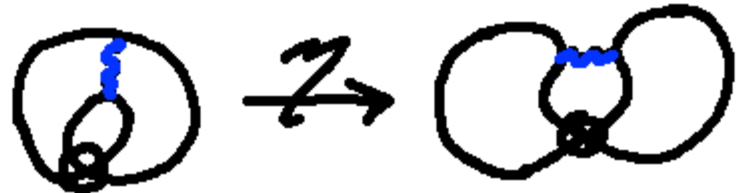
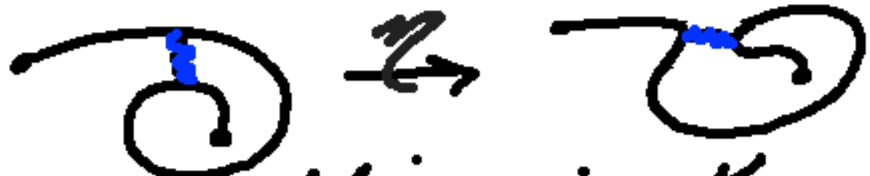
Lecture #6

These are the handwritten notes for this lecture. We also used the slides in Lecture #5.

Note: 

A re-smoothing in virtual category can be from one loop to one loop.

Note: 

A re-smoothing in Knotoid category can be from one long to one long.

These facts mean that Khovanov Homology for virtuals and knotoids requires special attention.

$\delta = q + q^{-1}$

$i=0$ $\qquad$ $i=1$ $\qquad$ $i=2$

$j = 0+\lambda$ $\qquad$ $j = 1+\lambda$ $\qquad$ $j = 2+\lambda$

$\langle H \rangle$
$= \langle \ \rangle - q \langle \ \rangle$
$= q^{-1}\delta - q(-q^{2})\delta$
$= (q^{-1}+q^{3})(q+q^{-1})$
$= 1 + q^{4} + q^{-2} + q^{2}$

$$\boxed{\langle H \rangle = \sum_{i,j} (-1)^i q^j}$$

$i \backslash j$	-2	0	2	4
0	$x \otimes x$	$x \otimes 1$ $1 \otimes x$	$1 \otimes 1$	
1		$(x,0)$ $(0,x)$	$(1,0)$ $(0,1)$	
2		$x \otimes x$	$x \otimes 1$ $1 \otimes x$	$1 \otimes 1$

$\downarrow \partial$

$+(q^{-2} + \cancel{2} + q^{2})$
$-(\cancel{2} + \cancel{2q^{2}})$
$+(1 + \cancel{2q^{2}} + q^{4})$ ✓

$\partial(x \otimes x) = 0 + 0 = 0$

$\partial(x \otimes 1) = (x,0) - (0,x) = (x,-x)$

$\partial(1 \otimes x) = (x,0) - (0,x) = (x,-x)$

... do this homology as an exercise!

Other examples

+ 8 states
+ more enhanced states

min 16 states
+ more enhanced states

Knotoids

ends not nec. in same region.

Include RM's as usual but not allowing passage across an end.

No!

Knot type knotoids

$\langle \bigcirc \rangle = q + q^{-1}$

$\langle$  $\rangle = q + q^{-1}$

Usual $\langle\ \rangle$ poly for knotoids

$\langle$  $\rangle = A\,\langle \asymp \rangle + A^{-1}\langle)(\rangle$

$\langle O K \rangle = (-A^{2} - A^{-2})\langle K \rangle$

$\langle$  $\rangle = 1$

R2 + R3

$f_K(A) = (-A^{3})^{-wr(K)}\langle K \rangle$

K

$\langle$  $\rangle = A\,\langle$  $\rangle + A^{-1}\langle$  $\rangle$ neg kink

$\langle$  $\rangle = A\,\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= (A + A^{-1})$

$= A(A + A^{-1}) + A^{-1}(-A^{-3})$

$= A^{2} + 1 - A^{-4}$

$f_K = (-A^{3})^{-2}\langle K \rangle = A^{-4} + A^{-6} - A^{-10}$

$\Rightarrow K \not\simeq K^{*}$ non trivial

Usual $\langle\ \rangle$ poly for knotoids

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$\langle O K \rangle = (-A^2 - A^{-2}) \langle K \rangle$

$\langle$  $\rangle = 1$

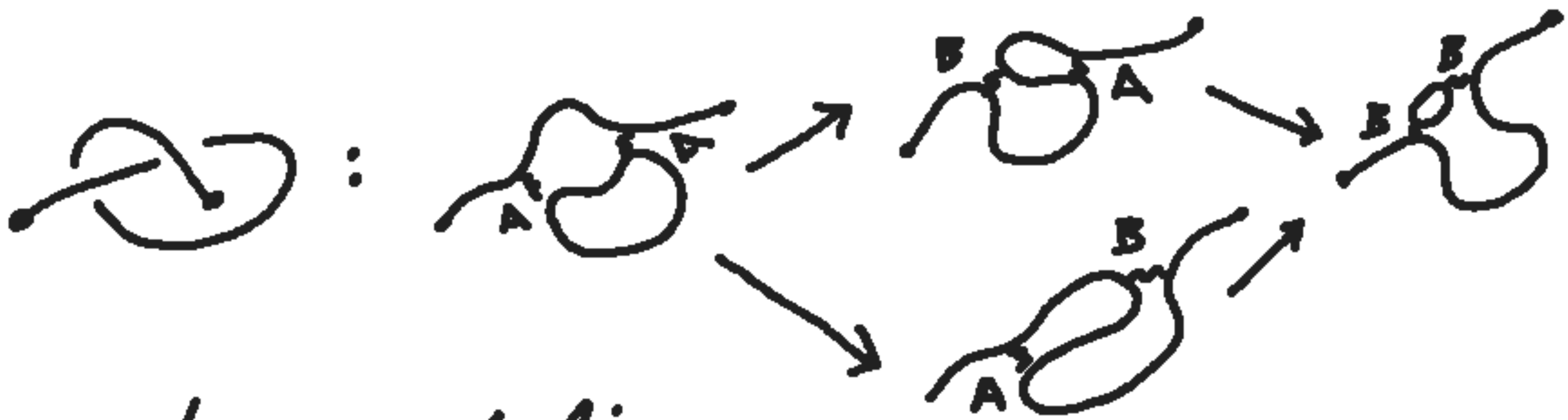
We can also take $\langle \overline{K} \rangle$ where $\overline{K}$ = virtual closure of the knotoid.

e.g. $\langle$  $\rangle = \langle$  $\rangle$

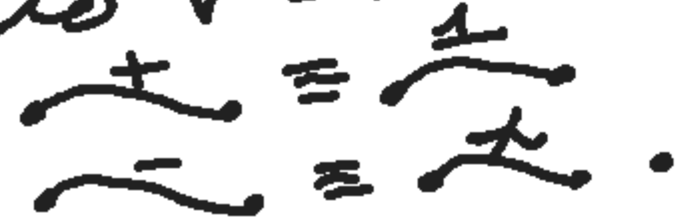
$= A \langle$  $\rangle + A^{-1} \langle$  $\rangle = A(A+A^{-1}) + A^{-1}(-A^{-3})$

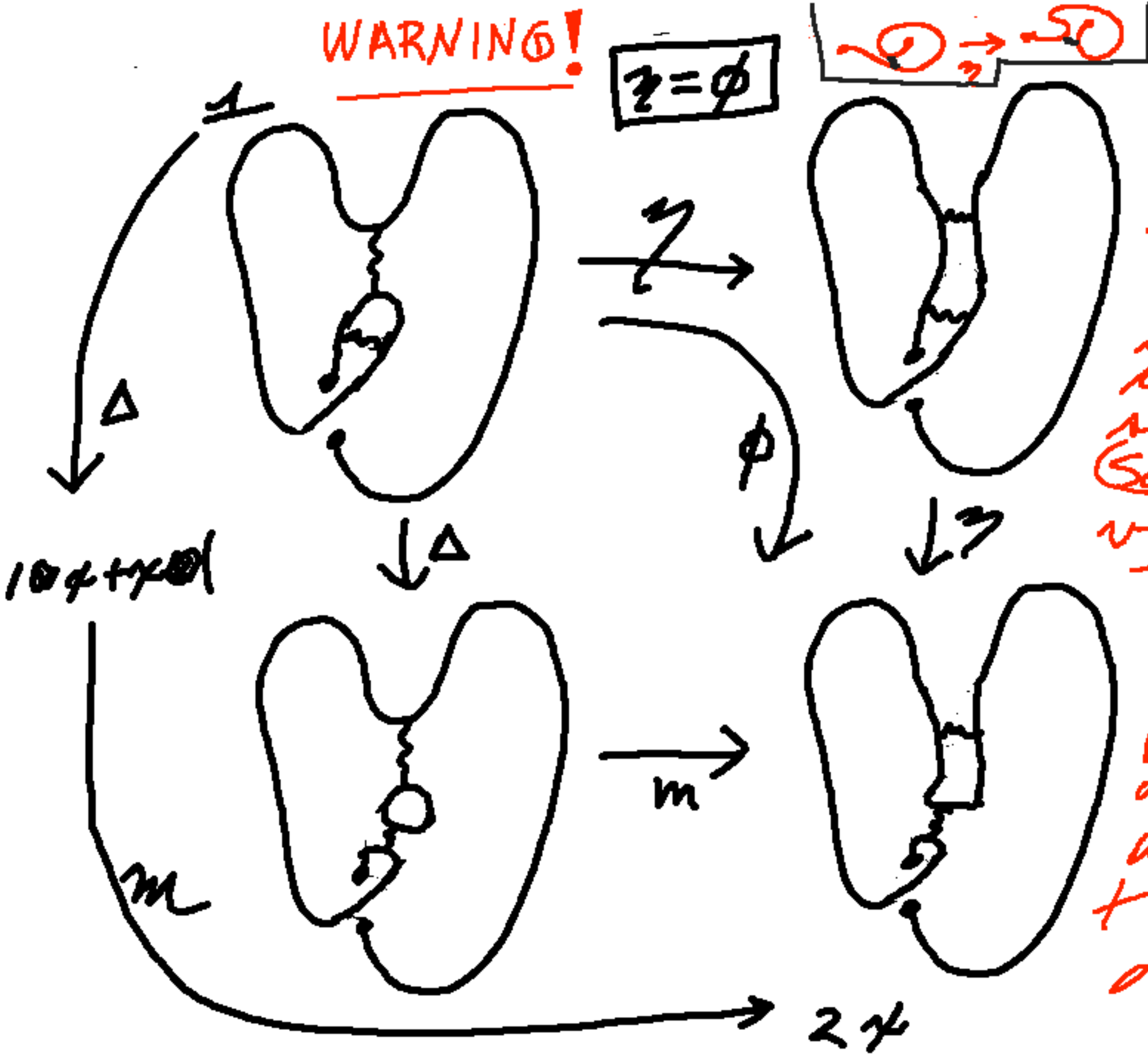
$= A^2 + 1 - A^{-4}$.

Exercise: $\langle K \rangle$ (as defined above) $= \langle \overline{K} \rangle$ (as defined for virtual knots)



We can define Khovanov Homology for knotoids exactly as we have defined Khovanov Homology for classical knots. For the long state  we associate the same module $V = k[x]/(x^2)$ and generators





For Knotoids we have same problem with single component smoothing (See left) as for virtual knots. Thus we will discuss Khovanov Homology of Knotoids along with the virtual discussion!

Conjecture. $\langle K \rangle$ detects the unknotoid.

k 

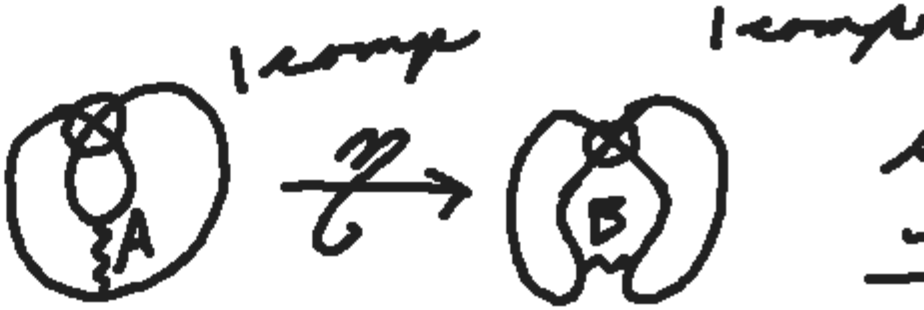
Exercise
compute $\langle k \rangle$.

$\langle k \rangle =$ some $\pm$ power of A

Conj $\Rightarrow k \sim$ 

Kho Homology for knotoids might detect the unknotoid. For classical knots this has been shown by Kronheimer and Mrowka.

What about KhoH for virtuals?



single cycle re-smoothing

non-orientable saddle pt.

$\mathcal{A} = k[x]/(x^2)$

Khovanov algebra.

$\eta : \mathcal{A} \to \mathcal{A}$, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

Assume now $\boxed{\eta = 0}$.

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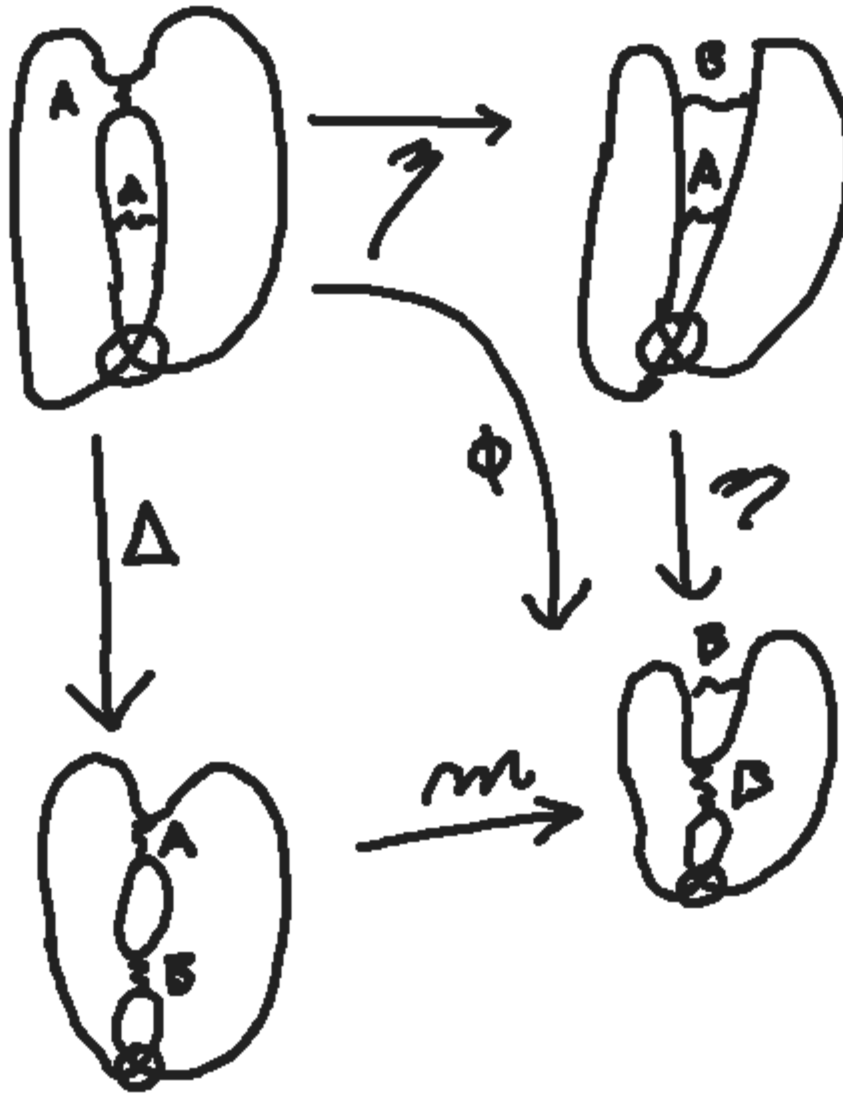


$\eta = \phi$

Fact

$KhoH^*$
OK mod 2
for virtuals.

$$m \circ \Delta(1) = m(1 \otimes x + x \otimes 1) = x + x = 2x$$

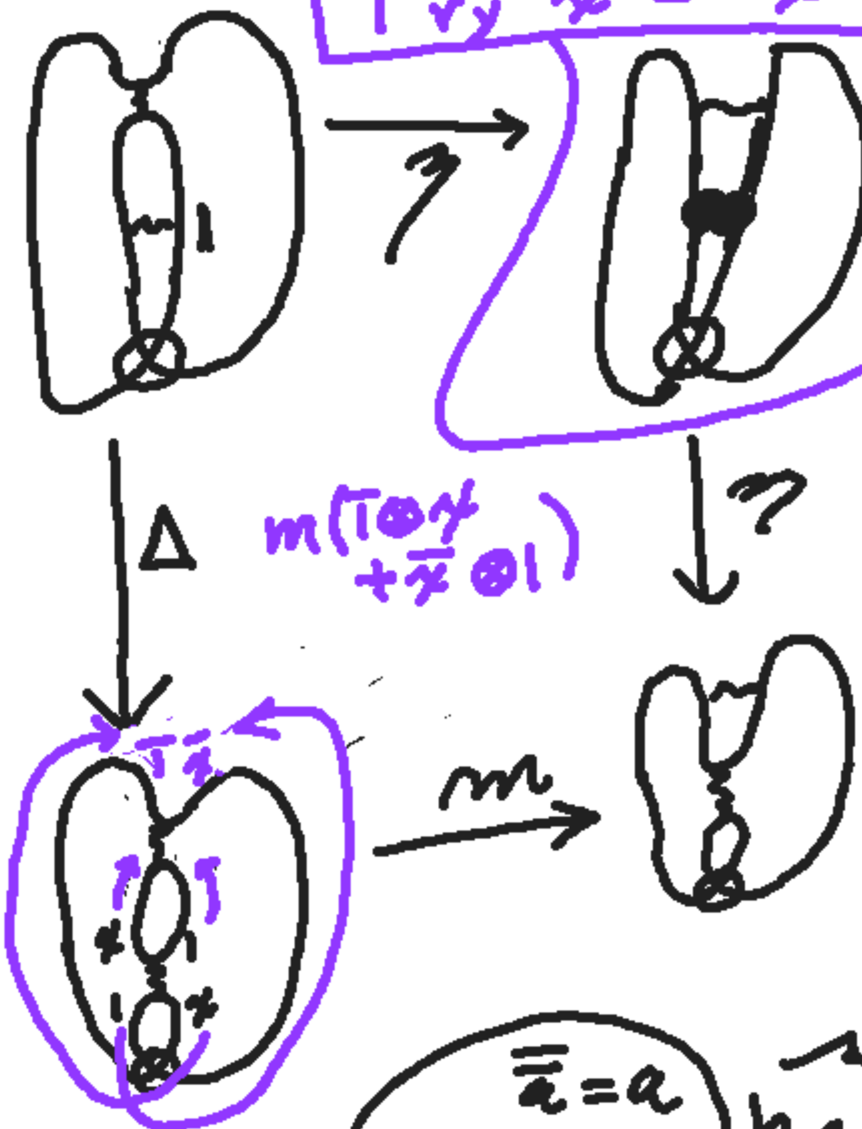
$$m \circ \Delta(x) = m(x \otimes x) = x^2 = 0 \checkmark$$

OK mod 2

Examples

Try $\bar{x} = -x$, $T = 1$

$$T\cdot x + \bar{x}\cdot 1 = 1\cdot x - x\cdot 1 = 0$$



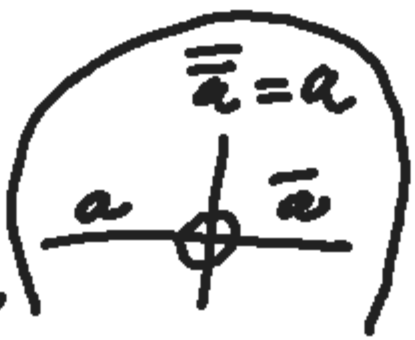
$x\otimes 1 + 1\otimes x$

$$\begin{tikzcd} 1 \arrow[d, "\xi"'] \arrow[dr, "\eta^{\tau}"] & \\ 1\otimes x + x\otimes 1 \arrow[r] & x + x \\ & 2x \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

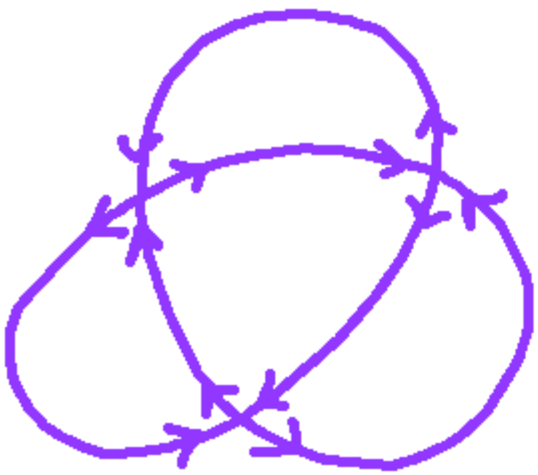
change to $-$?

use local coefficients so the algebra has location on diagram.

Try

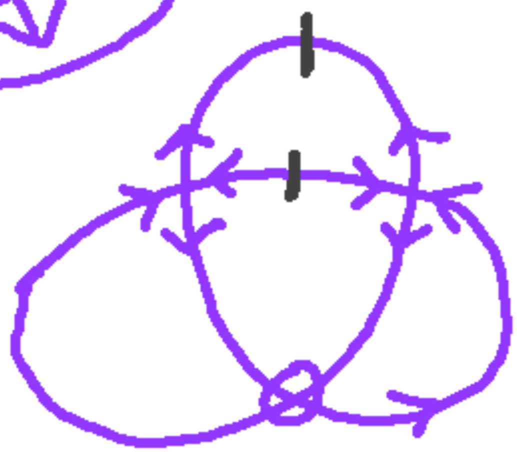


source sink orientation



Can always decorate a classical diagram with a source sink orientation.

Exercise!



It turns out that we can measure involutions across cut-points.

This will be the subject of the next lecture.