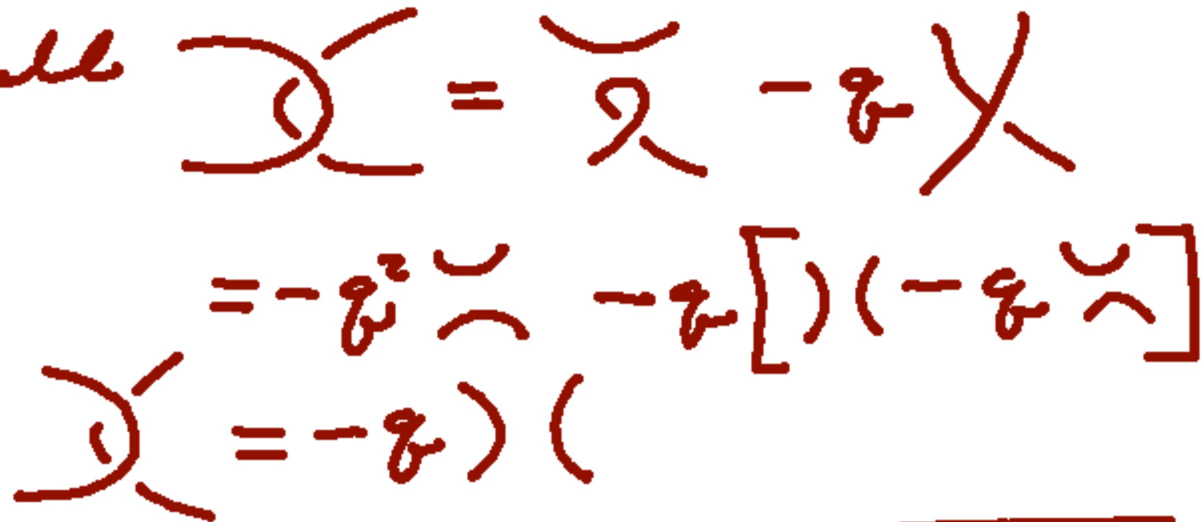
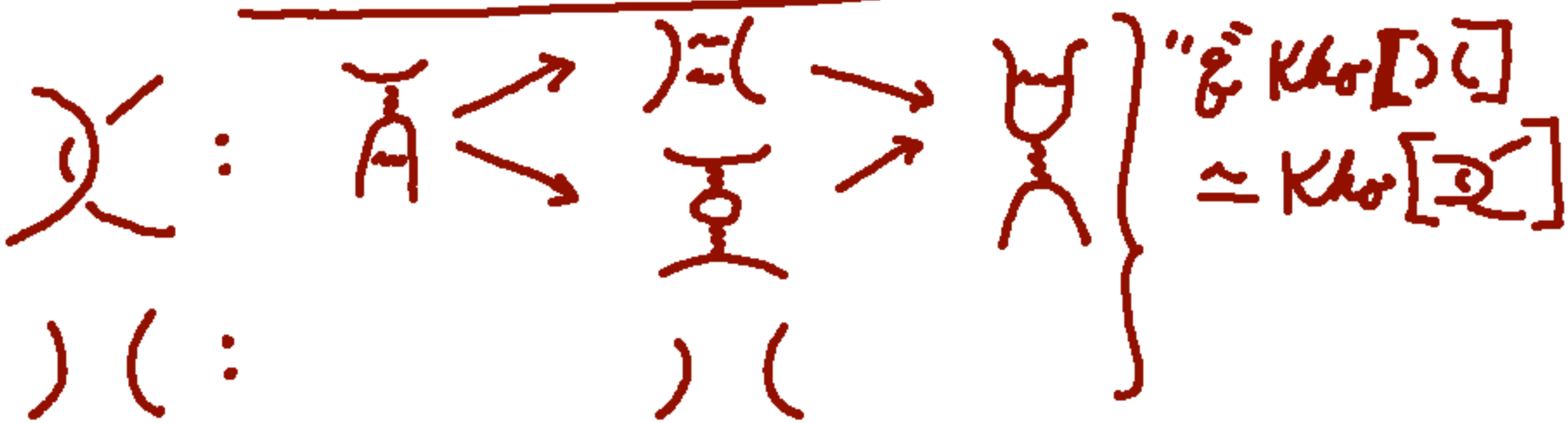
Lecture 8

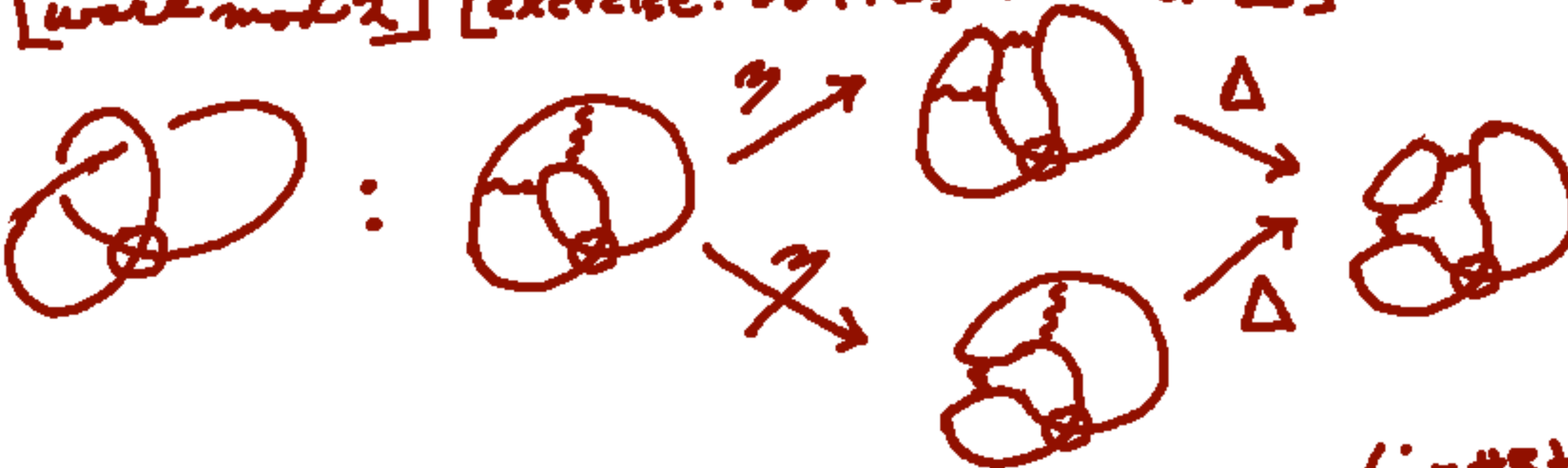
Examples

1. Recall





[work mod 2] [exercise: do it again over $\mathbb{Z}$]



$\left(\begin{array}{l} i = \#B + \lambda \\ j = i + \lambda \end{array}\right)$

$i \backslash j$	−1	0	1	2	3	4
0	$\mathbb{Z}/2$		1			
1		$(\mathbb{Z}/2, 0)$ $(0, \mathbb{Z}/2)$		(1, 0) (0, 1)		
2		$\mathbb{Z}/2 \oplus \mathbb{Z}/2$				$1 \oplus 1$

K Ex: $\boxed{\text{mod } 2}$

E

	−1	0	1	2	4
0	$\mathbb{Z}$		$\mathbb{Z}$		
1		$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}$	
2					$\mathbb{Z}$

$i \backslash j$	-1	0	1	2
0	x 4		1 4	
1		x 3		1 6

Family

K_n

$n = 1, 2, 3, \ldots$

Next week programs at least mod 2.

Lee's Algebra

$x^2 = 1$ Eun Soo Lee

$\varepsilon(x) = 1, \ \varepsilon(1) = 0$

$\Delta(1) = 1 \otimes x + x \otimes 1$

$\Delta(x) = x \otimes x + 1 \otimes 1$

$r = \frac{1+x}{2}$ $\qquad \varepsilon(r) = 1/2$

$g = \frac{1-x}{2}$ $\qquad \varepsilon(g) = -1/2$

$r^2 = r, \ g^2 = g$

$rg = \frac{1}{4}(1+x)(1-x) = \frac{1}{4}(1-x+x-1) = 0.$

$r + g = 1.$

$xr = (x + x^2)/2 = (x+1)/2 = r$

$xg = (x - x^2)/2 = (x-1)/2 = -g$

$$\Delta(r) = \frac{\Delta(1) + \Delta(x)}{2} = \frac{1 \otimes x + x \otimes 1 + x \otimes x + 1 \otimes 1}{2} = \frac{(1+x) \otimes (1+x)}{2} = 2r \otimes r$$

$$\Delta(g) = \frac{\Delta(1) - \Delta(x)}{2} = \frac{1 \otimes x + x \otimes 1 - x \otimes x - 1 \otimes 1}{2}$$

$$= -2\left(\frac{1-x}{2}\right) \otimes \left(\frac{1-x}{2}\right) = -2g \otimes g$$

Lee Algebra

gen by r, g with $r + g = 1$.

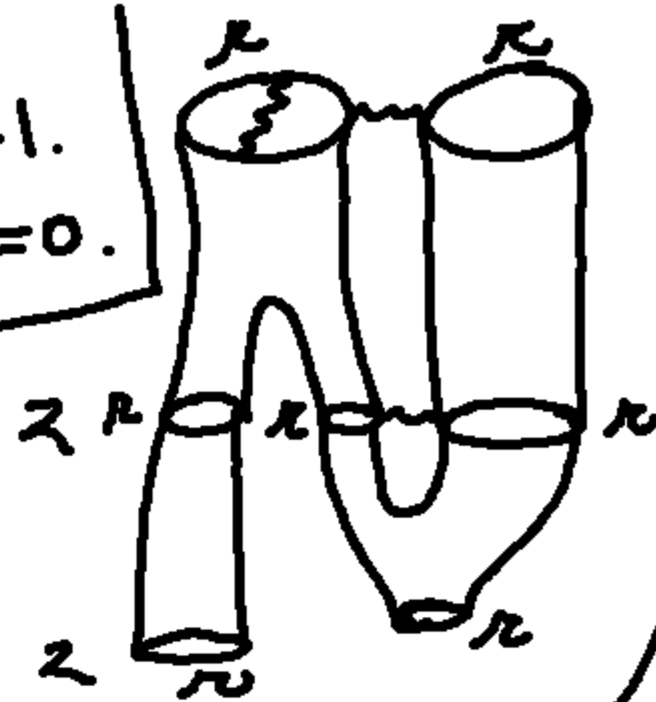
$\varepsilon(r) = \frac{1}{2}$, $\varepsilon(g) = -1/2$, $\varepsilon(1) = 0$.

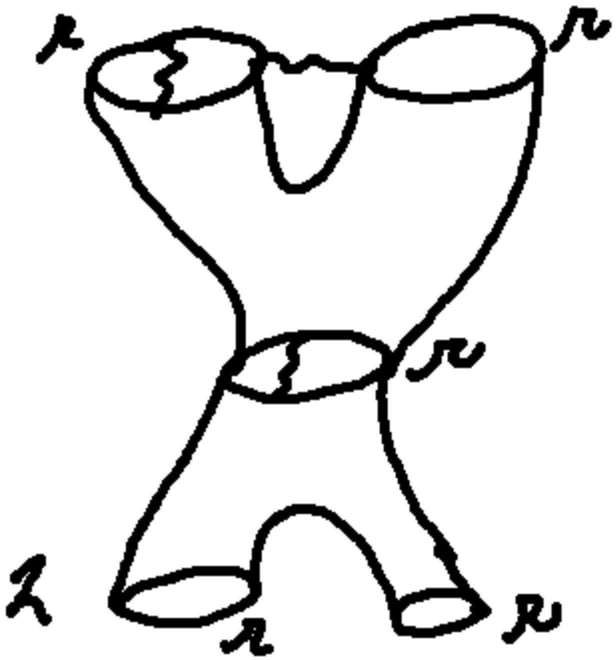
$r^2 = r$, $g^2 = g$

$rg = gr = 0$

$\Delta(r) = 2r \otimes r$

$\Delta(g) = -2g \otimes g$





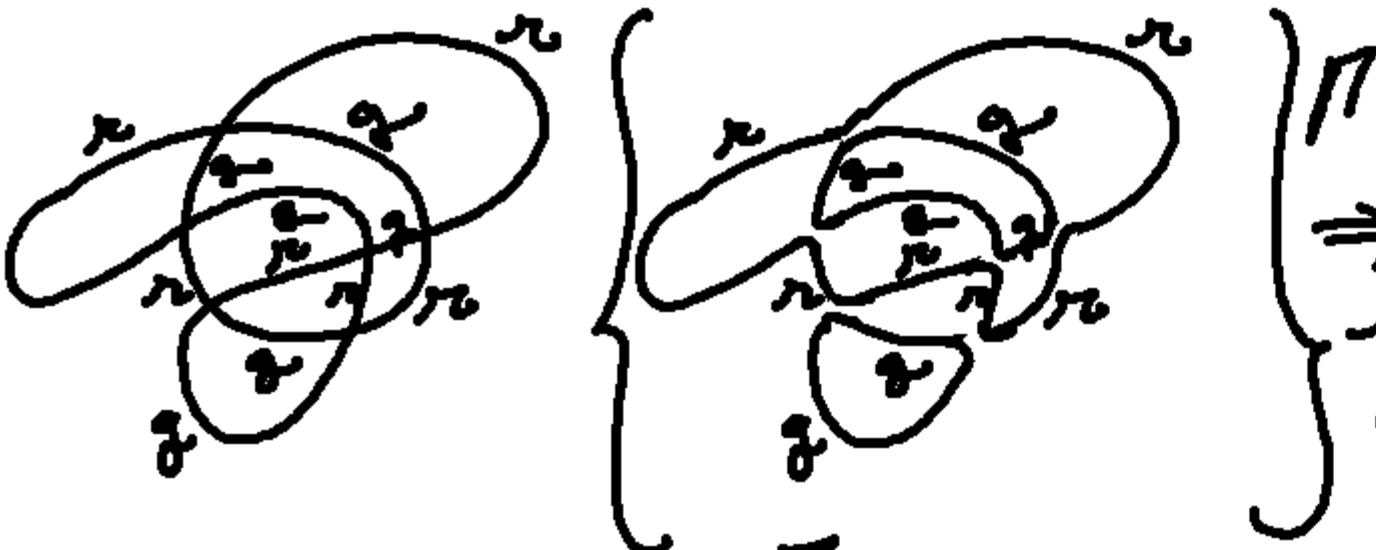
Lee Algebra

Gen by r, g with $r+g=1$.
$\varepsilon(r)=\frac{1}{2}$, $\varepsilon(g)=-1/2$, $\varepsilon(1)=0$.
$r^2=r$, $g^2=g$
$rg=gr=0$
$\Delta(r)=2r\otimes r$
$\Delta(g)=-2g\otimes g$

Now let's consider classical diagrams and try making Lee cycles.
Consider this:

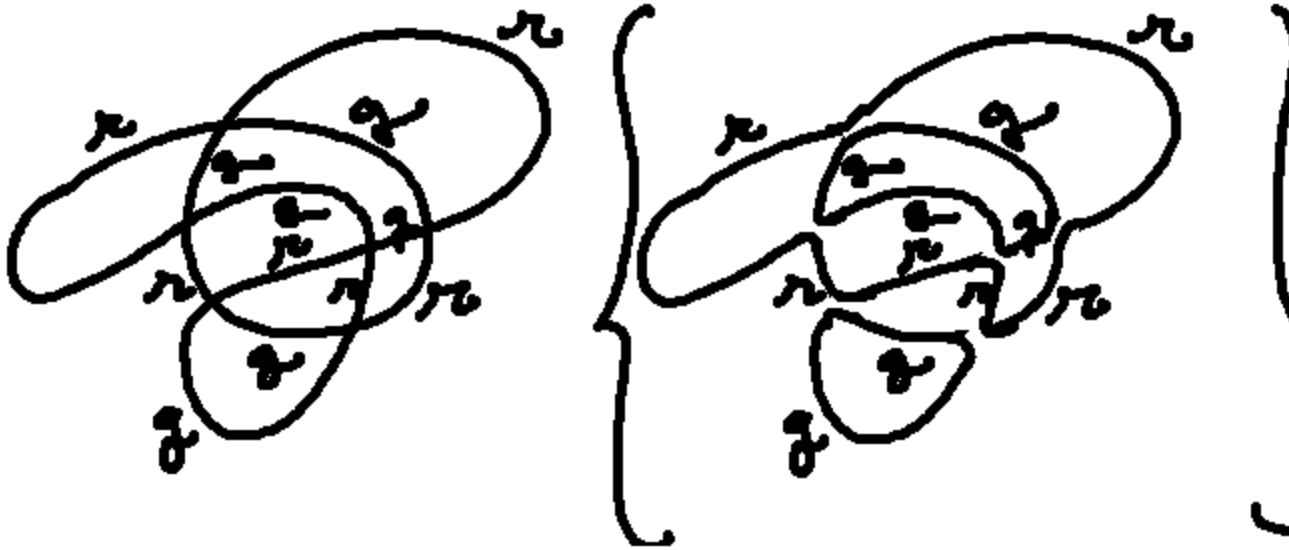
If these are on different loops,
then $\partial \leftrightarrow$ mult and $rg=0$.

So we can try coloring diagrams with 2 colors: r & g.

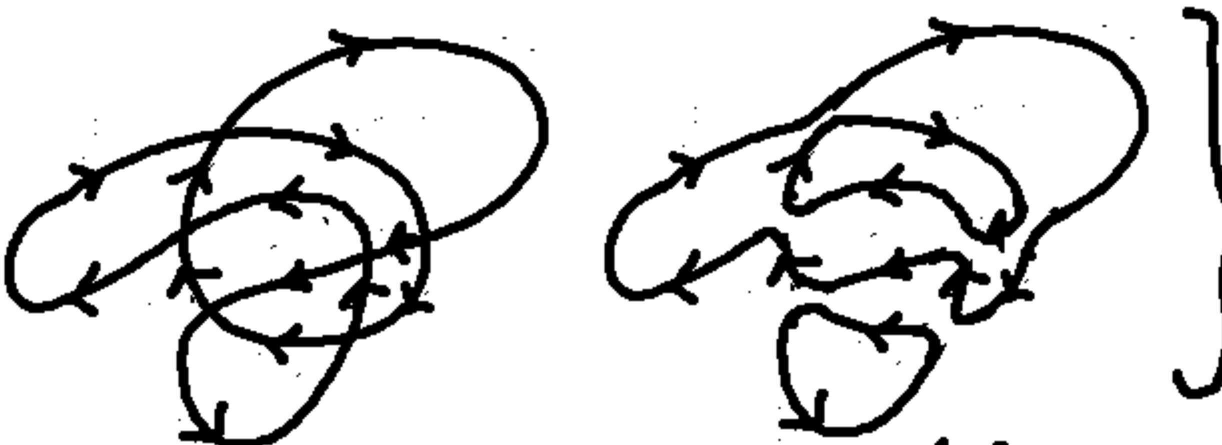


$\Rightarrow \partial\Gamma = \phi$ in Lee Homology.

Where this is in the complex depends on how the crossings are set & which are A or B smoothings.



$\Gamma \Rightarrow \partial\Gamma = \phi$ in Lee Homology.



Seifert Smoothing

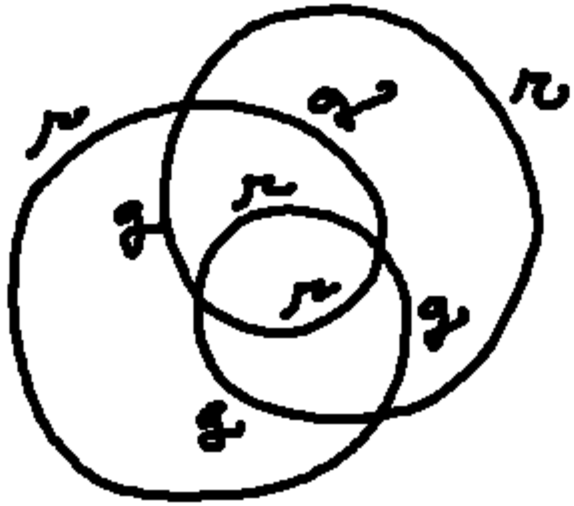
For classical diagrams the color smoothing is same as Seifert circles smoothing!
(to be proved in next lecture)

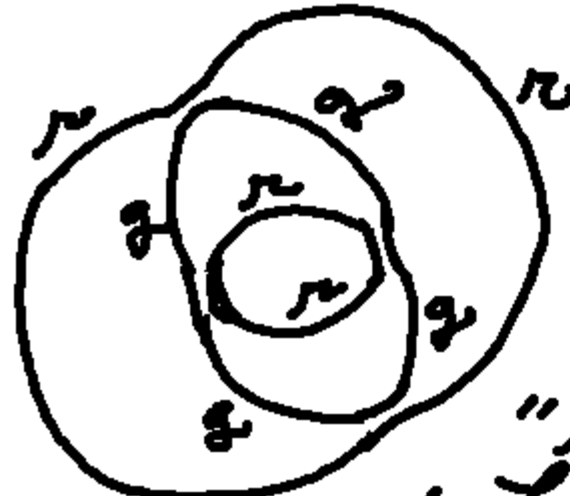
Note that a Seifert circle in a classical diagram cannot have a self-site because

at such a site we would have

opposite orientation and all Seifert smoothing sites are of the form

∴ Any 2-coloring of a classical diagram gives a Γ with $\partial\Gamma = \phi$ in Lee Homology?

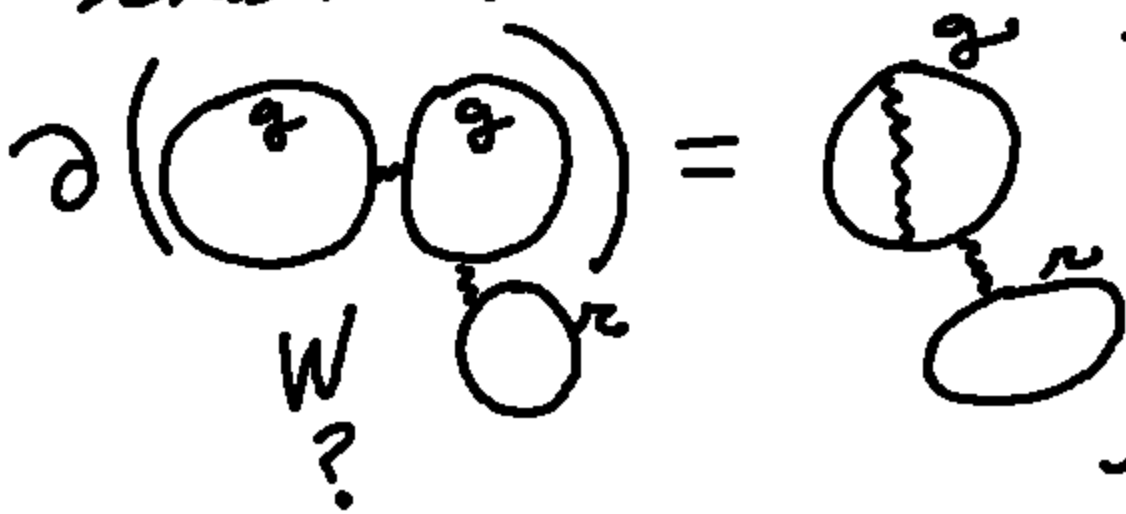




$\Gamma : \partial\Gamma = \phi$

We can construct "Seifert" Lee cycles Γ.

Could a Lee cycle Γ be $= \partial W$ for some element W in the Lee chains?



But Seifert Lee cycles have no self smoothings. So this is not an example.

Now recall how we uplevel to virtuals:

cut points.

And we shall define $\bar{r} = s(r) = g$

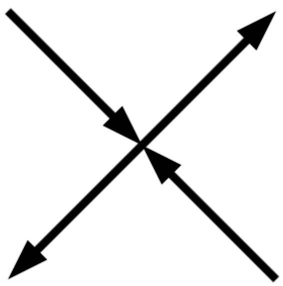
$\bar{g} = s(g) = r.$

across cut pts.

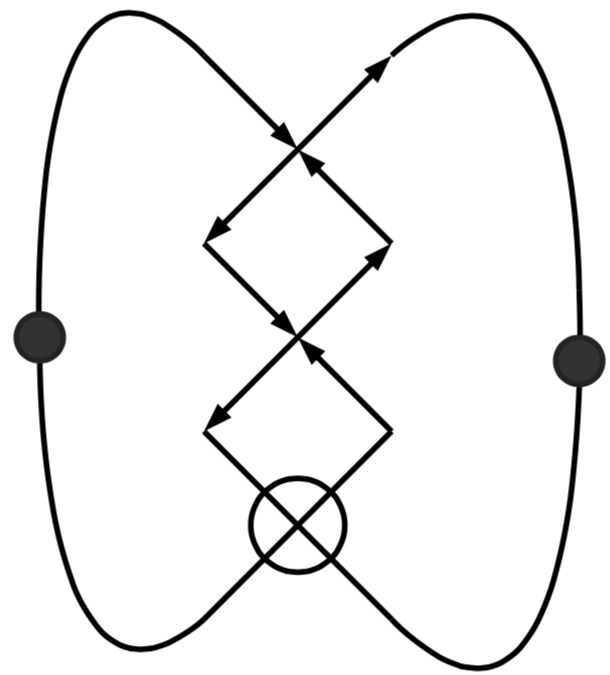
With the cut points in place these are the "virtual Seifert smoothings."

So we can generalize our results to virtuals <u>but</u> self-touching loops can occur.

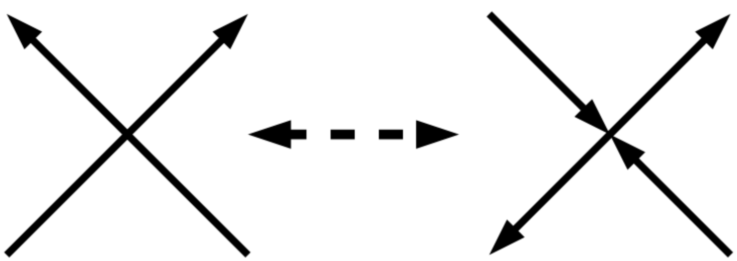
Remarks on Generalizing Khovanov Homology, Lee Homology and Rasmussen Invariant



: Source-sink orientation

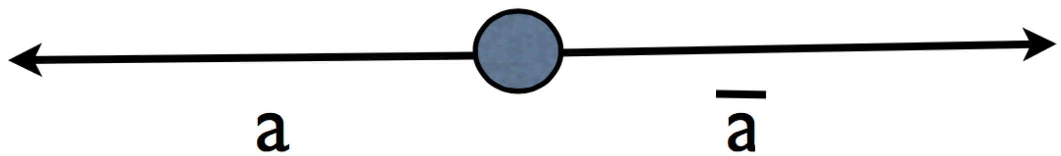


Cut loci for a two-crossing virtual knot



Canonical Source-Sink Orientation

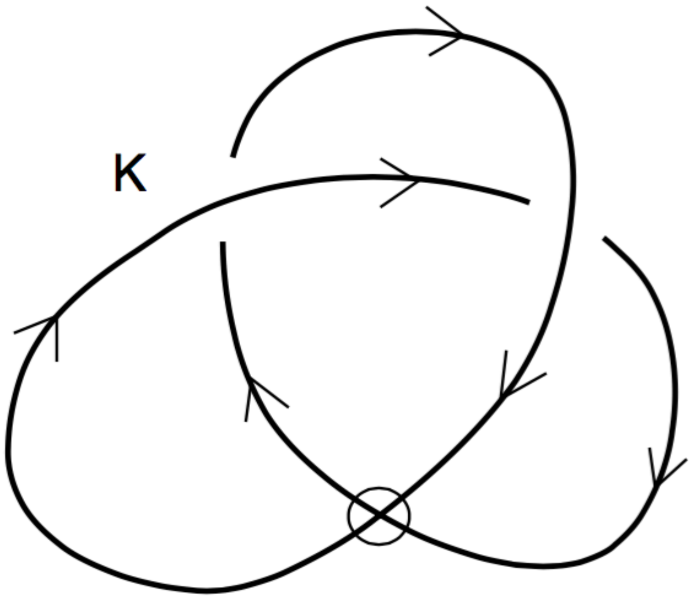
Cut Locus Involution

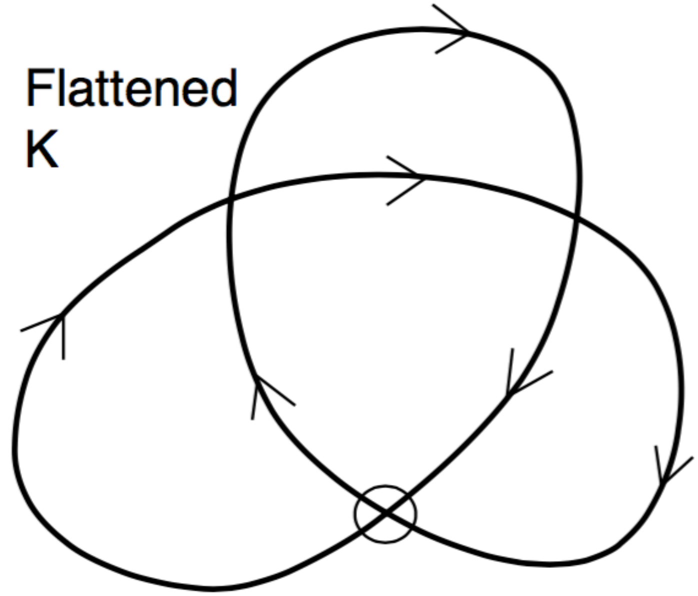


The Frobenius algebra controlling the Khovanov homology differentials has
an order two function

$$a \longrightarrow \bar{a}$$

that is applied whenever an algebra element is moved across a cut locus.





Lee Algebra

$rg = gr = 0$

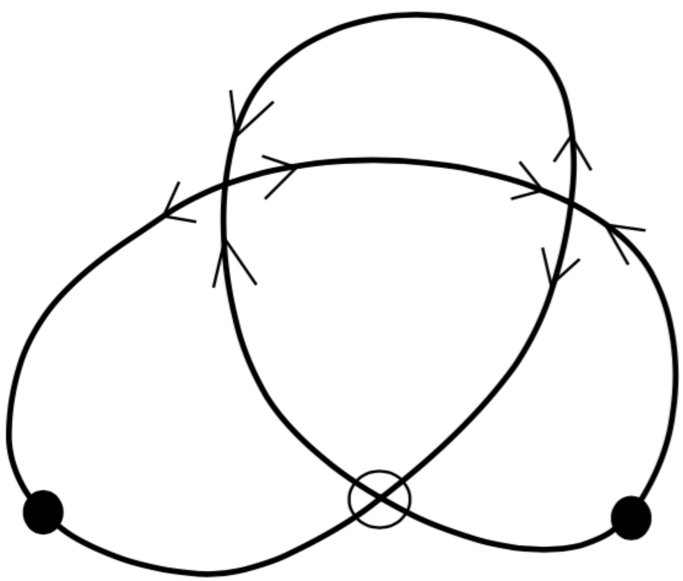
$rr = r$

$gg = g$

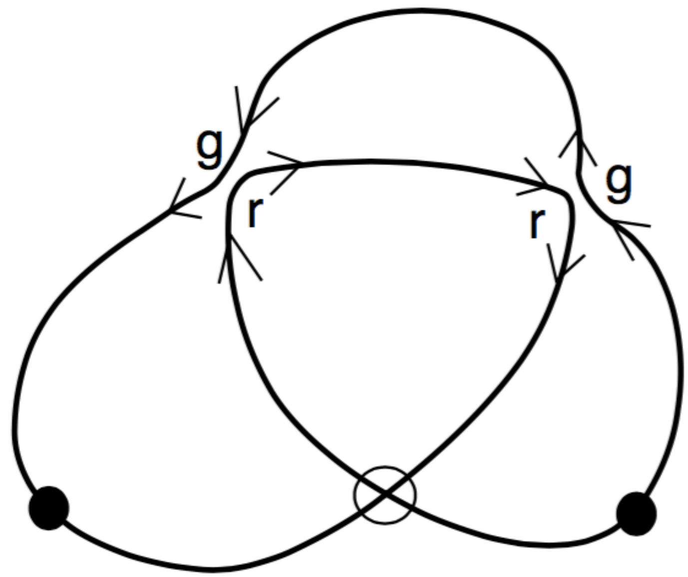
$r + g = 1$

$D(r) = 2r$

$D(g) = 2g$



K with canonical source sink orientations and cut loci

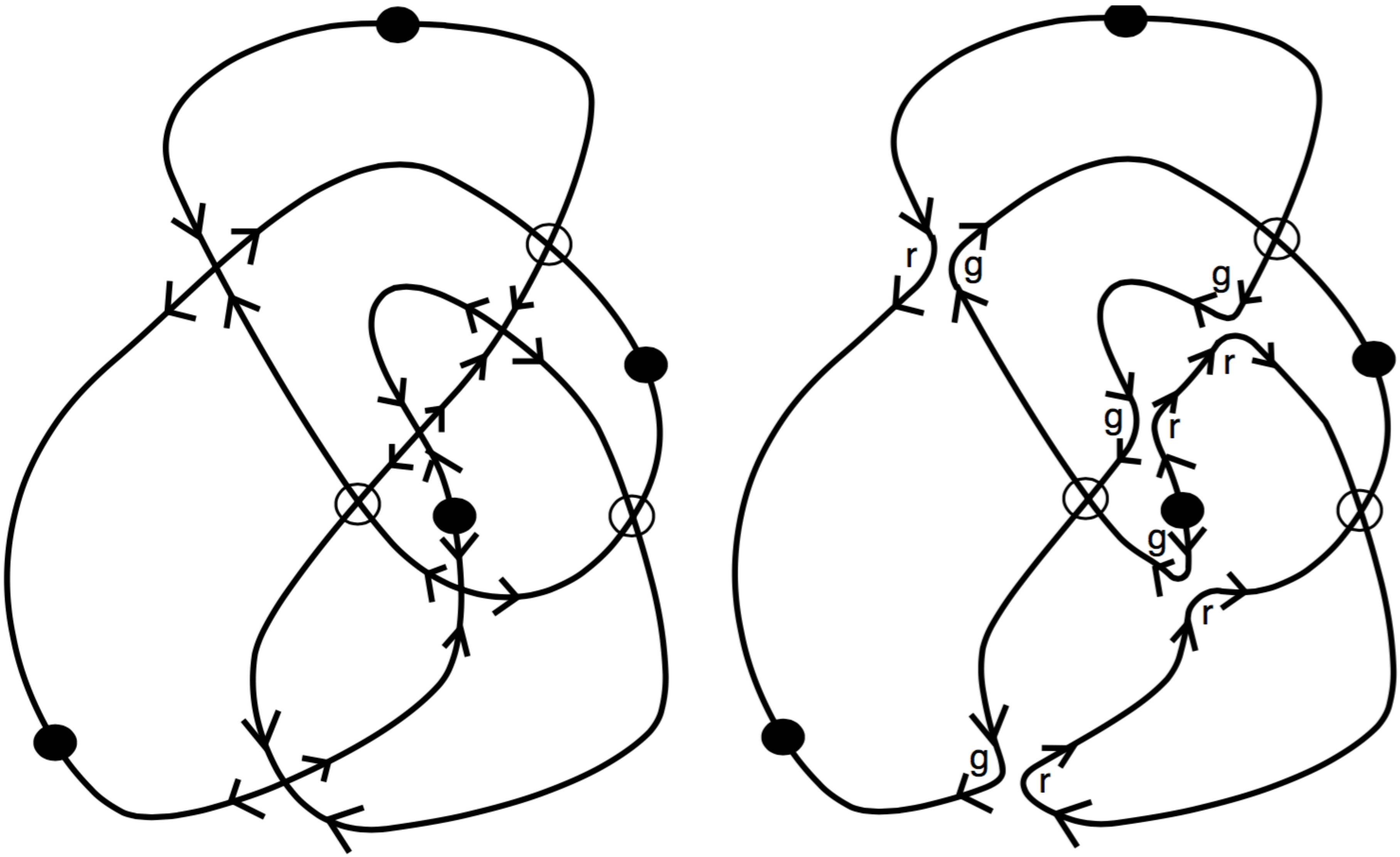


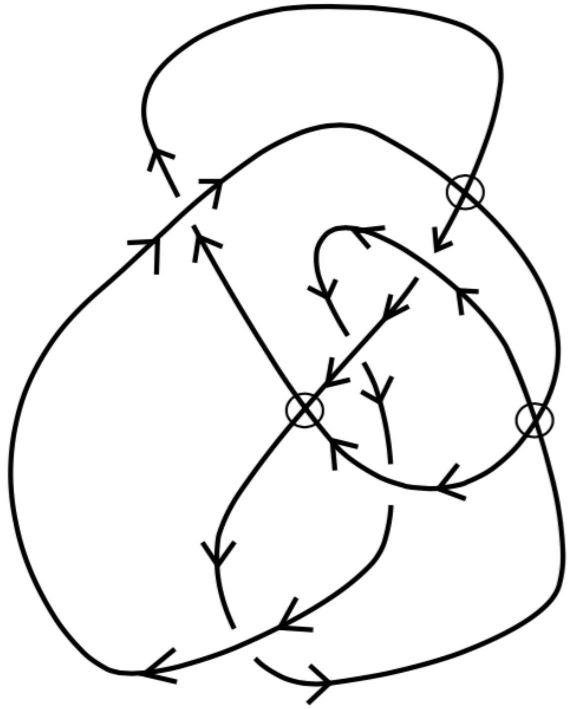
Seifert state labelled with Lee algebra is a non-trivial cycle.

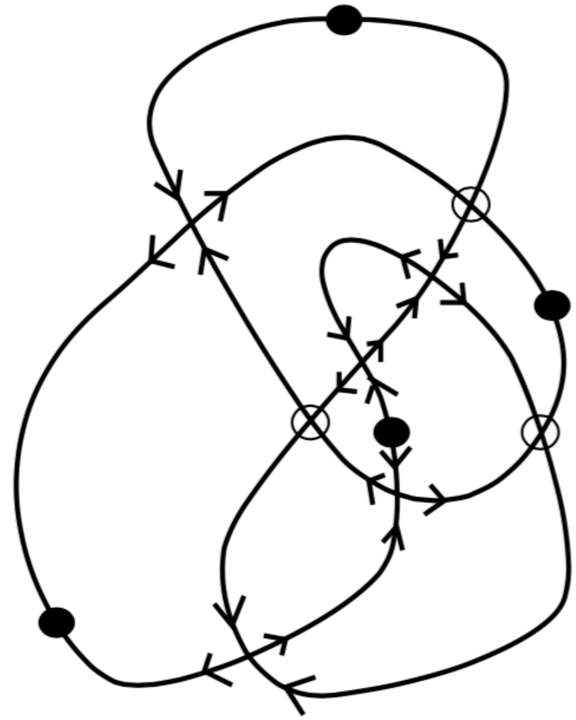
$\overline{r} = g$

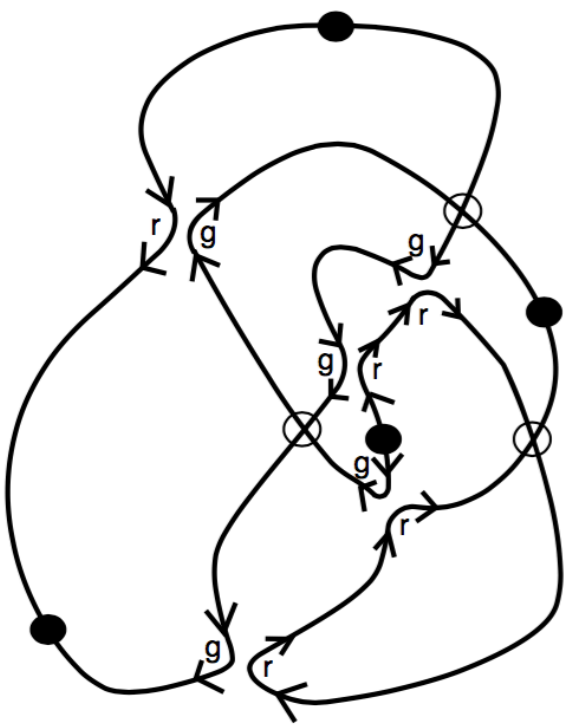
$\overline{g} = r$

Another Example of a Virtual Lee Cycle









genus = (1/2)(-r + n +1)
 = (1/2)(-2 + 5 +1)
 = 2.

Digression about a model for the Conway-Alexander Polyn $\nabla_K(z)$.

Conway's Axioms

1. There is a polyn $\nabla_K(z)$ assoc with oriented link K s.t. $K \underset{(R1,2,3)}{\sim} K' \Rightarrow \nabla_K = \nabla_{K'}$.

2. $\nabla_{\bigcirc} = 1$

3. 

Model for "Formal Knot Theory" by LK. $R = C+2$
(1 marker/region) regions crossings

K-states

$\bar{s} = s^{-1} = \frac{1}{s}$

S (a state)

$$\boxed{\nabla_K = \sum_S \langle K|S\rangle}\,, \qquad \underline{z = s - \bar{s}}$$

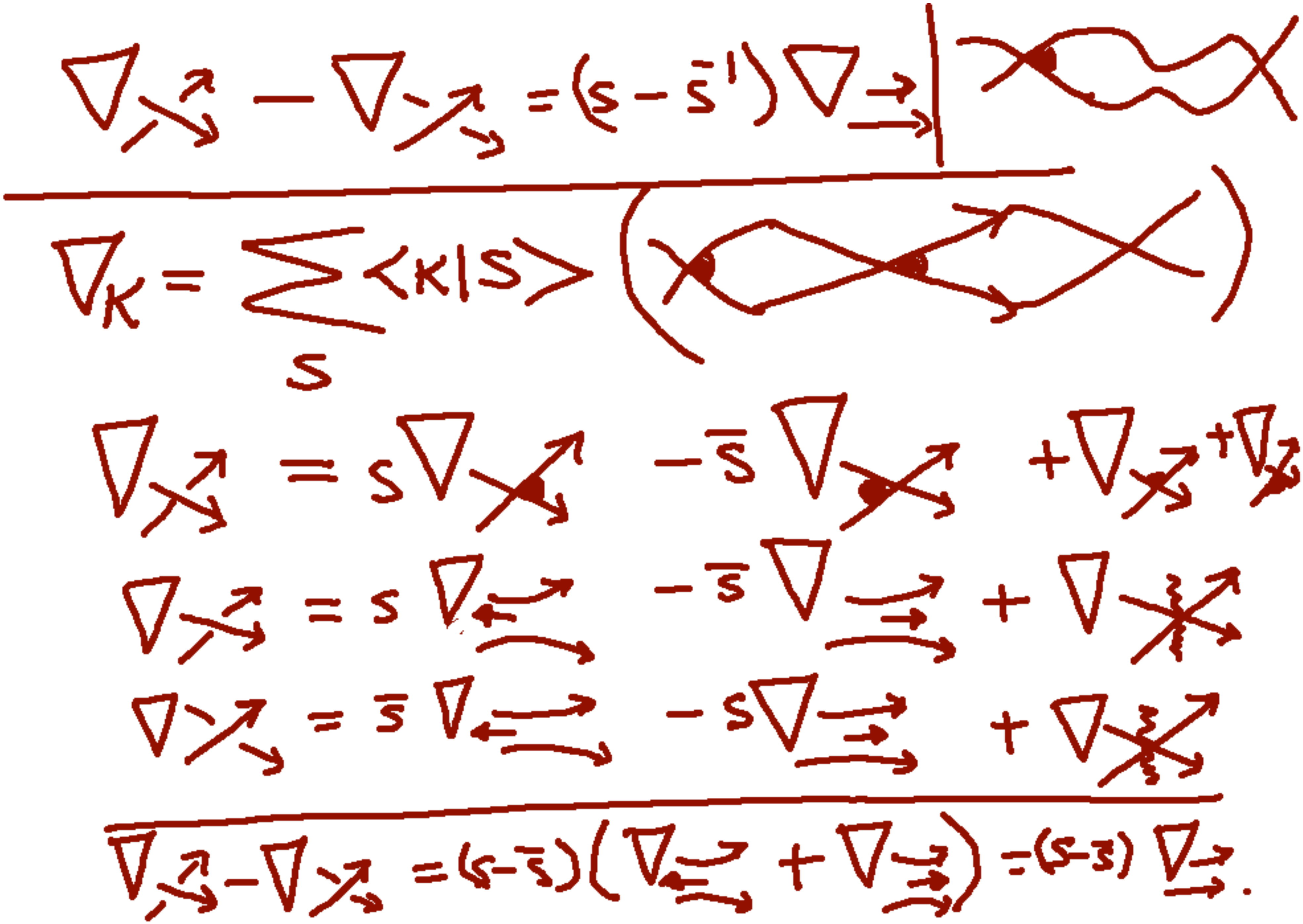
$\langle K|S\rangle$ = Product of bits of algebra touched by markers.

-1

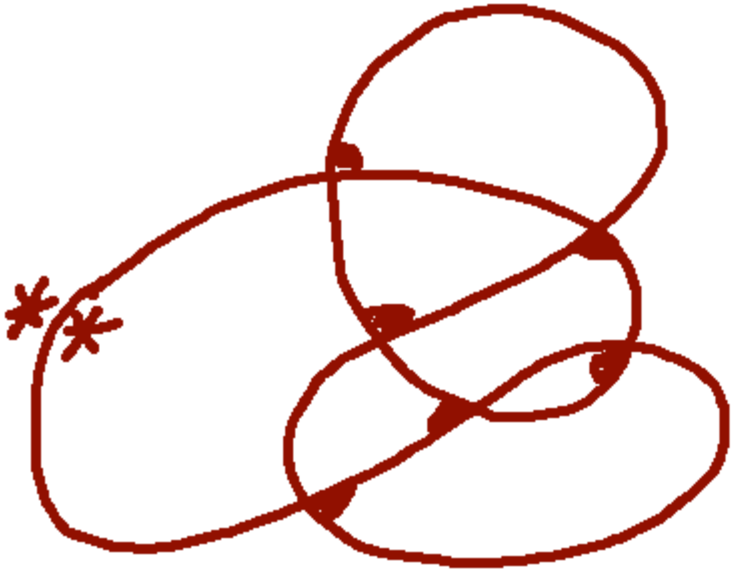
$+\bar{s}^2$

$+s^2$

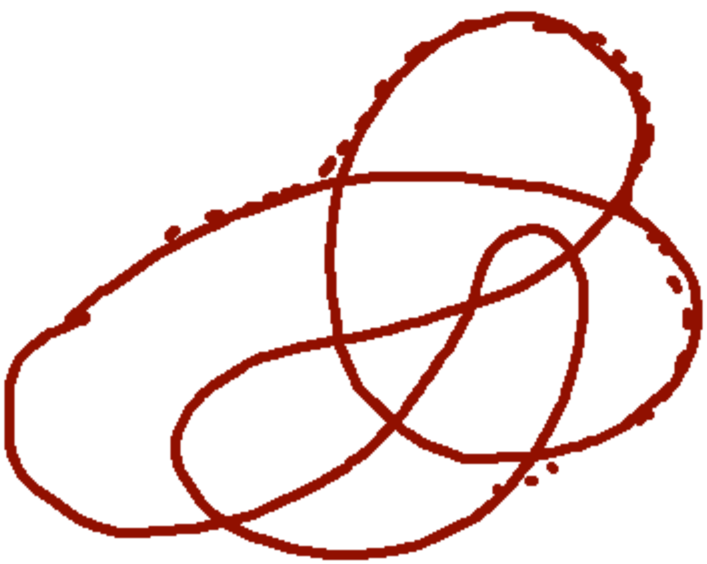
$$\nabla_K = -1 + \bar{s}^2 + s^2 = 1 + (s - \bar{s})^2 = 1 + z^2.$$



$\nabla_K = \sum_S \langle K|S \rangle$
$= (s - \bar{s}')\nabla$
$= s\nabla \quad -\bar{s}\nabla \quad +\nabla \quad +\nabla$
$= s\nabla \quad -\bar{s}\nabla \quad +\nabla$
$= \bar{s}\nabla \quad -s\nabla \quad +\nabla$
$= (s-\bar{s})(\nabla + \nabla) = (s-\bar{s})\nabla$



single cycle state



These states are a basis for the chain complex for the Ozsvath + Szabo Link Homology that categorifies $\nabla_K(z)$.

Let's $S(K)$ = $\overset{\text{Set of}}{\text{K-states}}$ $\boxed{\sum_A \xrightarrow{\partial} \sum_B (}$

forming a basis for
$C(K)$ the O.S. chains.

O.S. define $\partial : C(K) \circlearrowleft$
using high dim'l topology.

Give a comb definition!

This a "half-open" problem.
(soln using quite a bit of alg by Ozsvath.)

Still can we make a combinatorially
simple soln?

? 

can we make a ∂ from
this?

? Link Homology of O.S. type for knotoids.

O.S. = Knot Floer Homology has comb descrip via grid diags.

How are Marker States related to "grid combinatorics".