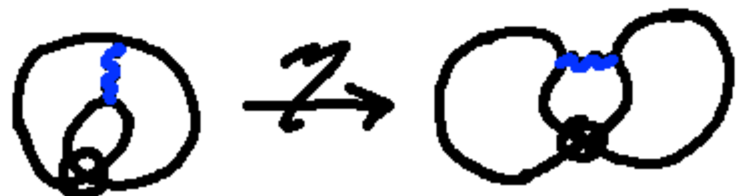
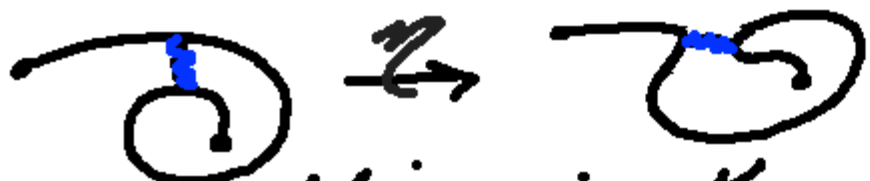
Lecture # 7

We begin with an overlap with lecture 6 and then continue further.

Note: 

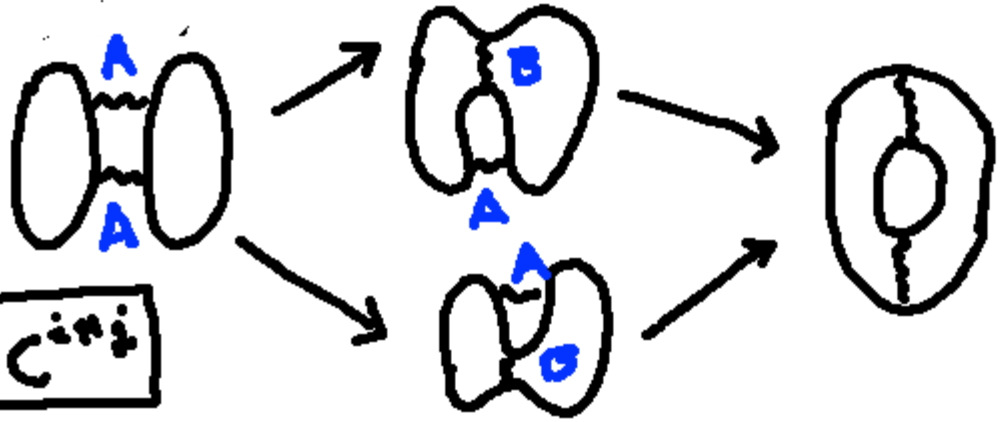
A re-smoothing in virtual category can be from one loop to one loop.

Note: 

A re-smoothing in Knotoid category can be from one long to one long.

These facts mean that Khovanov Homology for virtuals and knotoids requires special attention.

H : 



$\boxed{\partial : C^{i,j} \to C^{i+1,j}}$

$i \backslash j$	-2	0	2	4
0	$\mathbb{Z}$	$\mathbb{Z}$	0	
1	0	0	0	
2	0	0	$\mathbb{Z}$	$\mathbb{Z}$

$\downarrow \partial$

$i \backslash j$	-2	0	2	4
0	$x\otimes x$	$x\otimes 1$, $1\otimes x$	$1\otimes 1$	
1		$(x,0)$, $(0,x)$	$(1,0)$, $(0,1)$	
2		$x\otimes x$	$x\otimes 1$, $1\otimes x$	$1\otimes 1$ $\boxed{0}$

-2 $\boxed{\partial(x\otimes x) = 0}$

$\boxed{\begin{aligned}\partial(x,0) &= x\otimes x\\ \partial(0,x) &= x\otimes x\end{aligned}}$

$\partial(x\otimes x) = 0$

Im $\partial : C^{00} \to C^{10}$ is gen by $(x,-x)$.

What are cycles in C^{10}? gen by $(x,-x)$.
$\Rightarrow$ homol $= 0$

$\partial(x\otimes 1) = (x,0) - (0,x)$
$= (x,-x)$

$\partial(1\otimes x) = (x,-x)$

$\partial(x\otimes 1 - 1\otimes x) = 0$
cycles

$\boxed{2}$

$\partial(1\otimes 1) = (1,0) - (0,1) = (1,-1)$

$\partial(1,0) = 1\otimes x + x\otimes 1 = \partial(0,1)$

$[$$]$:

$i=0$

$i=1$

$j=0+\lambda \qquad\qquad 1+\lambda$

$$
\begin{aligned}
&= [\,\circ\circ\,] - q[\,\circ\,] \\
&= \delta^2 - q\delta \\
&= \delta(\delta - q) \\
&= \underline{\underline{(q+q^{-1})(q^{-1})}} \\
&= 1 + q^{-2}
\end{aligned}
$$

$i \backslash j$	-2	0	2
0	$x\otimes x$	$x\otimes 1$, $1\otimes x$	$1\otimes 1$
1		x	1

$\partial(x\otimes x)=0$

$\partial(x\otimes 1)=x$

$\partial(1\otimes x)=x$

$\partial(1\otimes 1)=1$

$i \backslash j$	-2	0	2
0	$\mathbb{Z}$	$\mathbb{Z}$	0
1	0	0	0

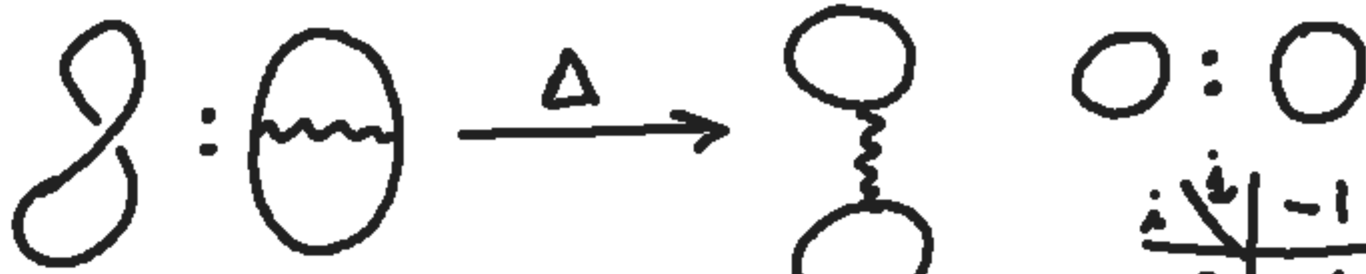
$q^{-2}+q^{0}=q^{-2}+1$

$\bigcirc$:

$i \backslash j$	-1	1
0	x	1

$i \backslash j$	-1	1
0	$\mathbb{Z}$	$\mathbb{Z}$

Note that grading shift is: "q" $\mathrm{KhoH}(\bigcirc) = \mathrm{KhoH}(\infty)$



$i \backslash j$	-1	1	3
0	x	1	
1	$x \otimes x$	$x \otimes 1$, $1 \otimes x$	$1 \otimes 1$

$\partial x = x \otimes x$
$\partial 1 = 1 \otimes x + x \otimes 1$

$i \backslash j$	-1	1
0	x	1

$i \backslash j$	-1	1
0	$\mathbb{Z}$	$\mathbb{Z}$

$q^{-1} + q$

$i \backslash j$	-1	1	3
0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$

$-q - q^3 = -q^2(q^{-1} + q)$

[] $= -q^2$ []

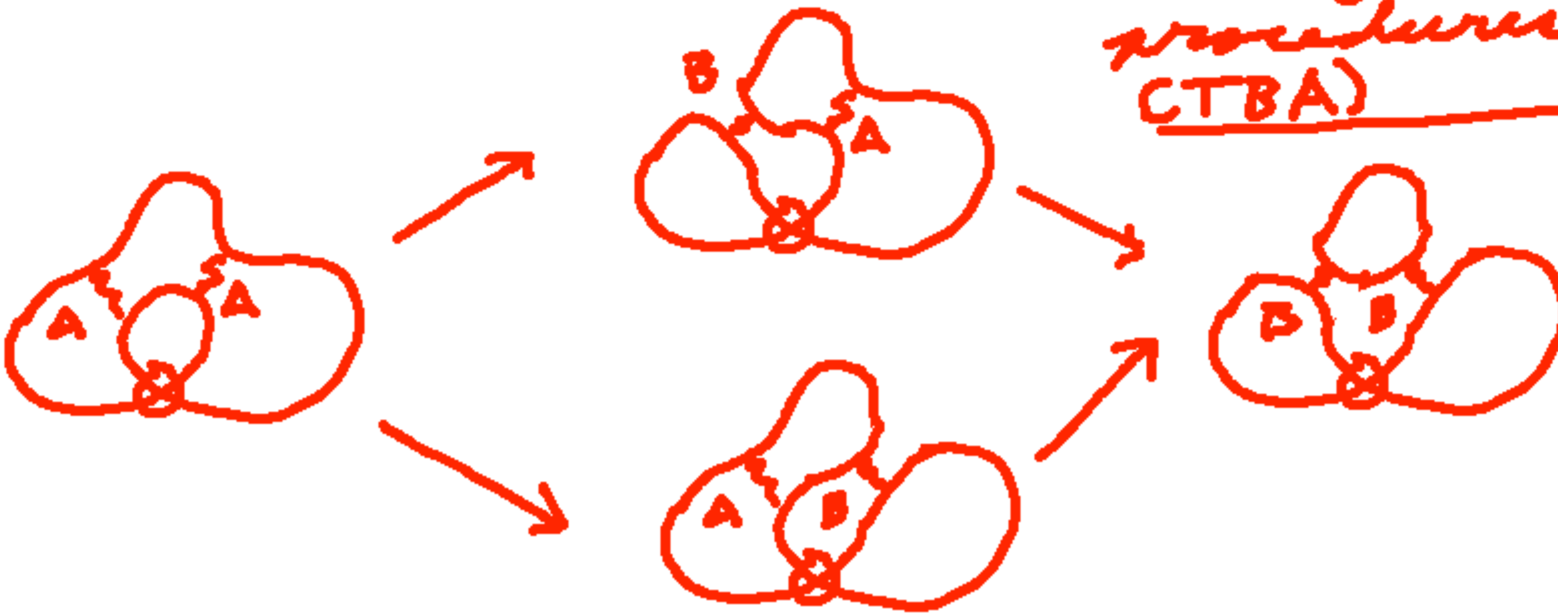
$=$

$-q^2$ KhoH() $=$

$=$ KhoH()

Exercise: 

1) do it mod 2
2) do it again using our procedures (TBA)



Exercise. 

Exercise. 

Other examples

:

+ 8 states
+ more enhanced states

min
16 states
+ more enhanced states

Knotoids

ends not
nec. in
same region.

Include RM's as usual but not allowing passage across an end.

Knot type knotoids

No!

$$\langle O \rangle = q + q^{-1}$$

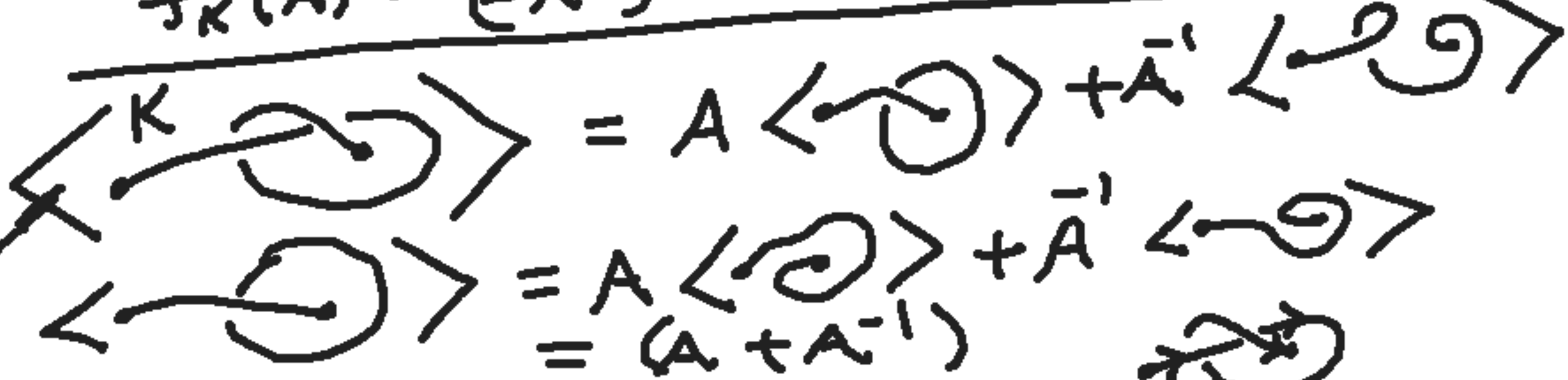
$$\langle \frown \rangle = q + q^{-1}$$

Usual $\langle\,\rangle$ poly for knotoids

$\langle O K \rangle = (-A^2 - A^{-2})\langle K \rangle$

$R2 + R3$

$f_K(A) = (-A^3)^{-wr(K)}\langle K \rangle$



$= A(A+A^{-1}) + A^{-1}(-A^{-3})$

$= A^2 + 1 - A^{-4}$

$f_K = (-A^3)^{-2}\langle K \rangle = A^{-4} + A^{-6} - A^{-10}$

$\Rightarrow K \not\simeq K^*$ non trivial

Usual $\langle\ \rangle$ poly for knotoids

$\langle$  $\rangle = A\,\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$\langle O K \rangle = (-A^{2}-A^{-2})\langle K \rangle$

$\langle$  $\rangle = 1$

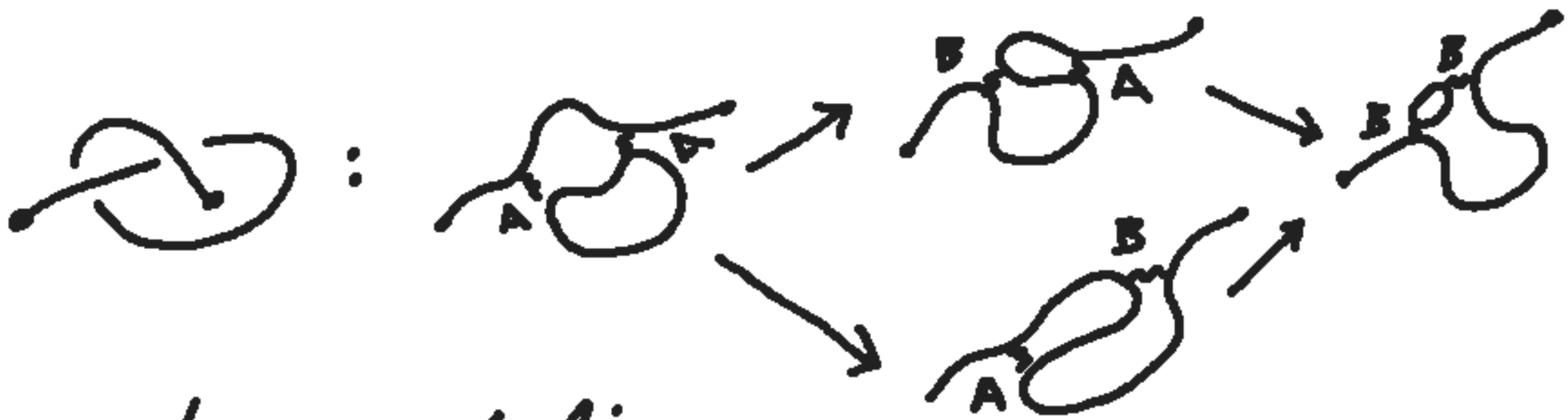
We can also take $\langle \bar{K} \rangle$ where $\bar{K}$ = virtual closure of the knotoid.

e.g. $\langle$  $\rangle = \langle$  $\rangle$

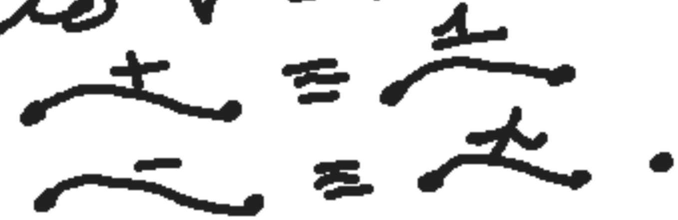
$= A\,\langle$  $\rangle + A^{-1}\langle$  $\rangle = A(A+A^{-1}) + A^{-1}(-A^{-3})$

$= A^{2}+1-A^{-4}.$

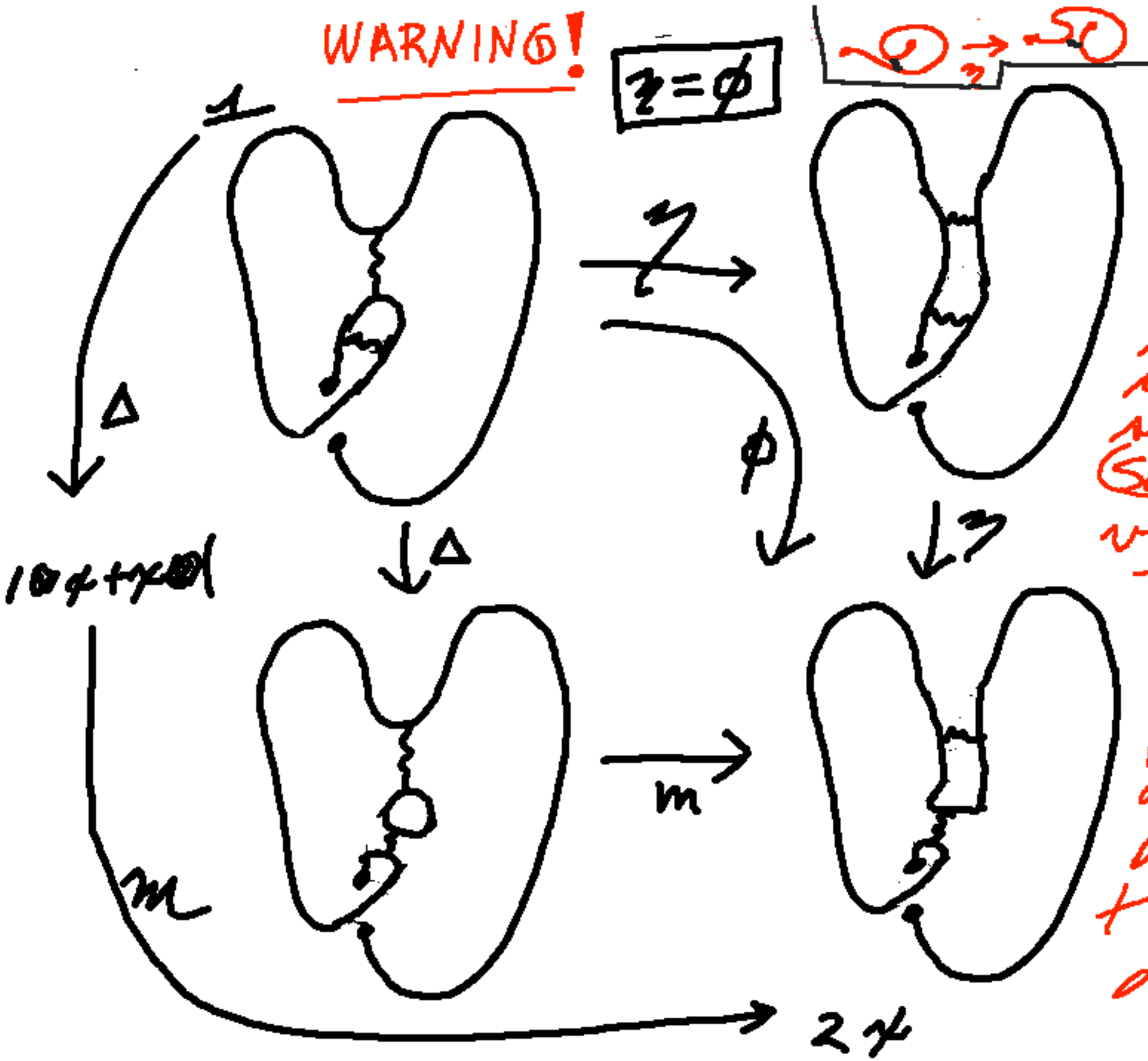
Exercise: $\langle K \rangle$ (as defined above) $= \langle \bar{K} \rangle$ (as defined for virtual knots)



We can define Khovanov Homology for knotoids exactly as we have defined Khovanov Homology for the long classical knots. For the long state  we associate the same module $V = k[x]/(x^2)$ and generators



WARNING!



For Knotoids we have same problem with single component smoothing (See left) as for virtual knots. Thus we will discuss Khovanov Homology of Knotoids along with the virtual discussion!

Conjecture. $\langle K \rangle$ detects the unknotoid.

K 

Exercise: compute $\langle K \rangle$.

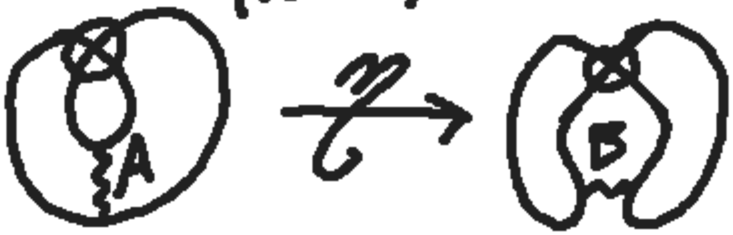
$\langle K \rangle$ = some $\pm$ power of A

Conj $\Rightarrow K \sim$ 

Kho Homology for knotoids might detect the unknotoid. For classical knots this has been shown by Kronheimer and Mrowka.

What about KhoH for virtuals?

1 comp 1 comp



single cycle re-smoothing

non-orientable saddle pt.

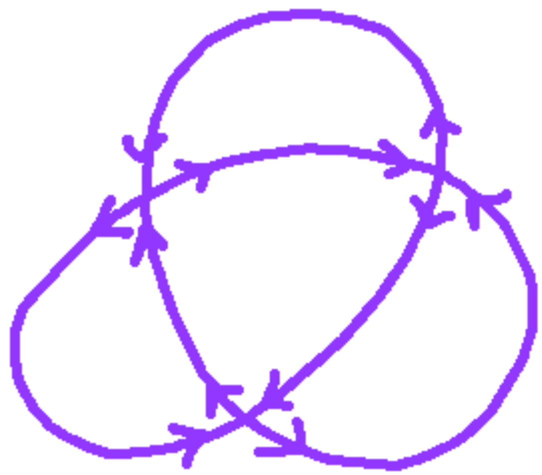
$\mathcal{A} = k[x]/(x^2)$

Khovanov algebra.

$\eta : \mathcal{A} \to \mathcal{A}$, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

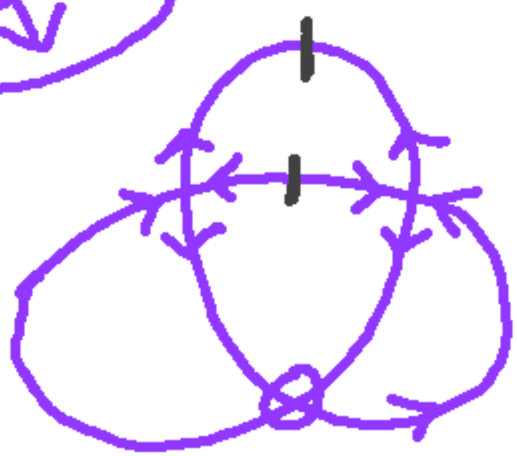
$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

Assume now $\boxed{\eta = 0}$.



source sink orientation

Can always decorate a classical diagram with a source sink orientation. Exercise!



It turns out that we can measure involutions across cut-points.

This will be the subject of the next lecture.

Khovanov Homology for Virtual Links

LK

1. $\mathcal{A} = \mathbb{Z}[x]/(x^2)$, $\Delta(1) = 1 \otimes x + x \otimes 1$
$\Delta(x) = x \otimes x$

$s : \mathcal{A} \longrightarrow \mathcal{A}$, $s(x) = \overline{x} = -x$
$s(1) = \overline{1} = 1$

__Note__: $\overline{(a+bx)}\,\overline{(c+dx)} = (a-bx)(c-dx)$
$= ac - bcx - adx$
$= \overline{ac + bcx + adx} = \overline{(a+bx)(c+dx)}$.

Thus $\overline{z}\,\overline{w} = \overline{zw}$ for all $z, w \in \mathcal{A}$.

__Note__: $\Delta(\overline{1}) = \Delta(1) = 1 \otimes x + x \otimes 1$
$\overline{\Delta(1)} = \overline{1 \otimes x + x \otimes 1} = -1 \otimes x - x \otimes 1$
$\therefore \overline{\Delta(1)} = -\Delta(\overline{1})$.

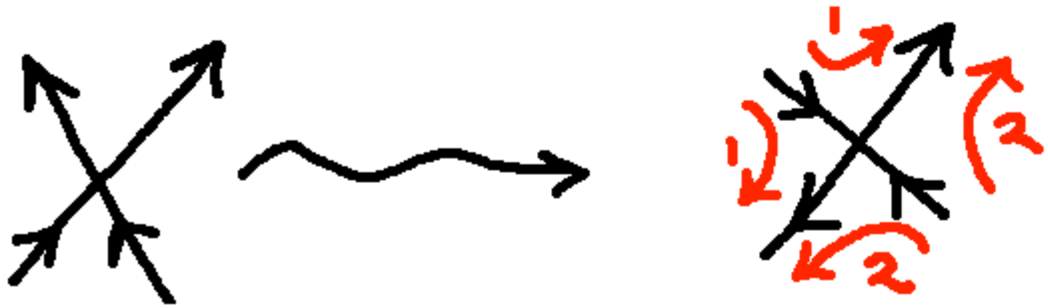
$$\Delta(\overline{\chi}) = \Delta(-\chi) = -\Delta(\chi) = -\chi\otimes\chi$$

$$\overline{\Delta(\chi)} = \overline{\chi\otimes\chi} = (-\chi)\otimes(-\chi) = \chi\otimes\chi$$

$$\therefore\ \overline{\Delta(\chi)} = -\Delta(\overline{\chi})$$

Thus $\forall z \in \mathcal{A}$, $\overline{\Delta(z)} = -\Delta(\overline{z})$.

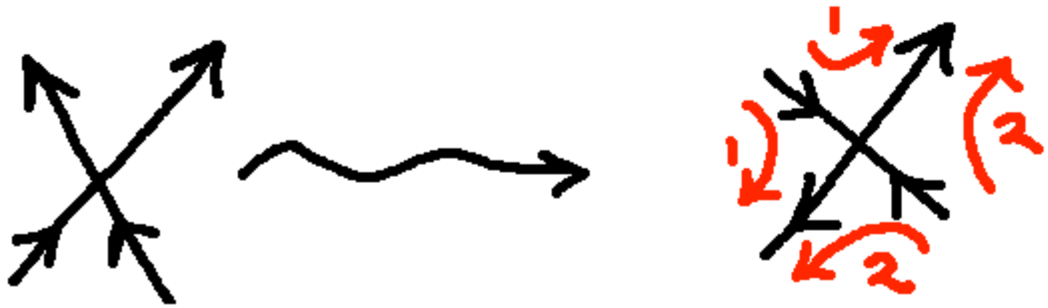
2° We work with an oriented link diagram K and associate a source-sink orientation to each crossing as shown below:



Local order on the cycles for each smoothing is determined by the labels 1 and 2 above.

The global order is an arbitrary choice of labels $1, 2, \ldots, n$ for the n states in the Khovanov complex. We will compare local and global orders.

This comparison can be expressed in terms of Grassmann algebra, but we shall define it in terms of permutations and signs of permutations.



Along with these orders, the source-sink orientations on the diagram induce cut pts on the diagram as in

. We apply $a \rightsquigarrow \overline{a}$ when moving algebra past a cut point.

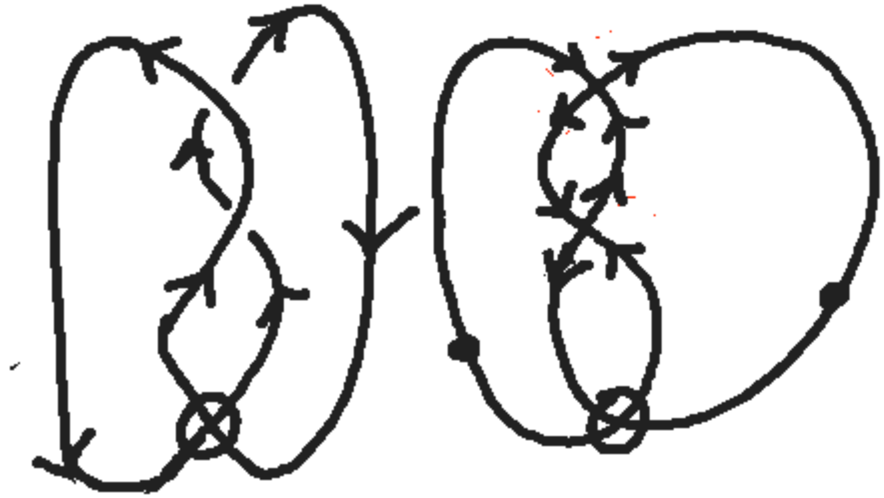
.

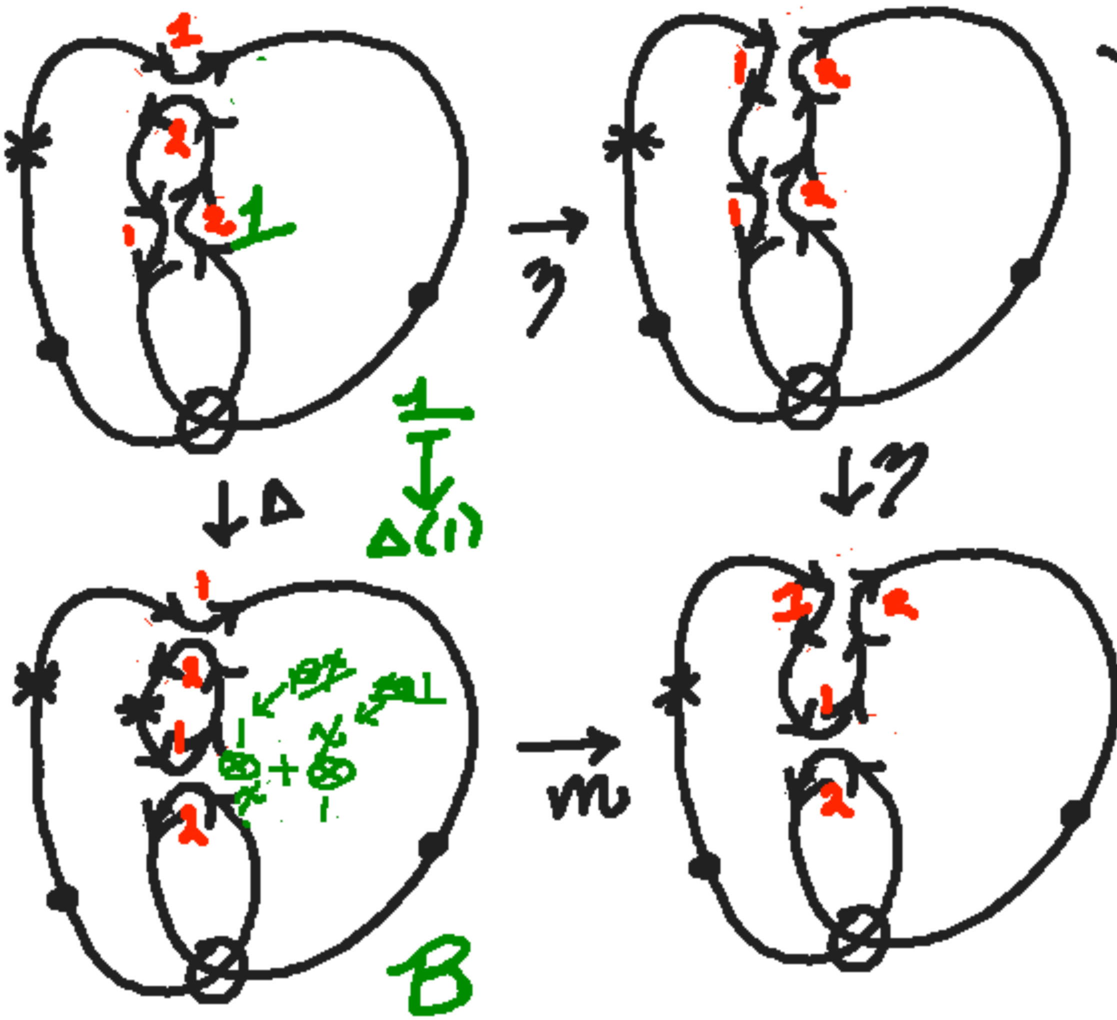
algebra resides at a basepoint and is transported in a state to the site of an operation (m or Δ) and then transported back to base-point.

(a) If labels 1 and 2 appear on the same cycle, permute global order so that this cycle appears in the first place. Leave remaining elts in same relative order.

(b) If labels 1 and 2 appear on different cycles, permute global order so they occupy 1st and 2nd place resp. Leave other cycles in same relative order.

In working with signs for maps, use the signs of the permutations for (a) and (b) along with the signs that occur from applying $S(z) = \bar{z}$ to algebra elements that transport across cut loci. Then all squares in the complex anti-commute.





• at B, $1 \otimes x + x \otimes 1$

transports from

$(1,2) \rightsquigarrow (2,1)$

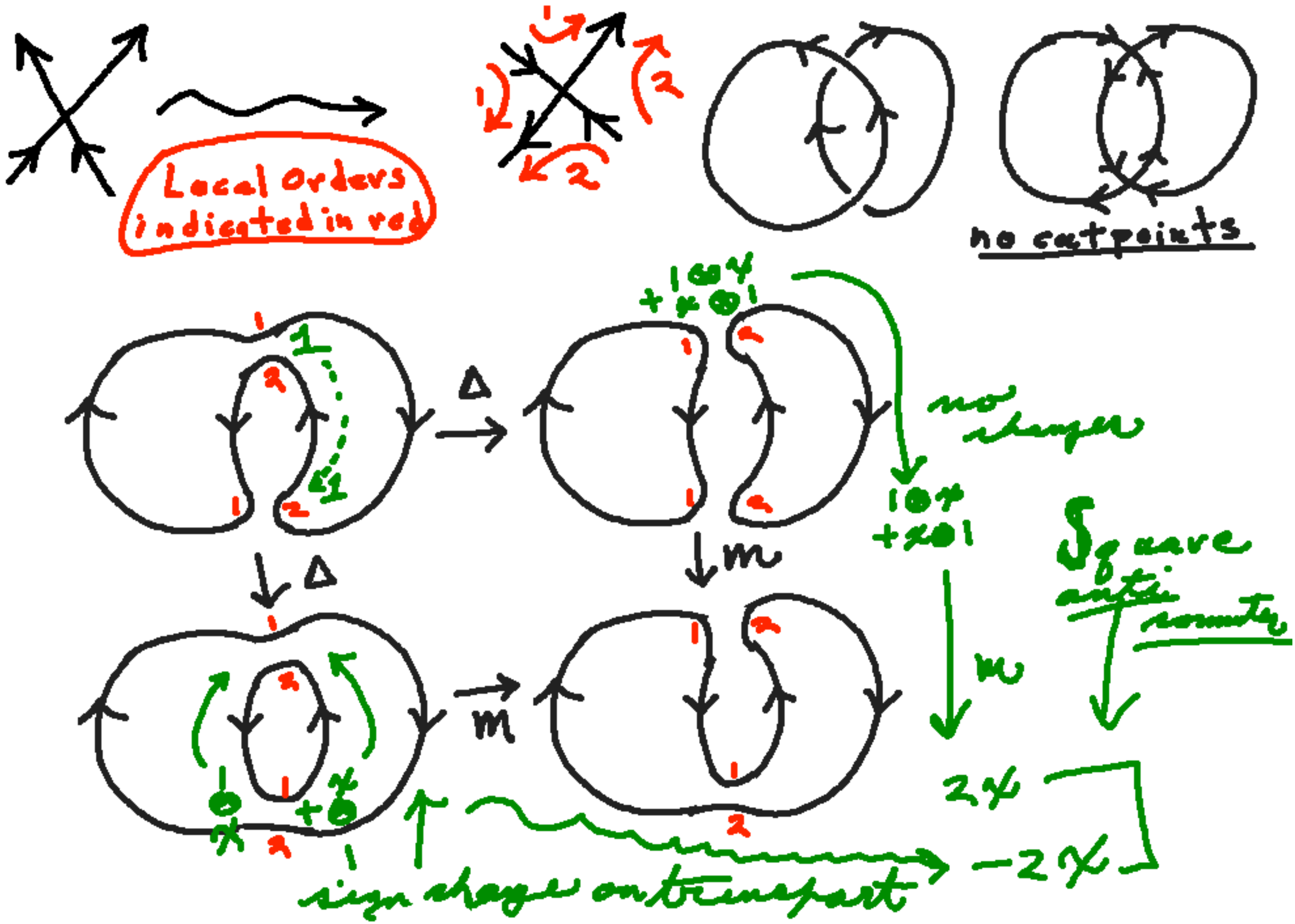
making global sign change.

$x \otimes 1 \rightsquigarrow -(1 \otimes x)$

$(1 \otimes x) \rightsquigarrow -(x \otimes 1)$

Apply m to get

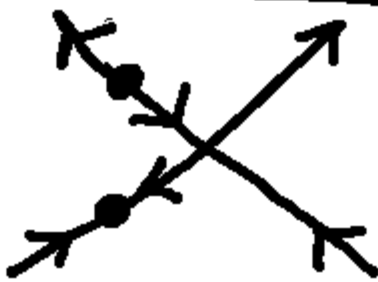
$-x - (-x) = \emptyset$ ✓.

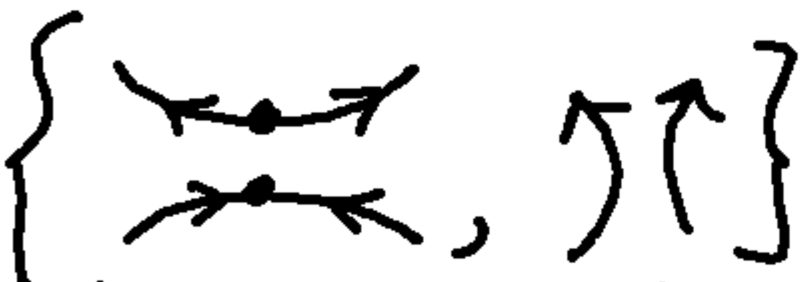

Local Orders indicated in red
no cutpoints
Δ
m
1⊗x
+x⊗1
no change
Square anti commute
2x
−2x
sign change on the one part

These examples illustrate how the local/global order permutations combine with the x ——•—— $\bar{x}$ barring transformations across cut points to give anti-commuting squares in the Khovanov complex.

or

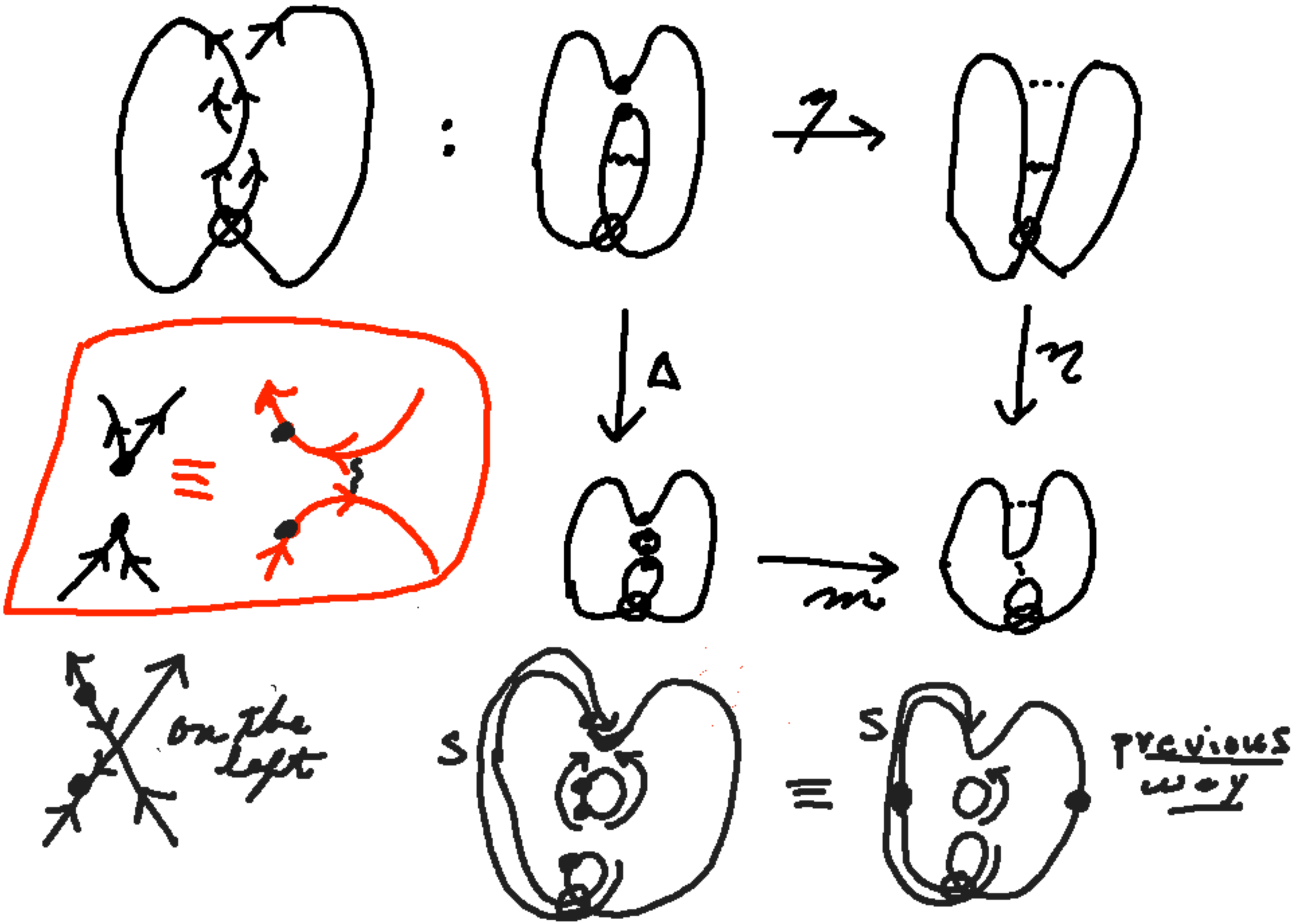
You + I want as explicit a formula as possible for ∂.

__Note__:  Can place 2 cut points at each crossing.



__reverse smoothing__, __standard smoothing__

smoothing str.


η
Δ
η
m
S
S
on the left
previous way

Formalizing the Complex

The diagram above is the key to the entire construction.

Each state σ has a chosen base pt and some cut-points.

Any algebra labeling cycles in the state is located at the base pt $*$.

To apply a partial boundary relative to surgery on a site s e.g.  s

find a path λ_s from $* \rightsquigarrow s$.

To apply a partial boundary relative to surgery on a site s e.g. 
find a path α_s from $* \rightsquigarrow s$.

Let $n(s) = \#$ of cut points on the path α from $*$ to s.

If algebra a labels $*$, then apply the operation m or Δ (denoted ∂_s) to $S^{n(s)}(a)$ where S is the involution on the algebra. ($S(x) = -x$, $S(1) = 1$). Recall that we also need the sign comes to the local/global order.

Let $sgn(s)$ = the permutation sign for site s wrt local/global order.

And let $sgn(s')$ = the permutation sign for site s' wrt local/global order.

Here

$$\begin{array}{ccc} s & \xrightarrow{\text{surgery}} & s' \\ \curvearrowright & & \\ \curvearrowleft & & \partial\uparrow \end{array}$$

Then the local boundary $d_s(a)$:

$$d_s(a) = sgn(s)\,sgn(s')\, S^{n(s')}(\partial_s(S^{n(s)}(a)))$$

and

$$\partial(a) = \sum_s d_s(a).$$

The difference between this formula and the original Khovanov boundary formula is entirely in the signs.

More to come ...