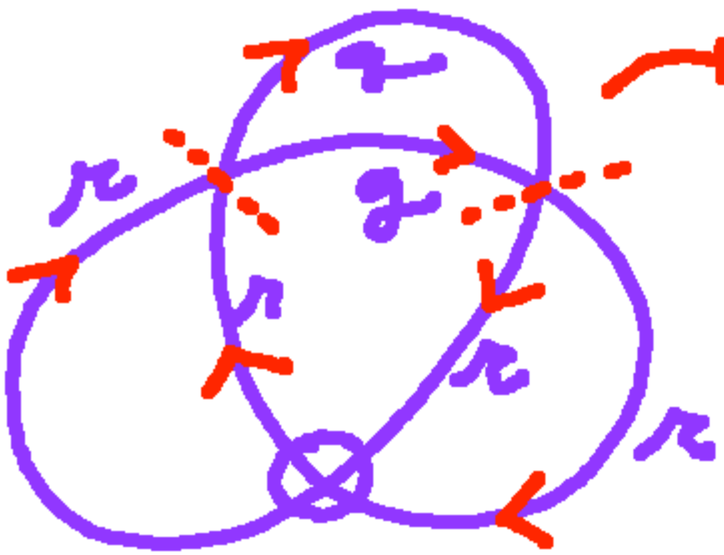
$$\boxed{j(s) = \#(\text{B smoothings}) + \#(1\text{'s}) - \#(x\text{'s})}$$

For **A**-state of a $K_{p,q}$ torus knot have (with all x's): • **p** loops

• $(p-1)q$ crossings

So
$$q(s) = (0 - p) + (p-1)q$$
$$= pq - q - p$$
$$q(s) = (p-1)(q-1) - 1$$

$$\Longrightarrow \boxed{s(K_{pq}) = (p-1)(q-1)}$$



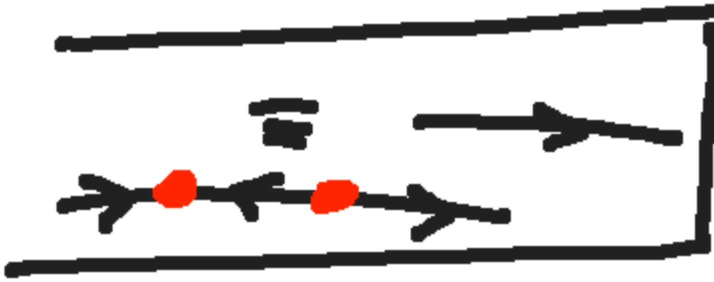
other invars Binar Brackets

rel to Rushworth's Doubled Kho Lee Homology for virtuals via same yoga with cut pts + involve as for Khov Homology for virtuals.

Color Smoothings = Seif Smoothings

canonical cut pt. assignment

(+ leads to global order)



After Cut Pts

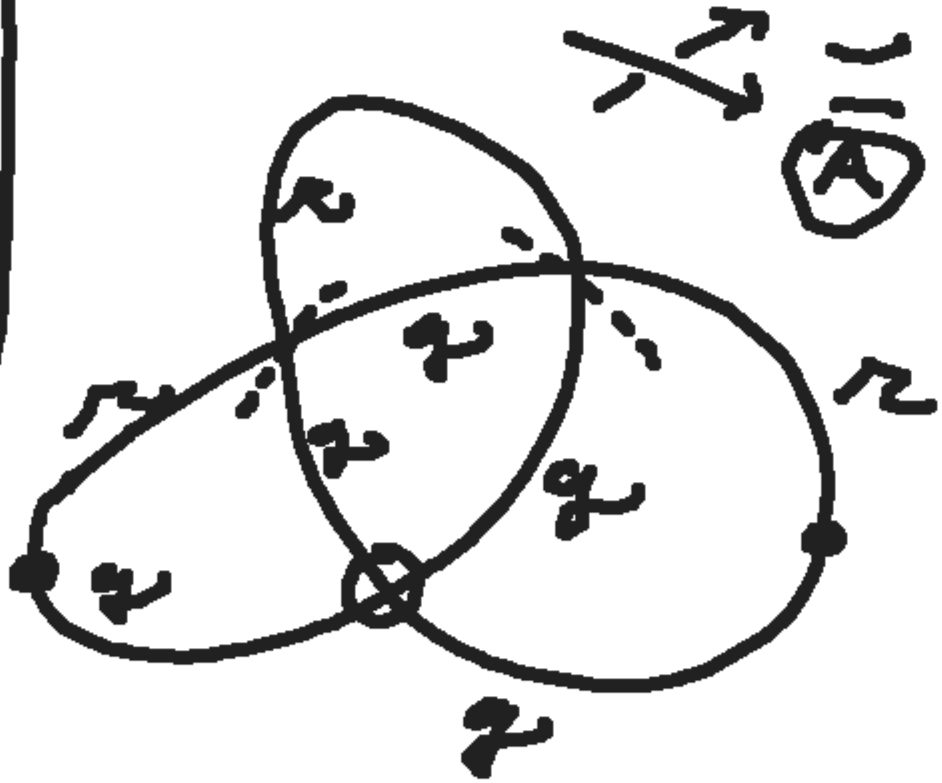
$\bar{r} = y$

$S\left(\frac{x+1}{2}\right)\bar{y} = r$

$= -\frac{x+1}{2}$

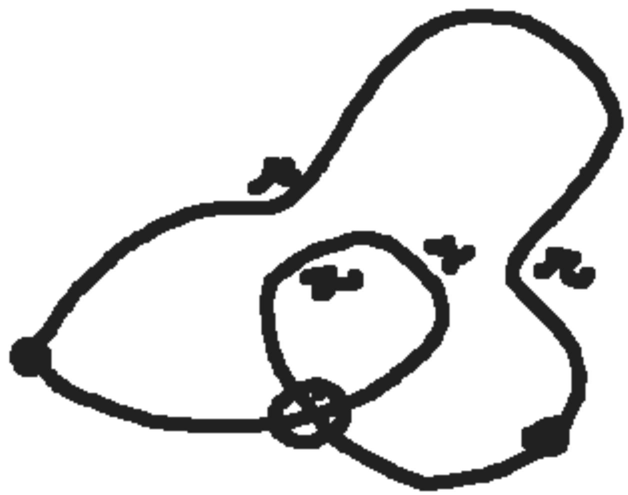
$S(r) = y$

Reassembly
Gen
to
"same"
result
for
pos
virtuals



= single
cycle map
= ϕ

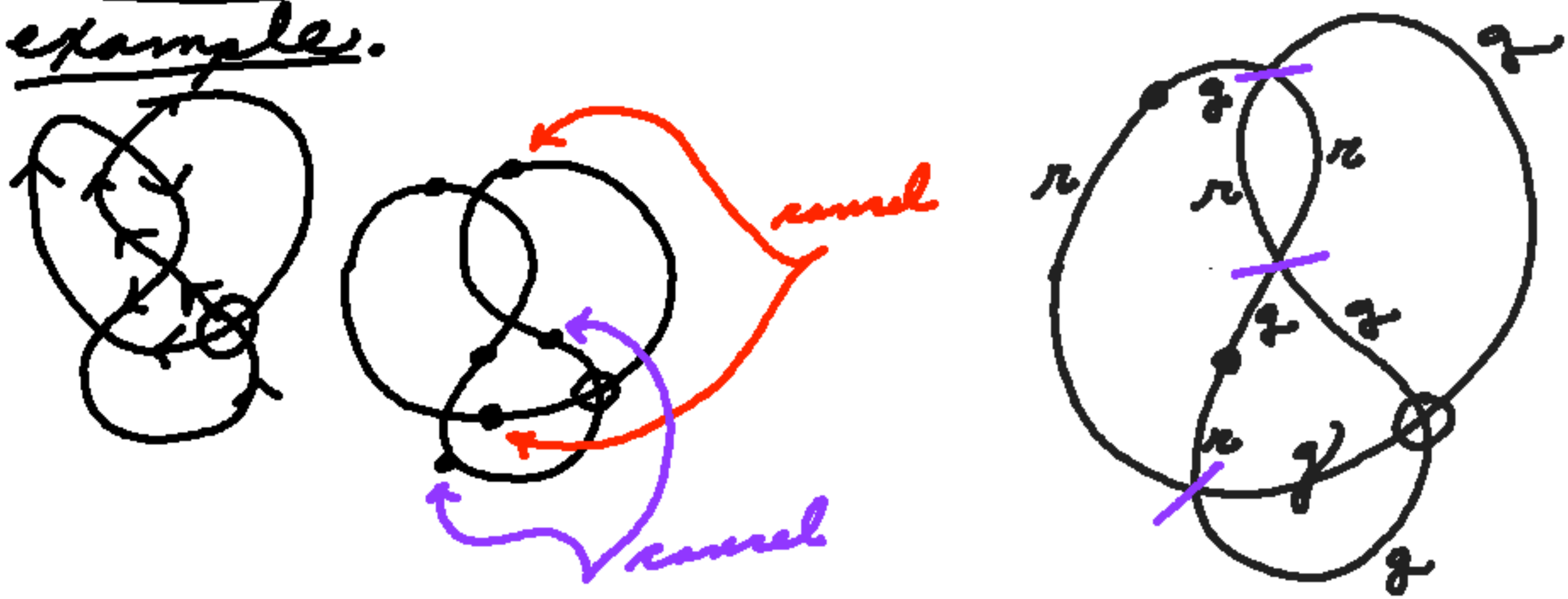
still
a cycle

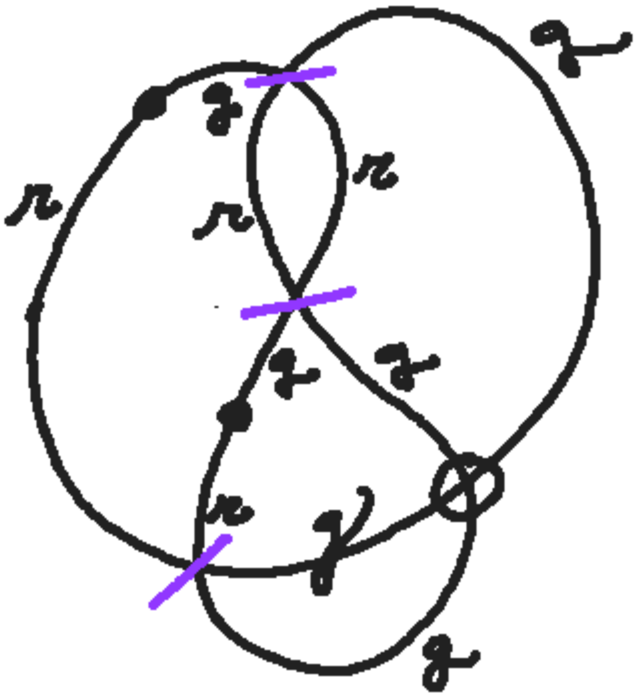


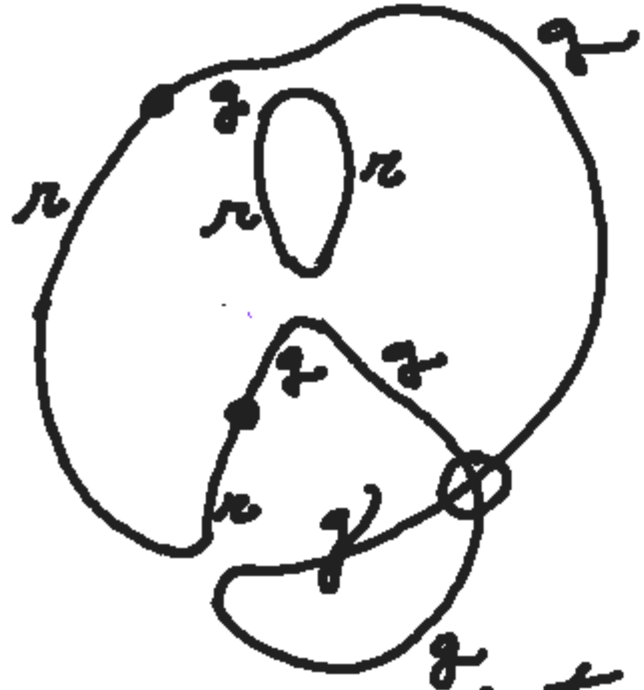
In a virtual diagram with cut-points added via 

the 2 color smoothing *is* the virtual Seifert smoothing.

example.







$$\boxed{\partial \gamma = \phi}$$

$\overset{\partial}{\longmapsto} \phi$

since $rg = \phi$.

Note that in the virtual case there <u>can</u> be self-touching sites in the Seifert state. When the link is <u>positive</u> then all A-smoothings are Seifert smoothings.

Thus the Lee cycle γ <u>as</u> <u>an A-state</u> is a non-trivial homology class.

Spanning Surfaces for Knots and Virtual Knots.

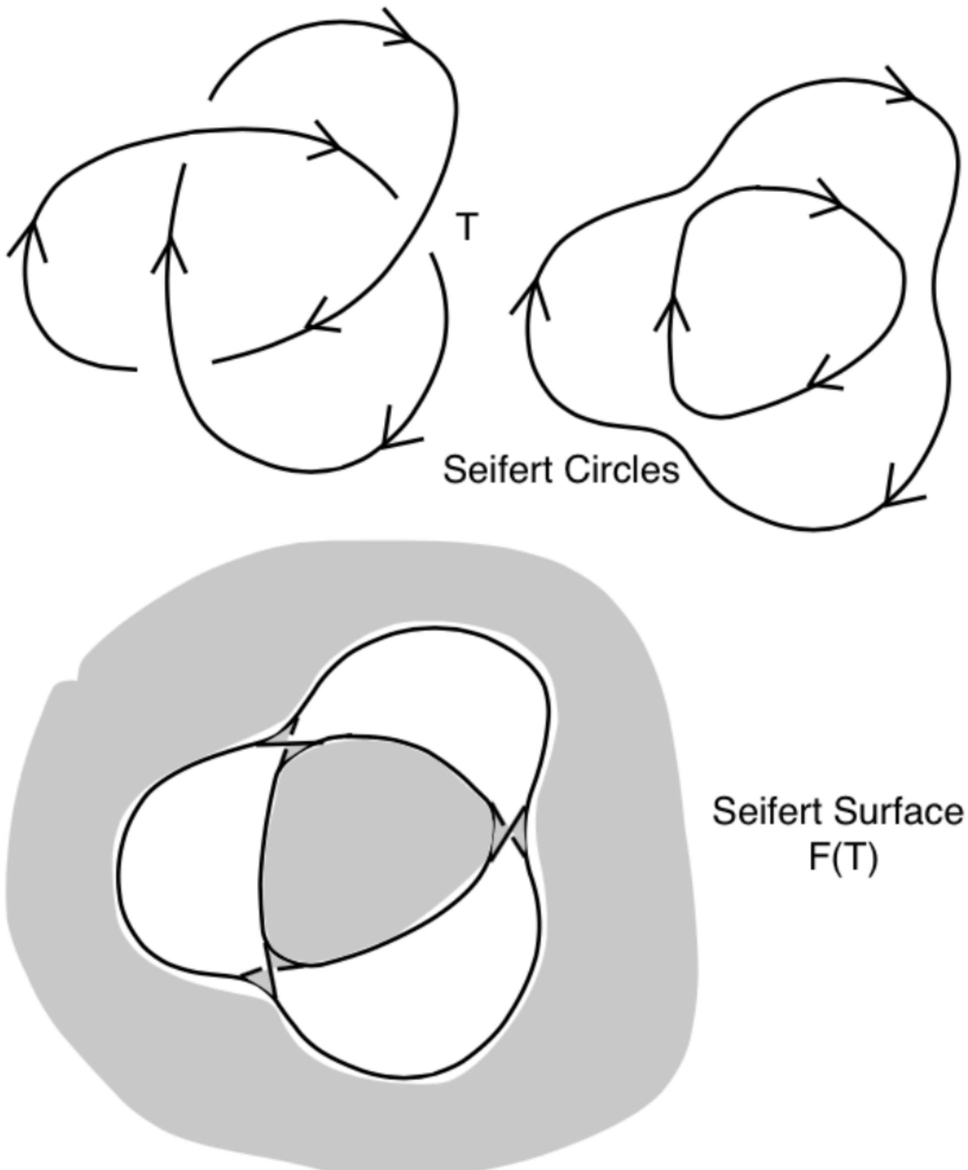
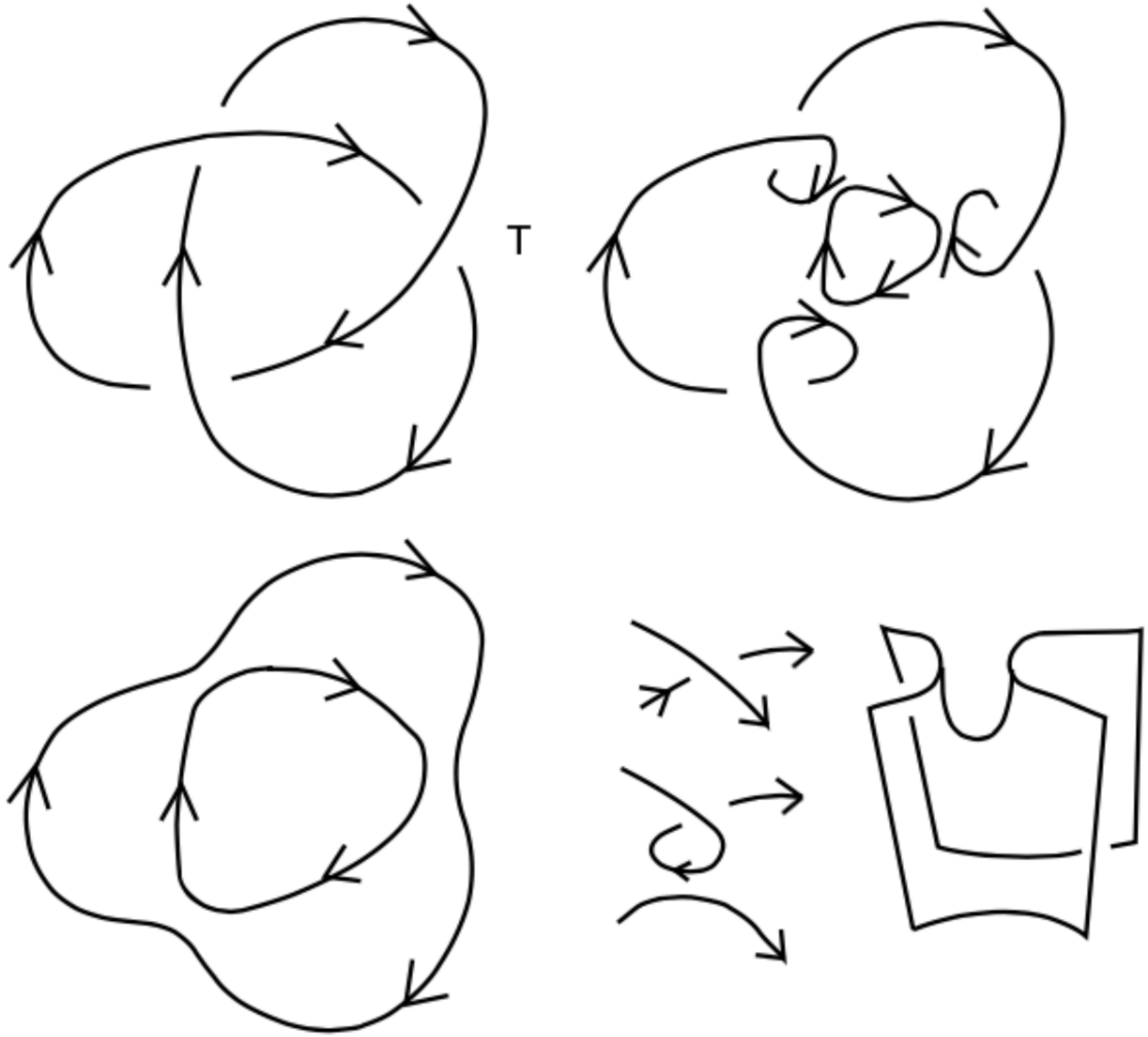
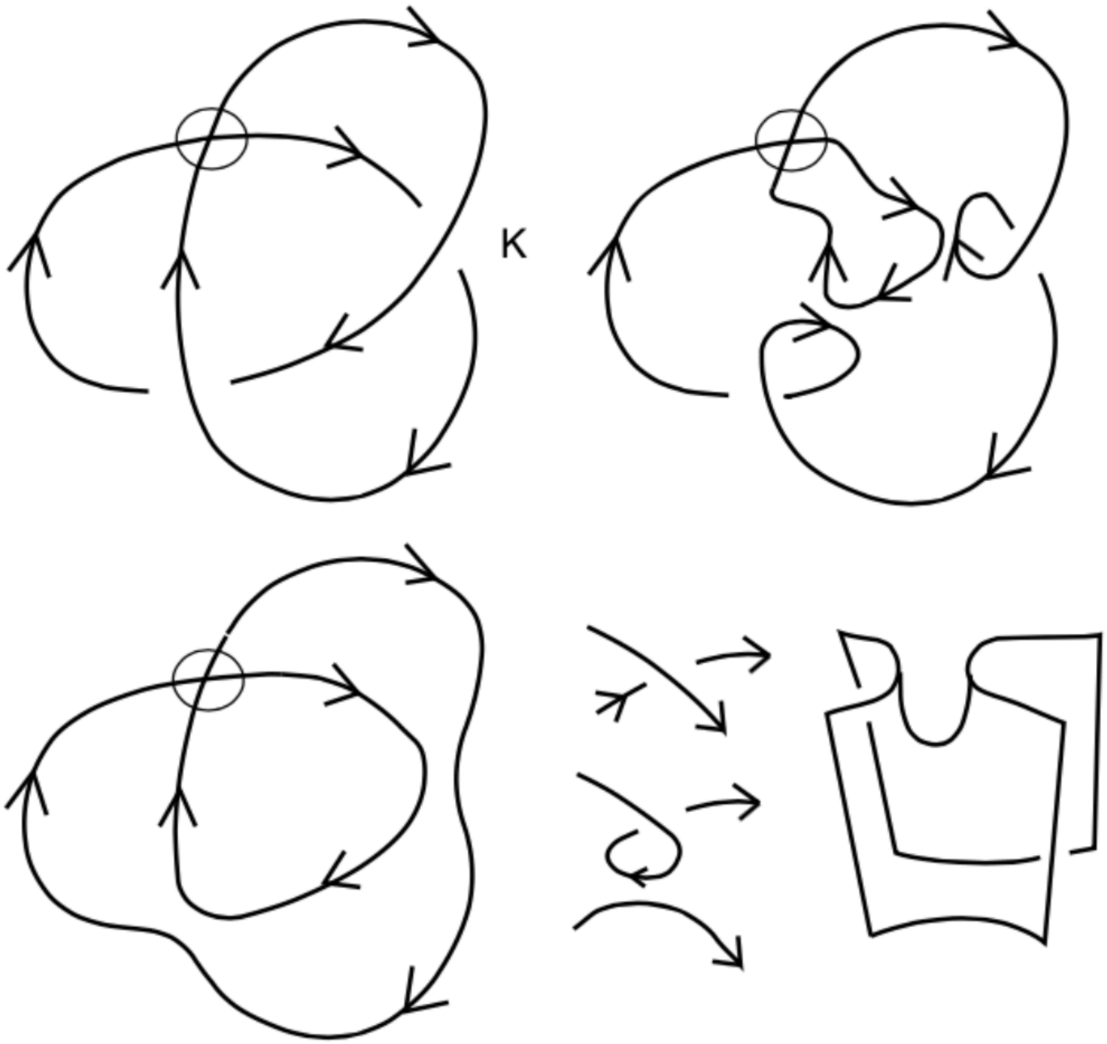


Figure 18: **Classical Seifert Surface**



Every classical knot diagram bounds a surface in the four-ball whose genus is equal to the genus of its Seifert Surface.

Figure 19: **Classical Cobordism Surface**

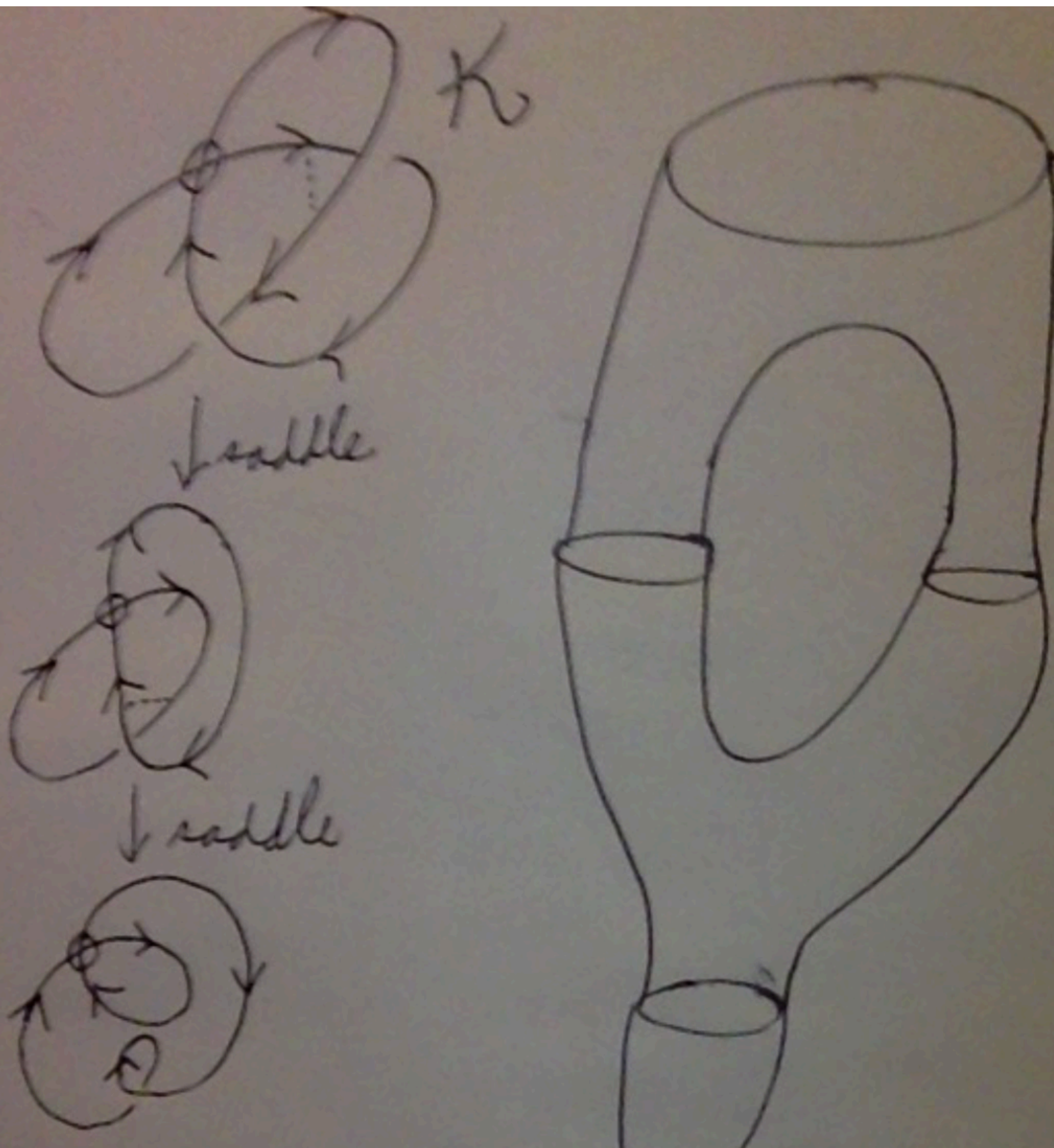


Seifert Circle(s) for K

$$r = 1,$$
$$n = 2,$$
$$g = (1/2)(-1 + 2 + 1) = 1$$

Every virtual diagram K bounds a virtual orientable surface of genus $g = (1/2)(-r + n + 1)$ where r is the number of Seifert circles, and n is the number of classical crossings in K. This virtual surface is the cobordism Seifert surface when K is classical.

Figure 20: **Virtual Cobordism Seifert Surface**



K bounds a surface of genus 1 in the virtual 4-ball.

Heather Dye, Aaron Kaestner and LK, prove the following generalization of Rasmussen's Theorem, giving the four-ball genus of a positive virtual knot.

Theorem [2]. Let K be a positive virtual knot (all classcial crossings in K are positive), then the four-ball genus $g_4(K)$ is given by the formula

$$g_4(K) = (1/2)(-r + n + 1) = g(S(K))$$

where r is the number of virtual Seifert circles in the diagram K and n is the number of classcial crossings in this diagram. In other words, that virtual Seifert surface for K represents its minimal four-ball genus.

The virtual Seifert surface for positive virtual K represents the minimal four-ball genus of K.

The Theorem is proved by generalizing both Khovanov and Lee homology to virtual knots and generalizing the Rasmussen invariant to virtual knots.