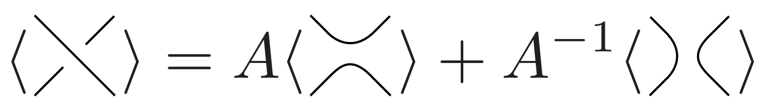
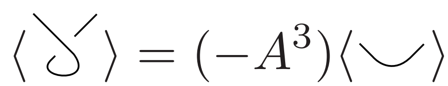
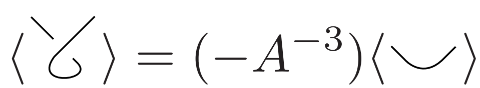
Khovanov Homology

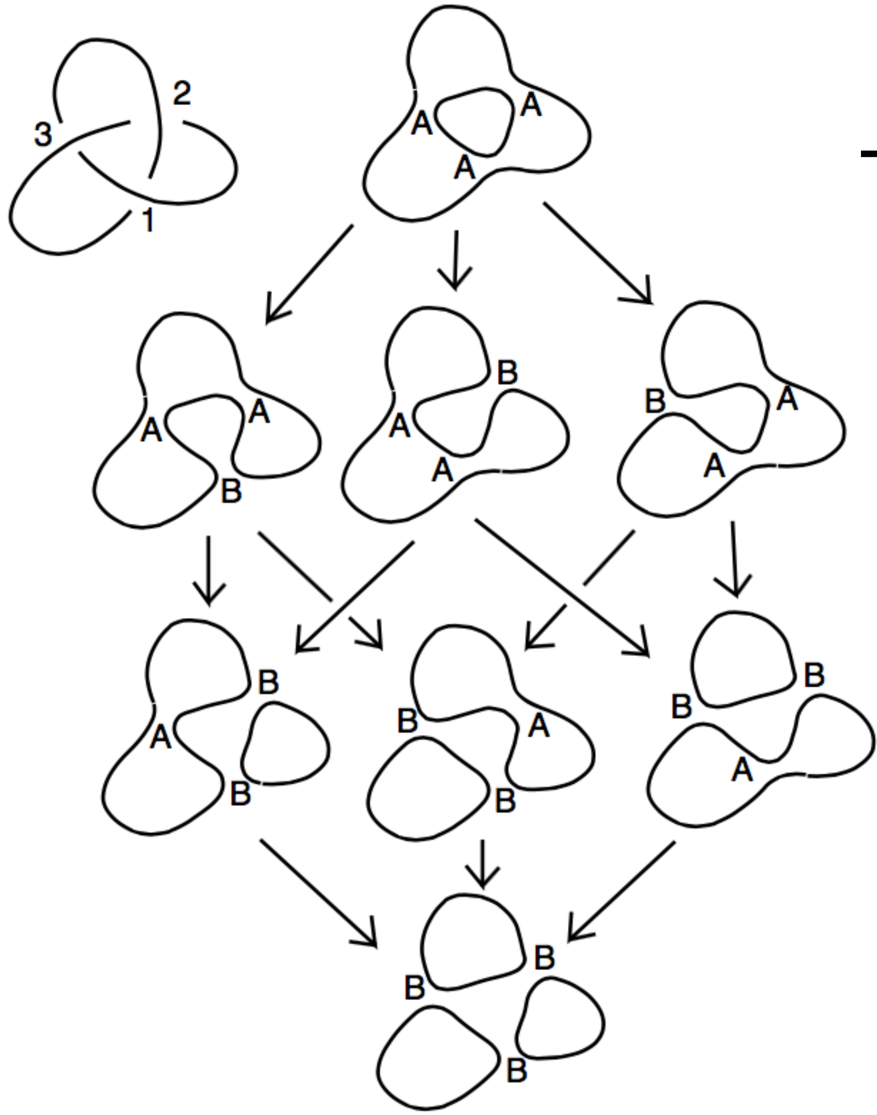
Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some homology theory associated with $\langle K \rangle$.



$$\langle K \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle$$







Cubism

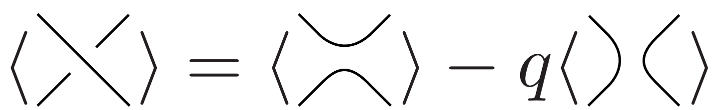
The bracket states form a category. How can we obtain topological information from this category?

Exploration: Examine the Bracket Polynomial for Clues.

Let c(K) = number of crossings on link K.

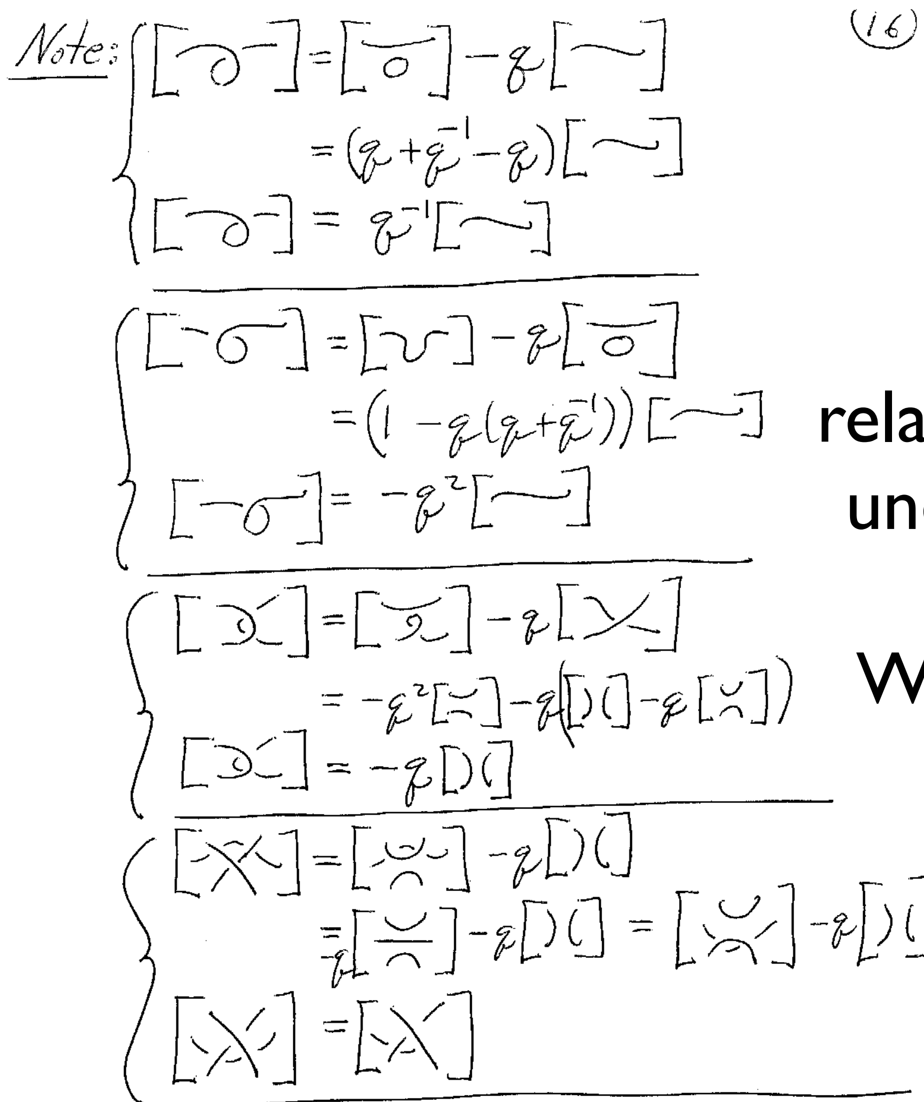
Form $A^{-c(K)}$ <K> and replace A by $-q^{-1}$.

Then the skein relation for <K> will be replaced by:



$$\langle \bigcirc \rangle = q + q^{-1}$$

$$\langle K \bigcirc \rangle = (q + q^{-1})\langle K \rangle$$



Grading Shifts for Khovanov Homology are related to these shifts under Reidemeister moves.

We will come back to this.

The Khovanov Homology is invariant under Reidemeister moves with the grading shifts that correspond to the behaviour of [K].

$$[\times] = [\asymp] - q\,[\supset\subset]$$

$$[\bigcirc] = q + q^{-1}$$

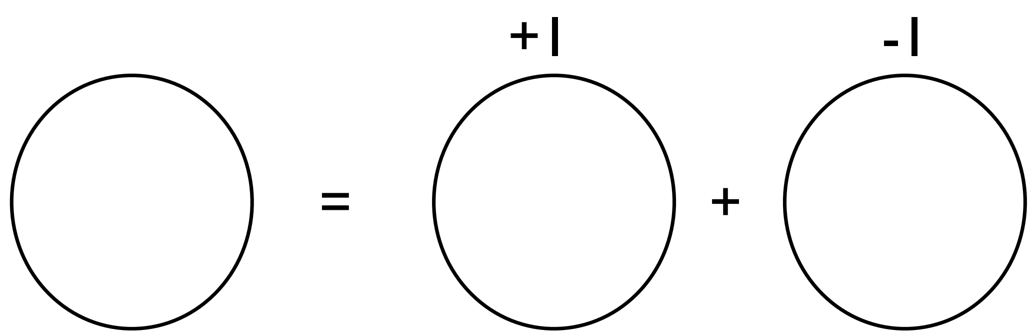
$$\boxed{J_K = (-1)^{n_-} q^{n_+ - 2n_-} [K]}$$

invar under all 3 RM's.

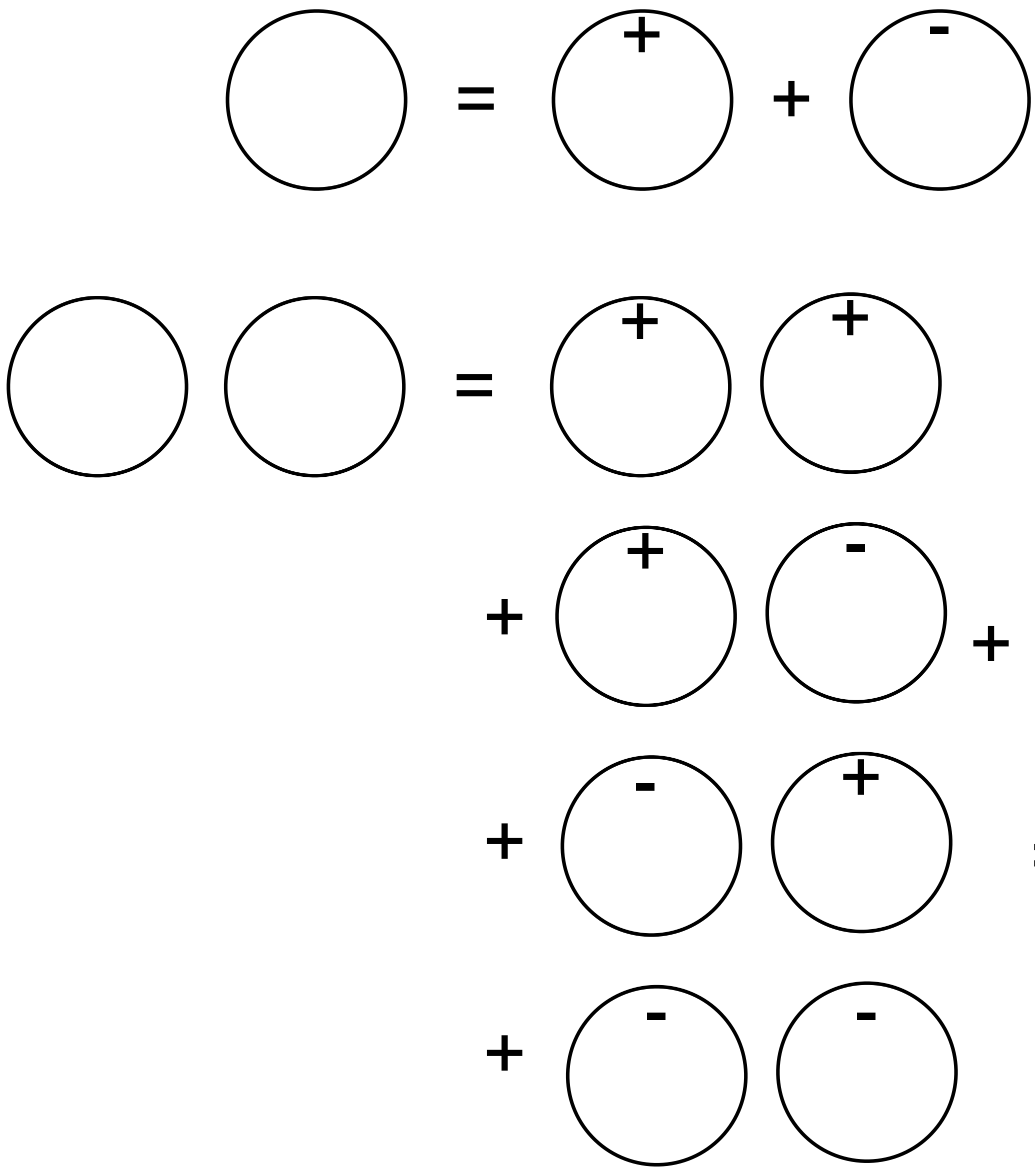
<u>Exercise</u>. Find out the formula that translates $J_K(q)$ to $V_K(t)$ formula

$$\boxed{t^{-1} V_{\nearrow\!\!\!\!\nwarrow} - t V_{\nwarrow\!\!\!\!\nearrow} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{\rightrightarrows} \qquad V_{\bigcirc} = 1}$$

Use enhanced states by labeling each loop with
+1 or -1.



$$\longleftrightarrow \quad q + q^{-1}$$



$$= q + q^{-1}$$

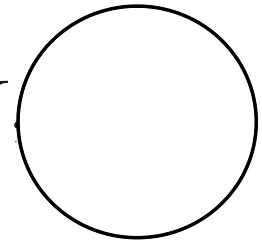
$$= qq + qq^{-1} + q^{-1}q + q^{-1}q^{-1}$$

$$= qq + 2 + (qq)^{-1}$$

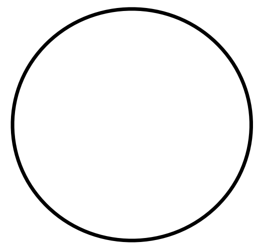
$$= (q + q^{-1})^2$$

Enhanced States

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X$$



$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.

$$\langle K \rangle = \sum_{s} (-1)^{n_B(s)} q^{j(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

$n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

C^{ij} = module generated by enhanced states with i $= n_B$ and j as above.

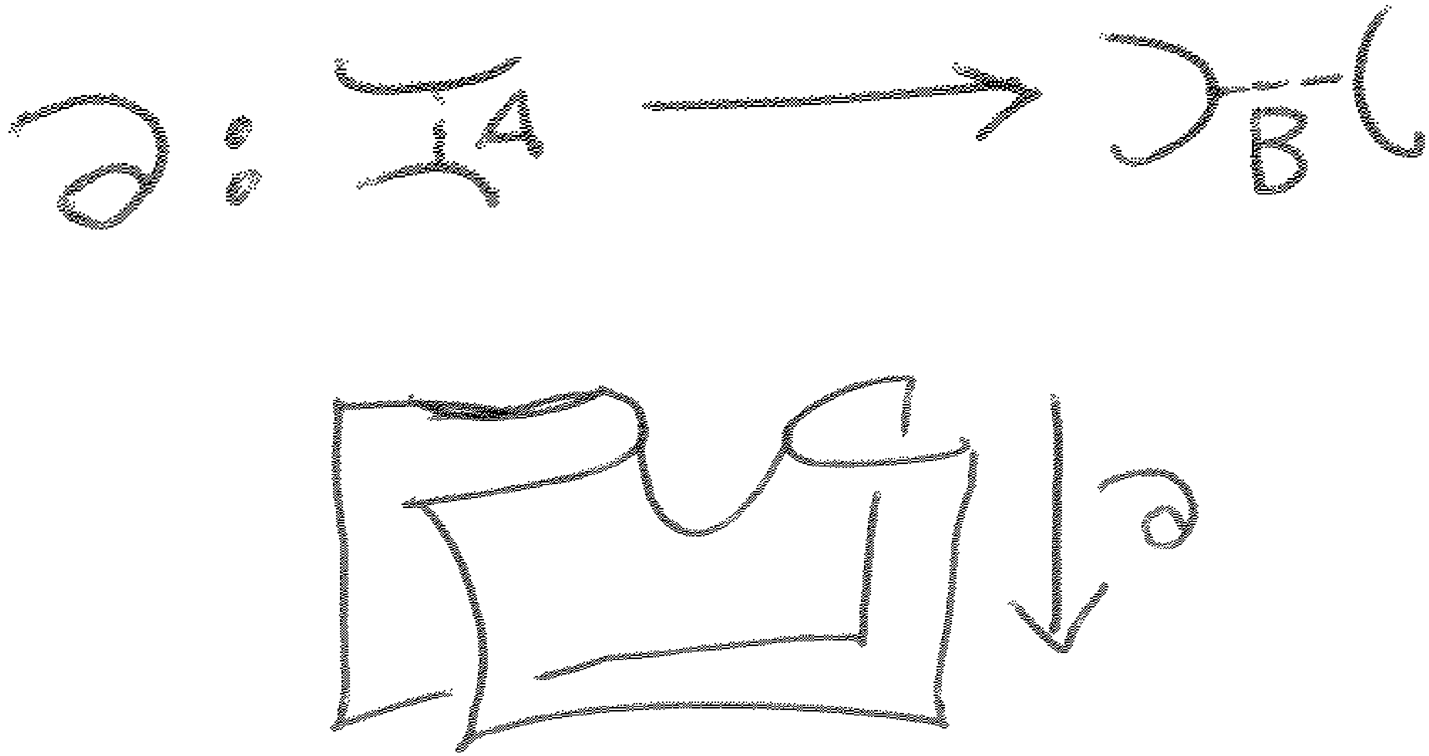
$$\langle K \rangle = \sum_{i\ ,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

Wanted: differential acting in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\, j}$$

For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.


∂ :
A
B
∂

The differential should increase the homological grading i by 1 and leave fixed the quantum grading j.

Then we would have

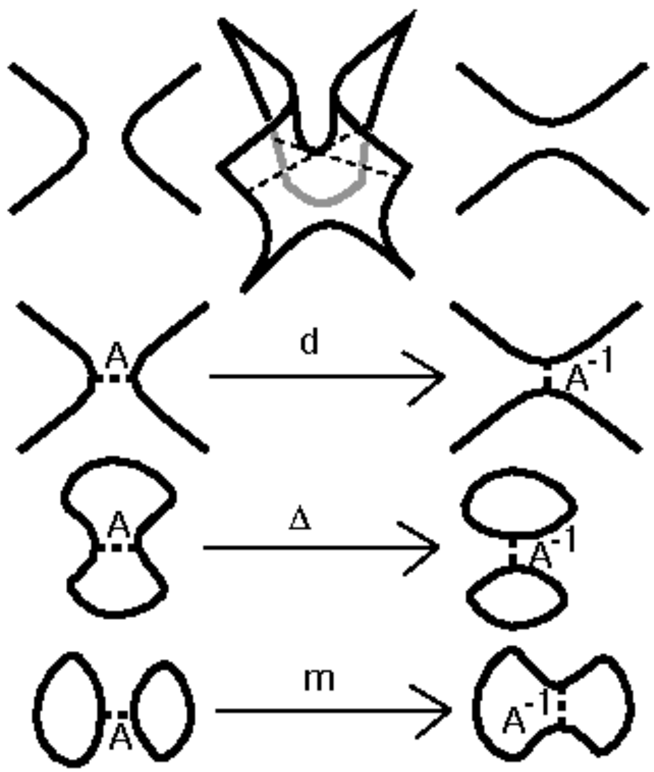
$$\langle K\rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet\, j})$$

$$\chi(H(\mathcal{C}^{\bullet\, j})) = \chi(\mathcal{C}^{\bullet\, j})$$

$$\langle K\rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

The boundary is a sum of partial differentials corresponding to resmoothings on the states.

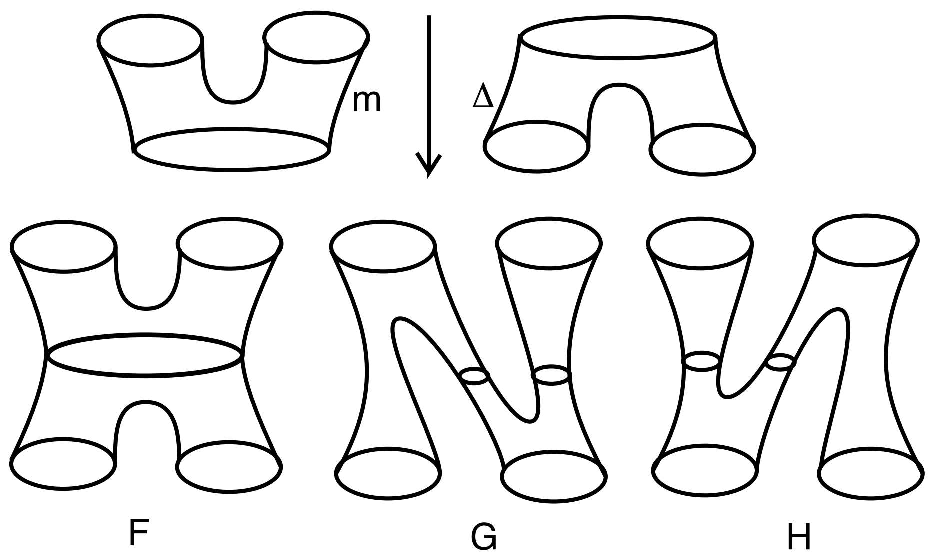


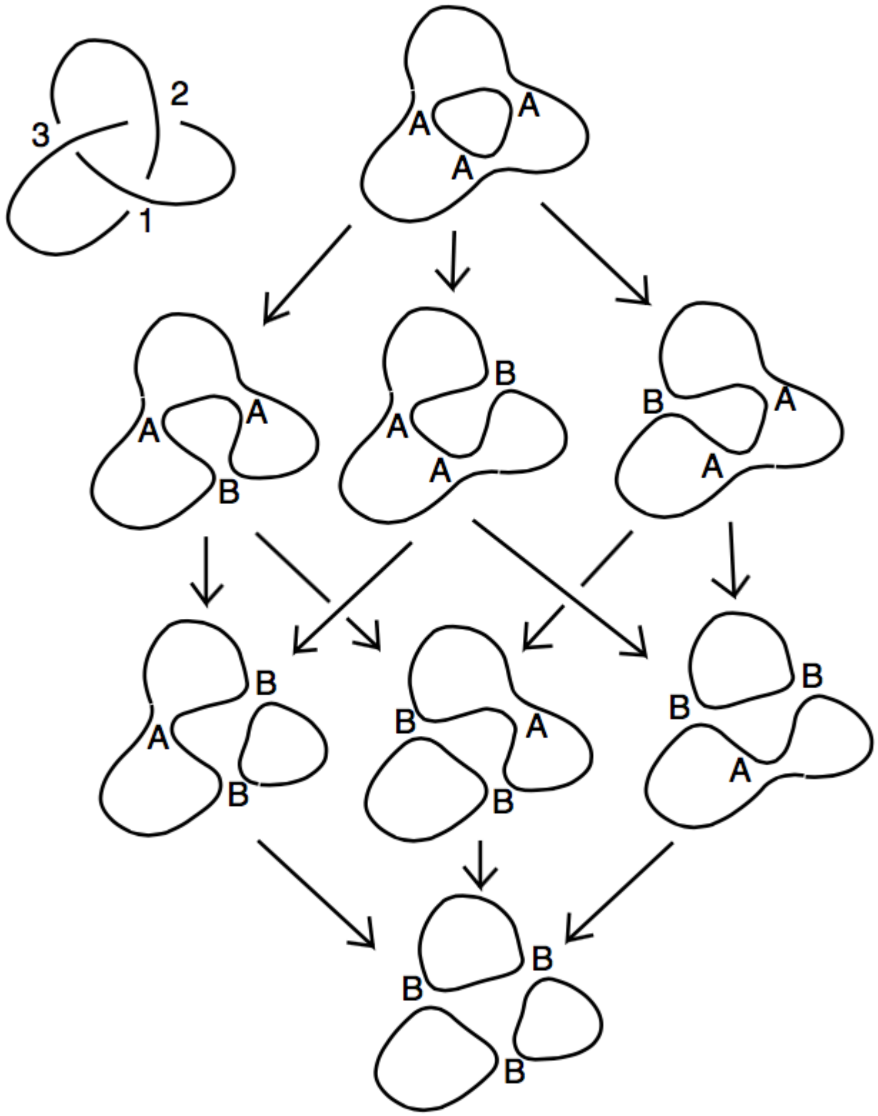
$\Delta(X) = X \otimes X$ and $\Delta(1) = 1 \otimes X + X \otimes 1.$

$X^2 = 0$

Proposition. The partial differentials $\partial_\tau(s)$ are uniquely determined by the condition that $j(s') = j(s)$ for all s' involved in the action of the partial differential on the enhanced state s. This unique form of the partial differential can be described by the following structures of multiplication and comultiplication on the algebra A = $k[X]/(X^2)$ where $k = Z/2Z$ for mod-2 coefficients, or $k = Z$ for integral coefficients.

1. The element 1 is a multiplicative unit and $X^2 = 0$.
2. $\Delta(1) = 1 \otimes X + X \otimes 1$ and $\Delta(X) = X \otimes X$.





$$C^0 \to C^1 \to C^2 \to C^3$$

Bracket states
form a
category that
assembles
itself into a chain
complex.

Levels in the chain
complex are
direct sums of modules
corresponding to
states with a constant
number of B
smoothings.

Note that signs in the boundary for an element in cube category follow the rule (-1)# where
= number of A's preceding that A to be smoothed.

Thus [AAA] $\longrightarrow$ [BAA]-[ABA]+[AAB]

and [BAA] $\longrightarrow$ [BBA]-[BAB]

$\partial : C^{i,j} \longrightarrow C^{i+1,j}$

For j to remain fixed, we need $\boxed{\lambda \overset{\partial}{\longmapsto} \lambda - 1}$

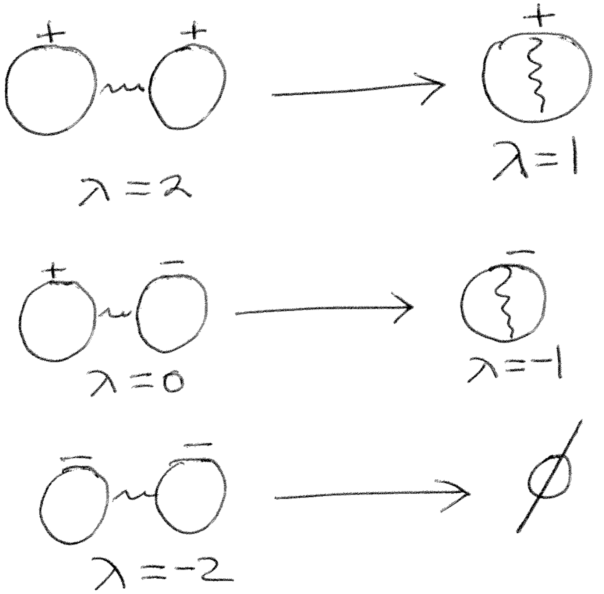
where

$$\lambda(s) = \#(+1 \text{ loops}) - \#(-1 \text{ loops}).$$

<u>The ∂ is determined</u>

<u>by this condition.</u>

③



$x \equiv -$

$1 \equiv +$

Then

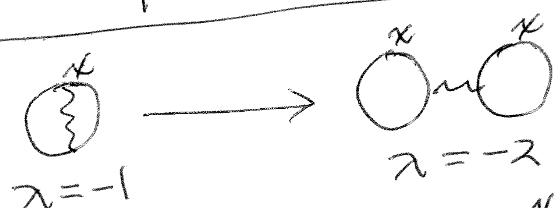
$x^2 = 0$

$1 \times 1 = 1$

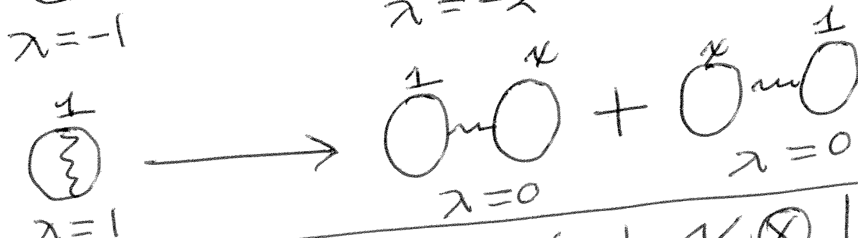
$1 \times x = x \times 1 = x$

So far $\underline{V = \mathbb{Z}[x]/(x^2)}$.

For product $m : V \otimes V \longrightarrow V$.



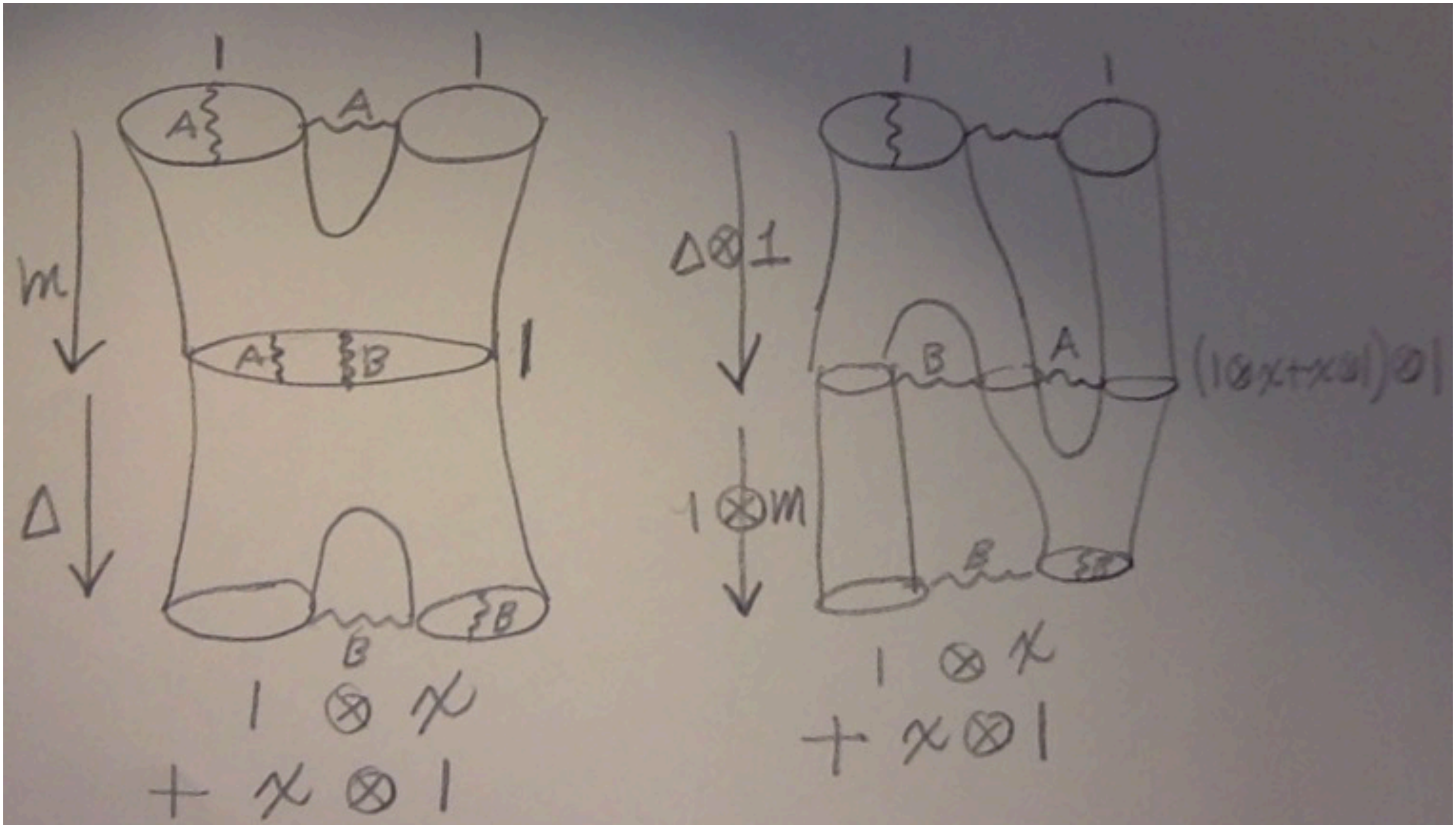
$\boxed{\Delta(x) = x \otimes x}$

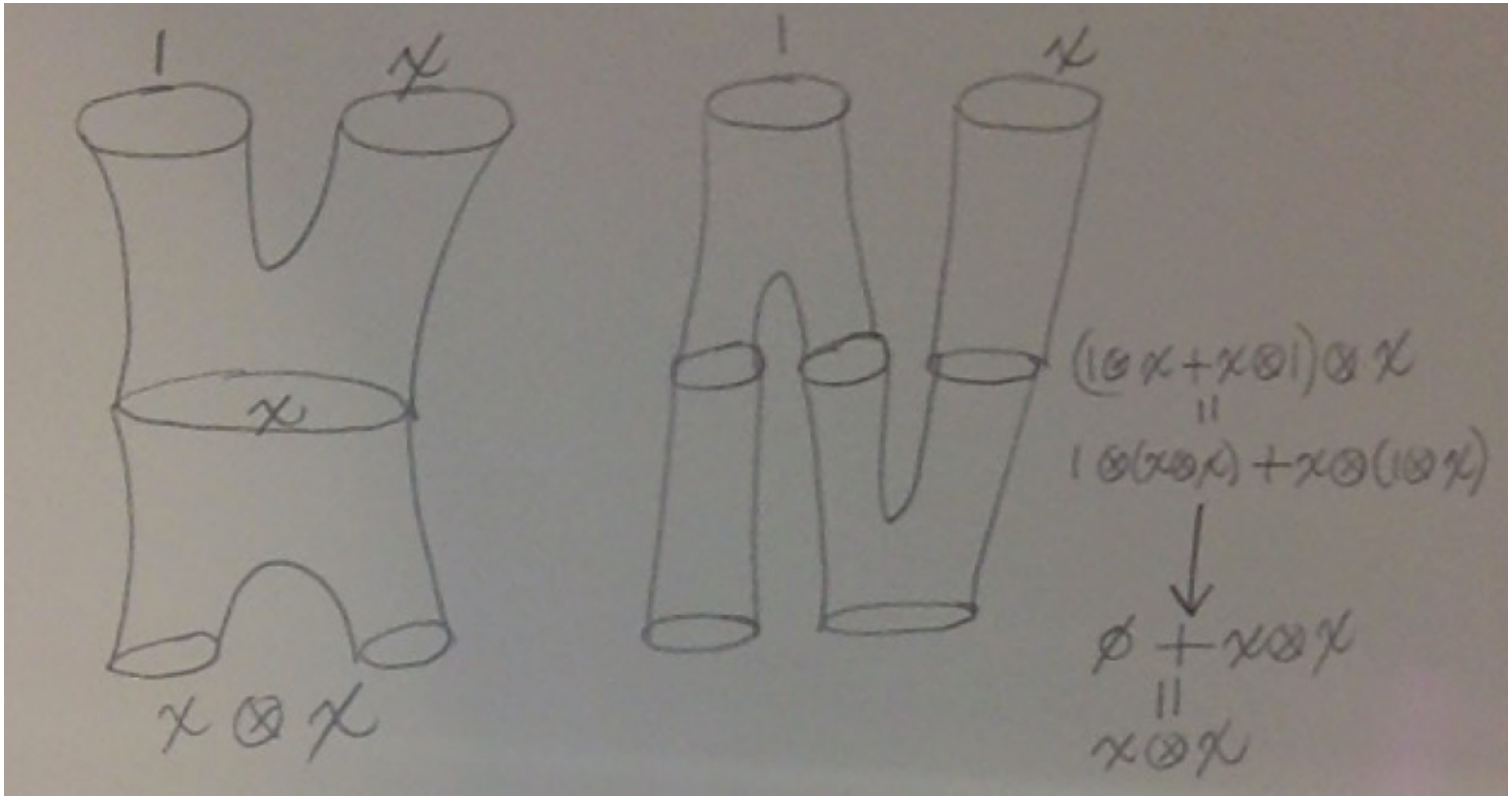


$\boxed{\Delta(1) = 1 \otimes x + x \otimes 1}$

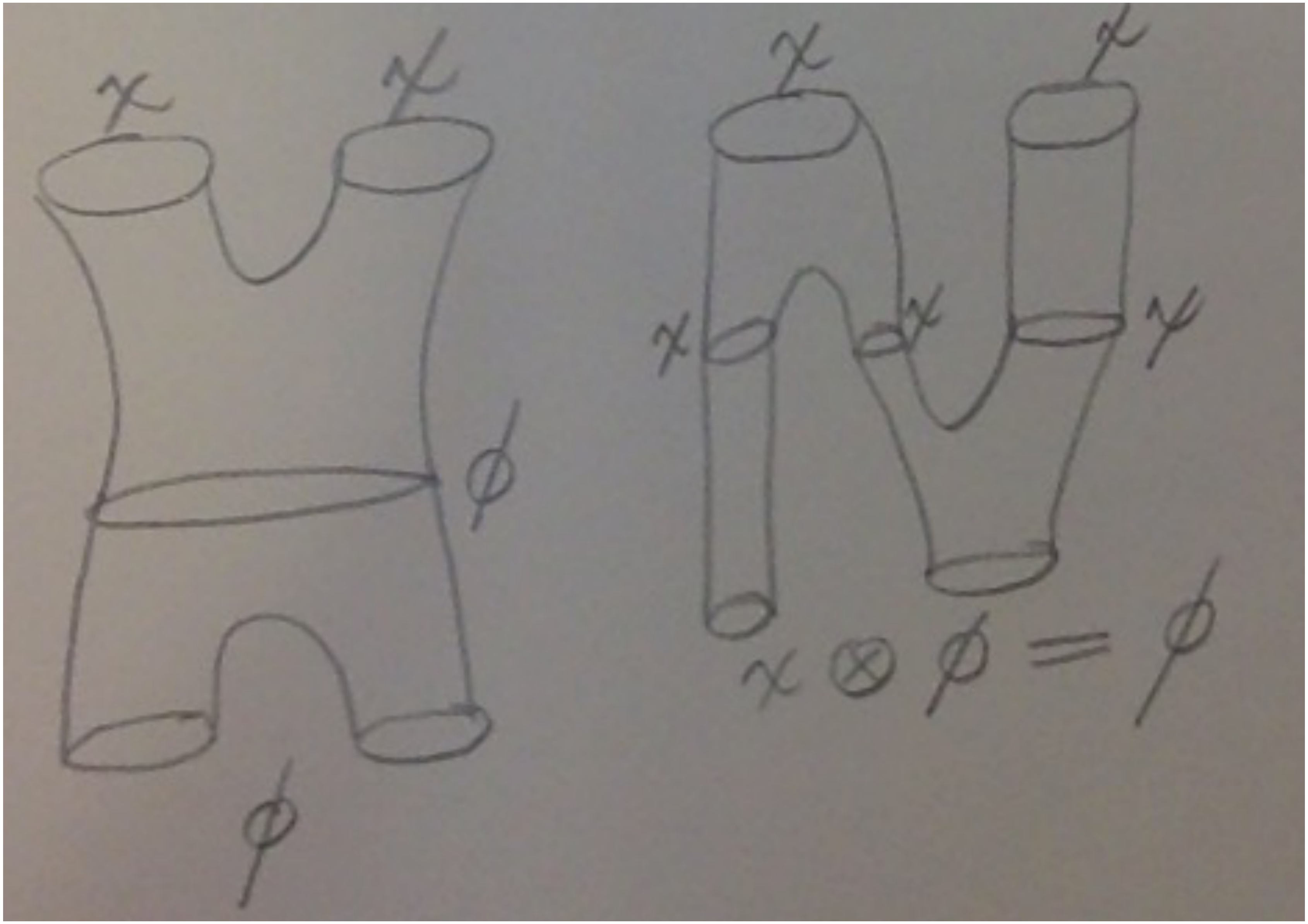
Enhanced States
Plus
Boundary
Requirement
Yields
Frobenius Algebra.

Checking Order Compatibility

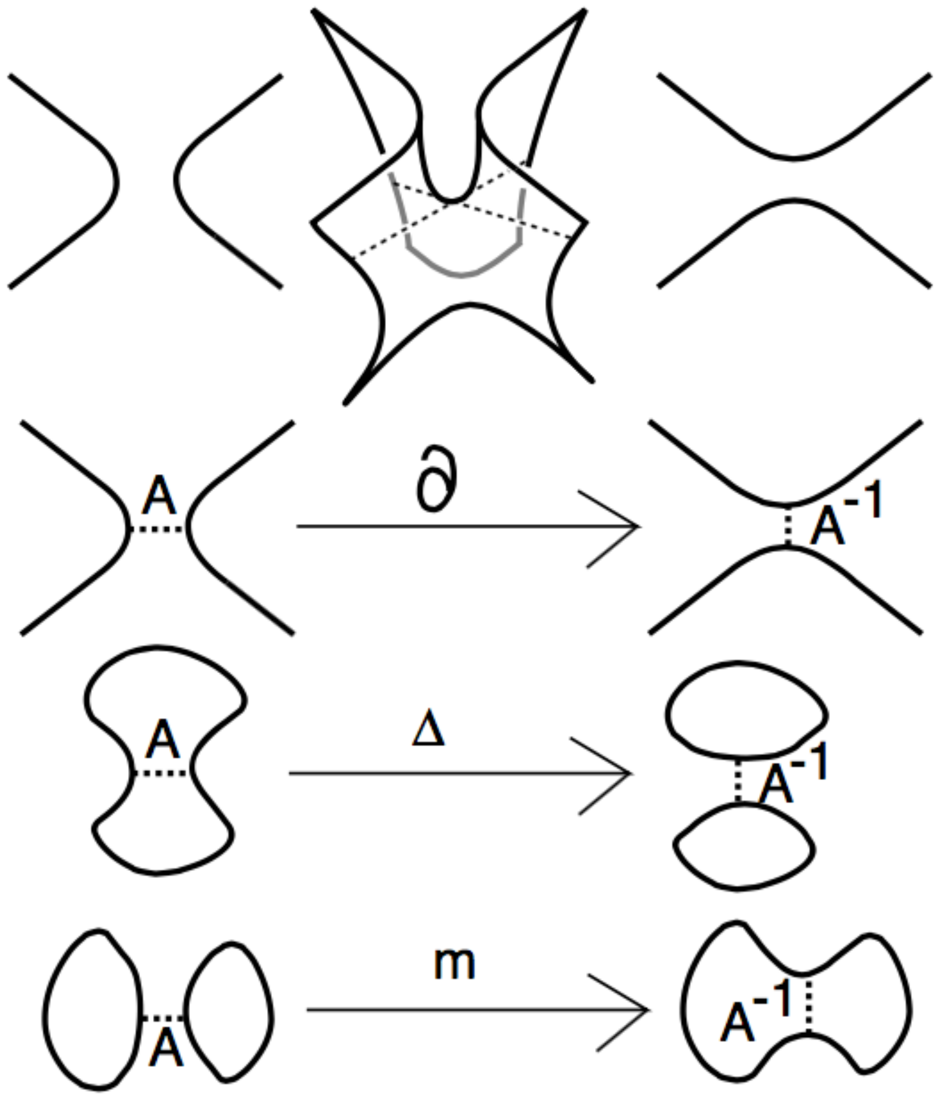




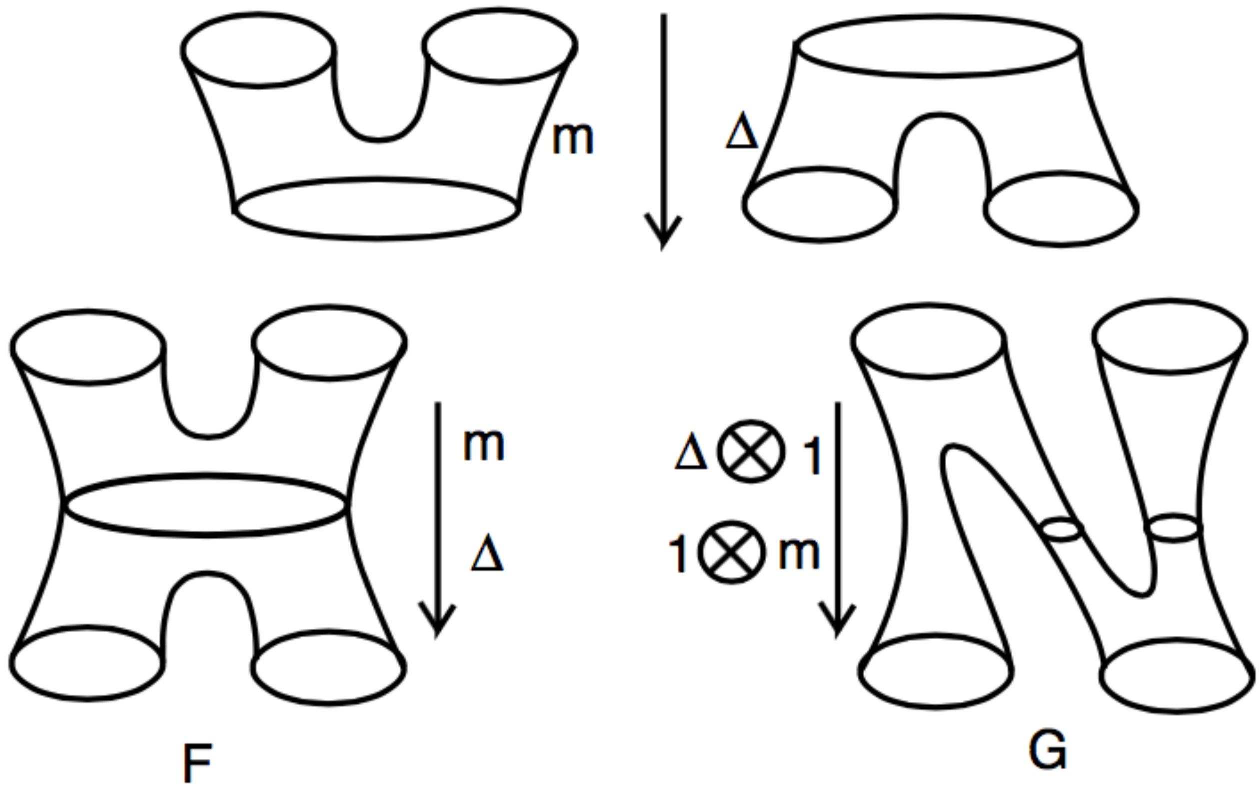
1
x
x
x ⊗ x
1
x
(1⊗x + x⊗1)⊗x
=
1⊗(x⊗x) + x⊗(1⊗x)
∅ + x⊗x
=
x⊗x



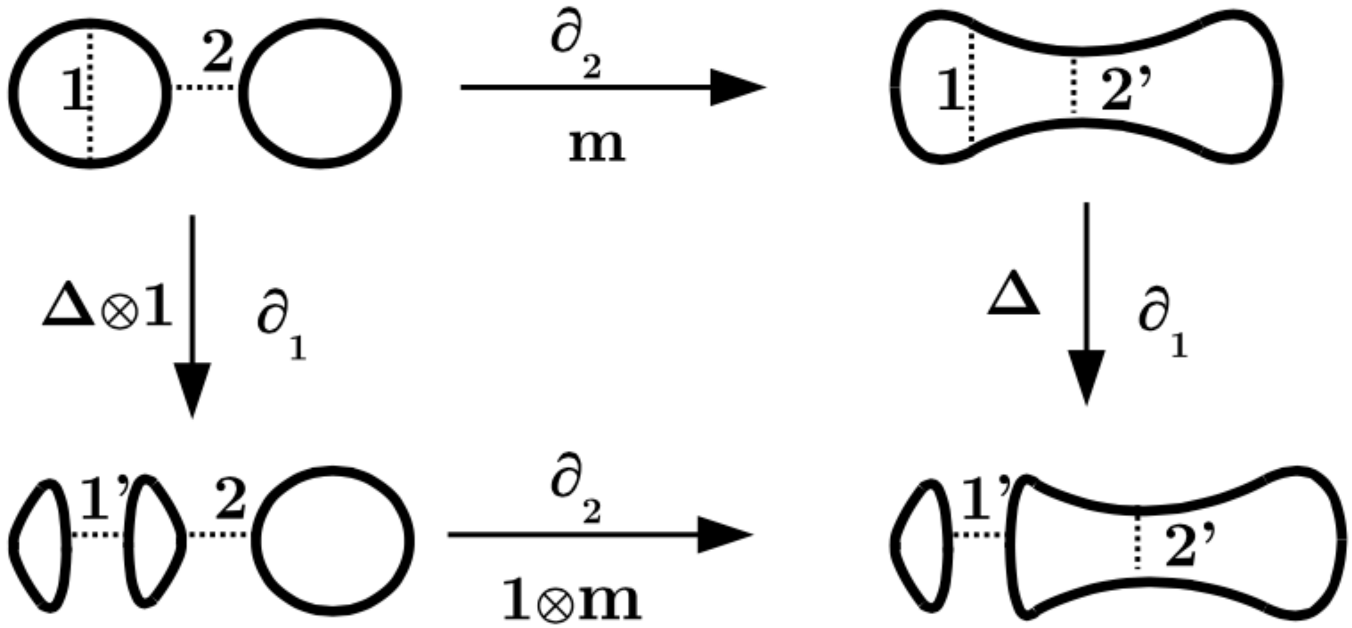
x
ψ
φ
φ
x
ψ
x
x
ψ
x ⊗ φ = ψ



A
∂
A-1
A
Δ
A-1
A
m
A-1



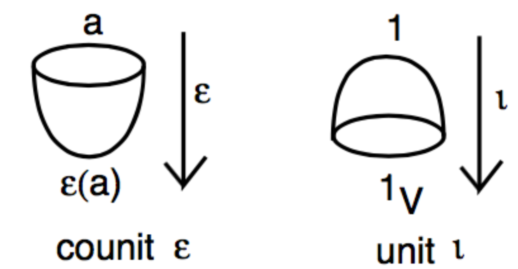
m
Δ
m
Δ
Δ ⊗ 1
1 ⊗ m
F
G



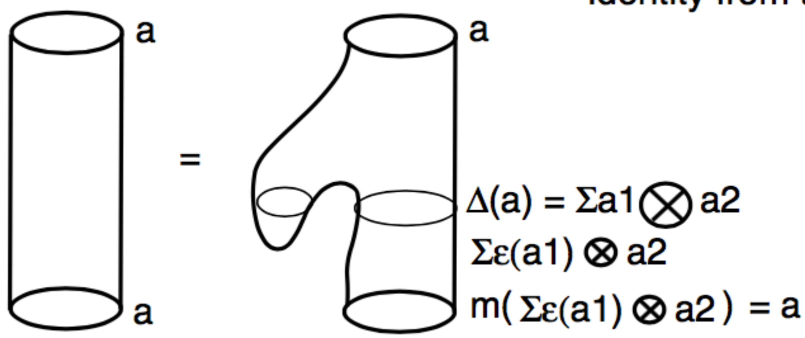
$$\partial_2\partial_1 = (1\otimes m)(\Delta\otimes 1)$$

$$\partial_1\partial_2 = (\Delta)(m)$$

$$\partial_1\partial_2 = \partial_2\partial_1$$



Evaluations at
successive levels.
Identity from topology.

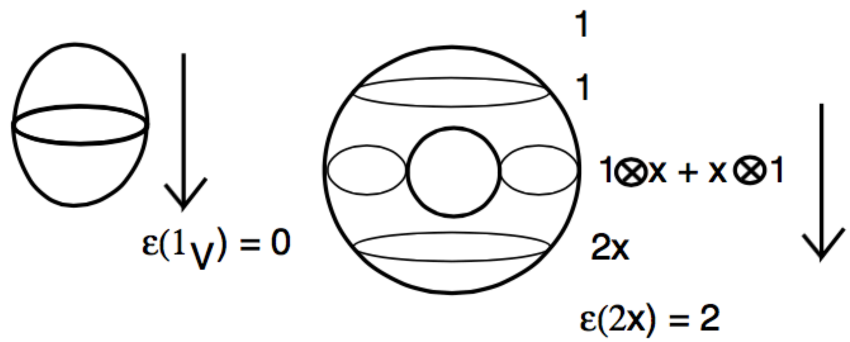


Using special case of a=1, we obtain:

$m(\varepsilon(1) \otimes x + \varepsilon(x) \otimes 1) = 1$

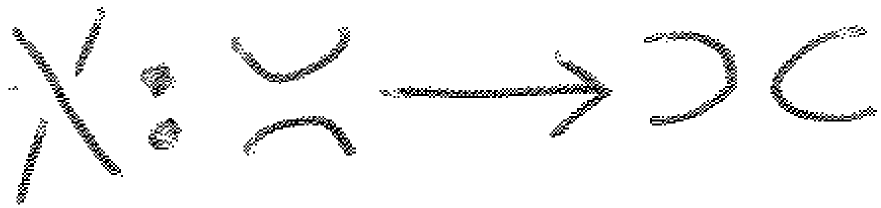
$\Longrightarrow \varepsilon(1)x + \varepsilon(x)1 = 1$

$\Longrightarrow \begin{array}{l} \varepsilon(1) = 0 \\ \varepsilon(x) = 1 \end{array}$



We have arrived at the Frobenius algebra, but there is still work to be done to see the invariance under ambient isotopy of knots and links.

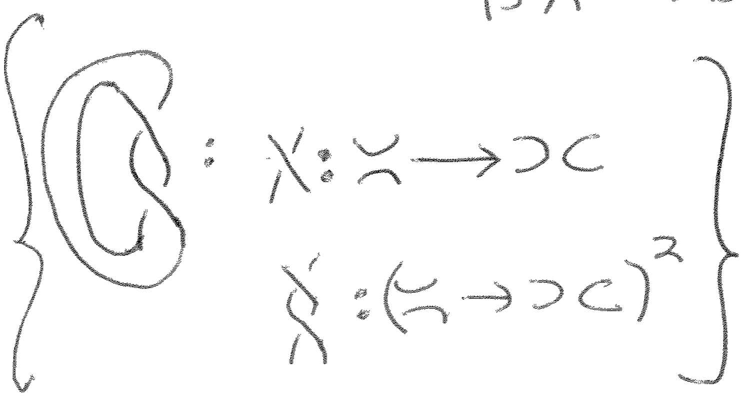
Categorification and the Morse Dream



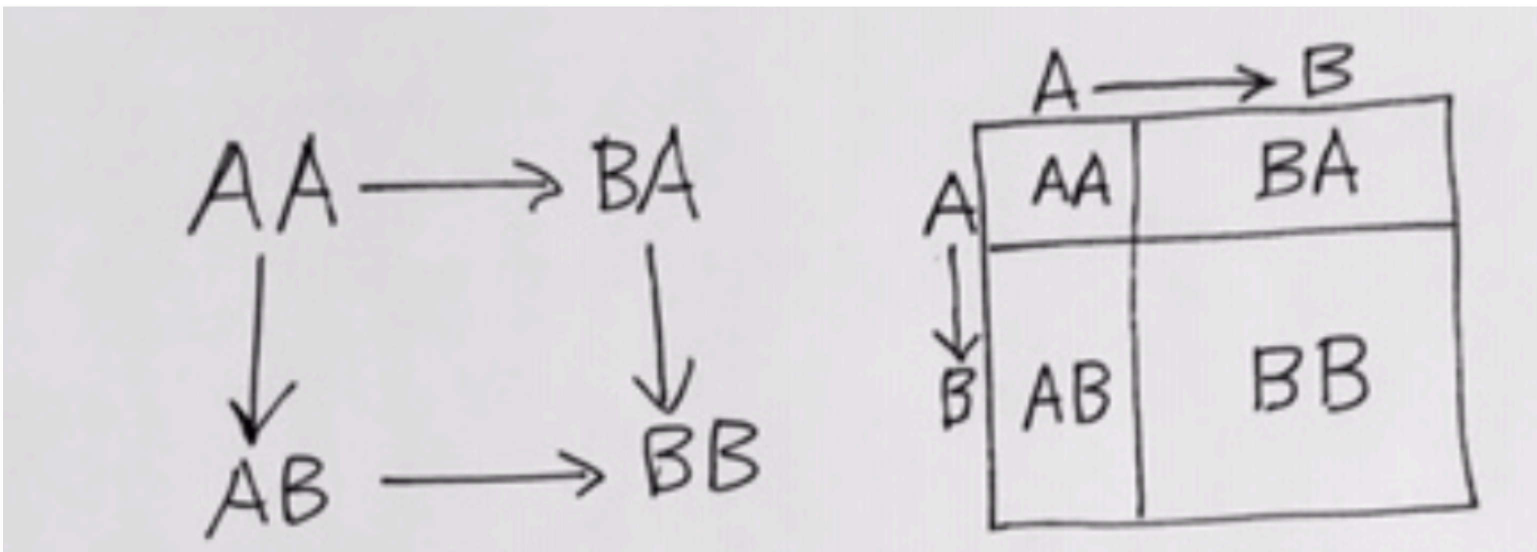
$$(A \to B)^2 = (A \to B)(A \longrightarrow B) = A(A \to B) \longrightarrow B(A \to B)$$

$$= (AA \to AB) \longrightarrow (BA \to BB)$$

$$= \begin{tikzcd} AA \arrow[r] \arrow[d] & AB \\ BA \arrow[r] & BB \end{tikzcd} = \begin{tikzcd} AA \arrow[r] \arrow[d] & AB \arrow[d] \\ BA \arrow[r] & BB \end{tikzcd}$$



Morse Diagram $\longrightarrow$ Cube Category

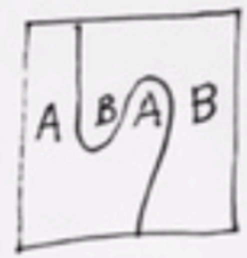

AA
BA
AB
BB
A
B
A
AA
BA
B
AB
BB

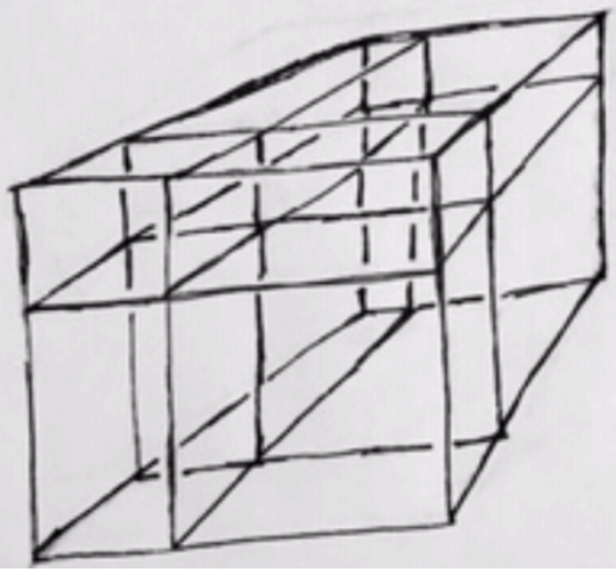
Flatten $(A \to B)^3$:

$$\begin{tikzcd}
 & AAB \arrow[rr] \arrow[dd] & & BAB \arrow[dd] \\
AAA \arrow[ur] \arrow[rr] \arrow[dd] & & BAA \arrow[ur] \arrow[dd] & \\
 & ABB \arrow[rr] & & BBB \\
ABA \arrow[ur] \arrow[rr] & & BBA \arrow[ur] &
\end{tikzcd}$$

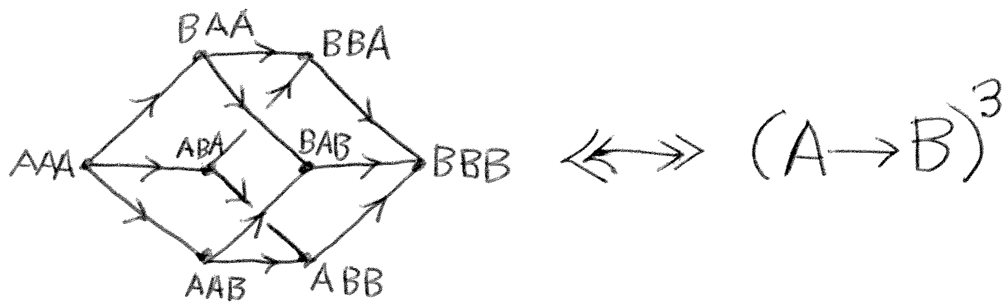
Region walls indicate morphisms.

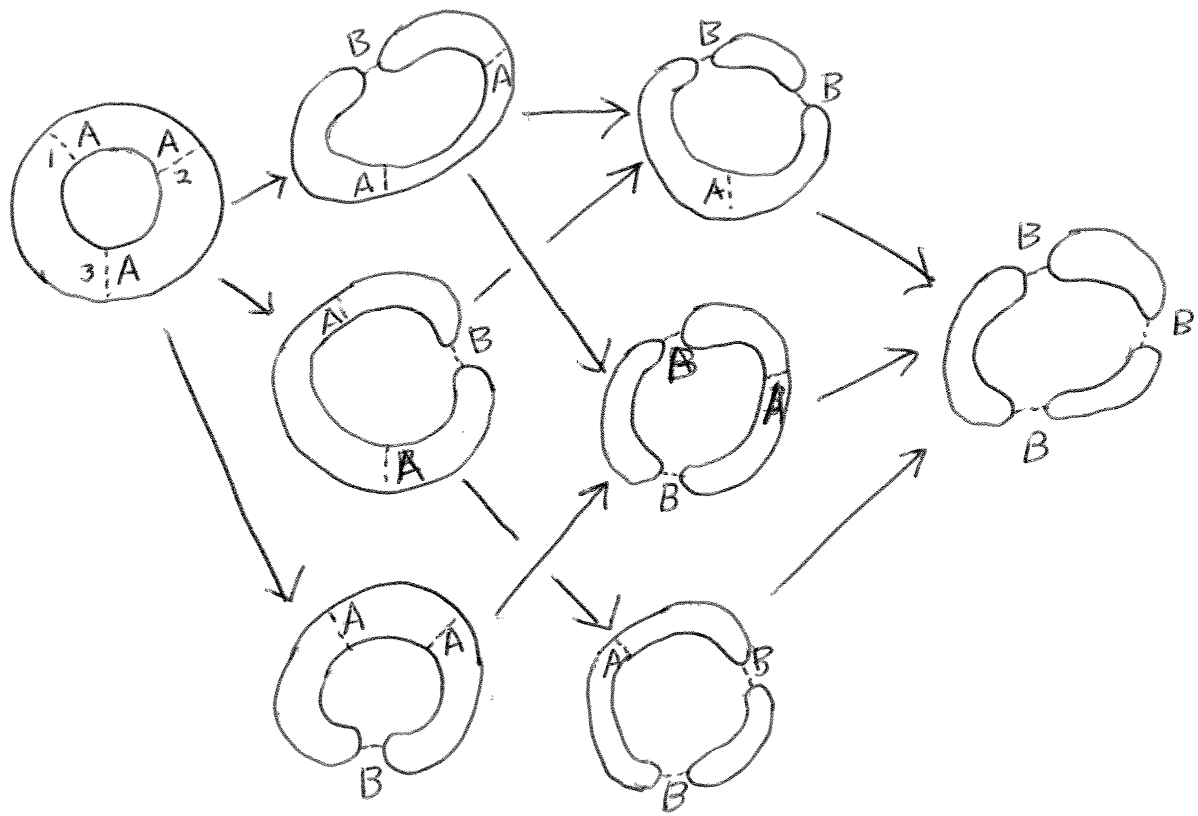
Compare with string diagrams.



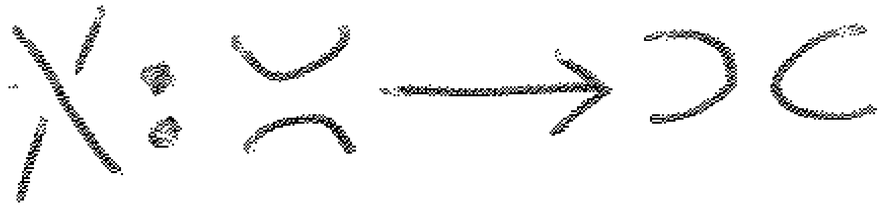


Cubism Again





Categorification and the Morse Dream

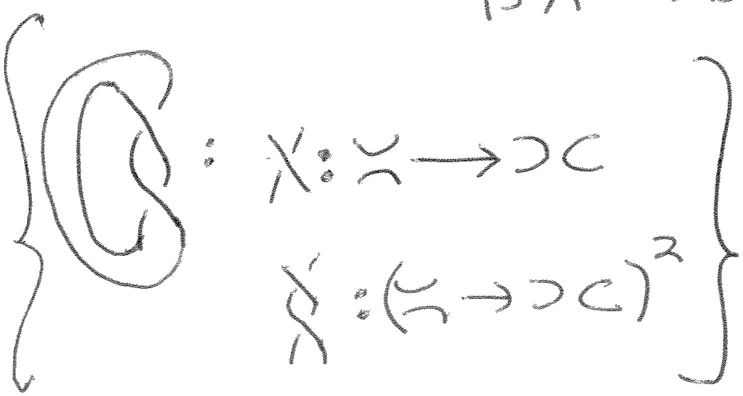


(flattening a higher category)

$$(A\to B)^2 = (A\to B)(A\longrightarrow B) = A(A\to B)\longrightarrow B(A\to B)$$

$$= (AA\to AB)\longrightarrow(BA\to BB)$$

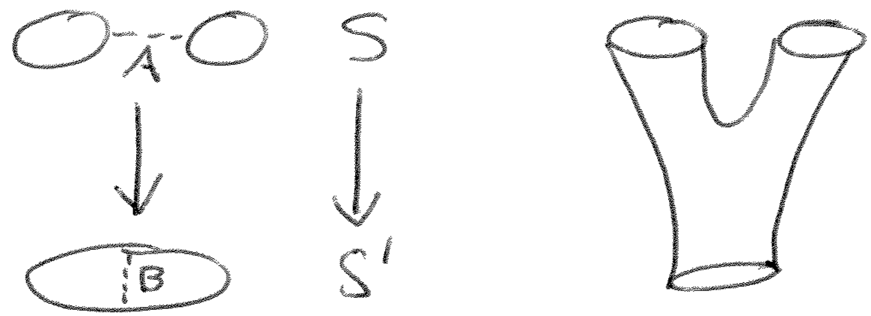
$$= \begin{array}{c} AA \longrightarrow AB \\ \downarrow \\ BA \longrightarrow BB \end{array} = \begin{array}{ccc} AA & \longrightarrow & AB \\ \downarrow & & \downarrow \\ BA & \longrightarrow & BB \end{array}$$



Morse Diagram $\longrightarrow$ Cube Category

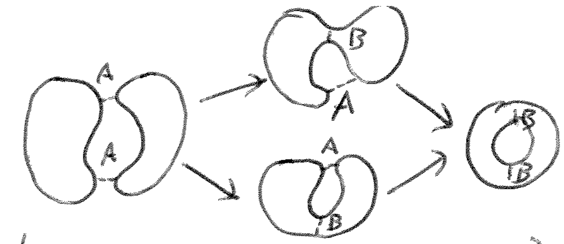
We have category $\mathcal{S}(K)$ whose objects are states S of K & morphisms gen by arrows $S \longrightarrow S'$, $b(S)+1 = b(S')$.

Regard the arrow as a surface cobordism.



Two surface cobs are $=$ as morphisms iff the corresponding surfaces are homeomorphic.

$$\underbrace{\begin{tikzcd} & AB \arrow[dr, "\partial_1"] & \\ AA \arrow[ur, "\partial_2"] \arrow[dr, "\partial_1"'] & & BB \\ & BA \arrow[ur, "\partial_2"'] & \end{tikzcd}}_{\mathrm{Cube}[2]} \xrightarrow[\mathcal{F}_H]{}$$

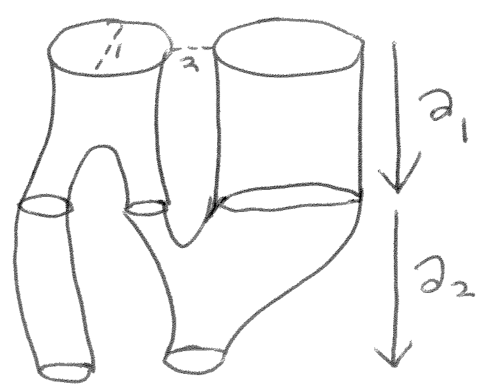


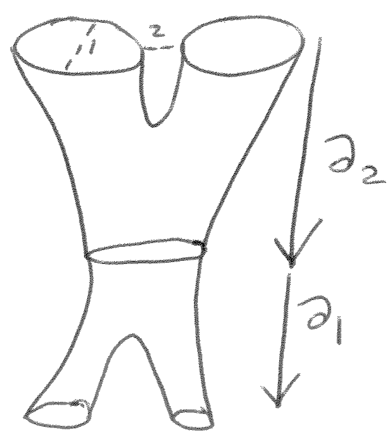
$\mathcal{S}$ (Hopf)  $= H$

$$\mathcal{S}(K) = \mathcal{F}_K\left(\mathrm{Cube}[c(K)]\right)$$

and $\mathcal{F}$ extends to a <u>functor</u> from the cube category to the category $\mathcal{S}(K)$.

This means that all relevant squares commute. e.g.





We make an abstract analogue of a chain complex from $\mathcal{S}(K)$ by extending to an additive category with dir sums.

$$A_1, \dots, A_n \rightsquigarrow A = \bigoplus_{i=1}^{n} A_i$$

$$f : A \longrightarrow B, \quad B = \bigoplus_{j=1}^{m} B_j$$

$$f = (f_{ij}), \quad f_{ij} : A_i \longrightarrow B_j .$$

$$g : B \longrightarrow C$$

$$A_i \xrightarrow{f_{ik}} B_k \xrightarrow{g_{kj}} C_j$$

$$(g \circ f)_{ij} = \sum_k g_{kj} \circ f_{ik}$$

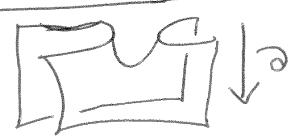
$$S \in \mathcal{O}bj(\mathcal{S}(K))$$

$$\mathcal{C}^i(K) = \bigoplus_{\substack{S \in \mathcal{O}bj(\mathcal{S}(K)) \\ b(S) = i}} S$$

$$\partial : \mathcal{C}^i(K) \longrightarrow \mathcal{C}^{i+1}(K)$$

$$\partial = \sum_{k=1}^{c(K)} \pm \partial_k$$

$$\partial : \Sigma^A \longrightarrow \Sigma_B$$



Dror's Canopoly

An abstract categorical analog of a chain complex.
That can be taken up to chain homotopy.
The maps are additive combinations of surface cobordisms.

We say $f \sim g$ iff $\exists\, H : \mathcal{C} \longrightarrow \mathcal{C}'$ s.t. $\partial H + H\partial = f - g$.

Work Mod 2 .

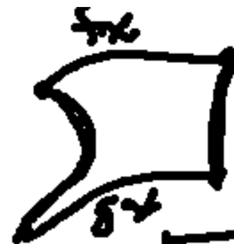
Categorical Chain Homotopy

Question: What is least equiv reln on $\mathcal{CS}(K)$ s.t. $[\mathcal{CS}(K)]$ (= chain homotopy equiv class of) is invar under RM's ?

We examine this question as though we had not seen the Frobenius algebra.

space

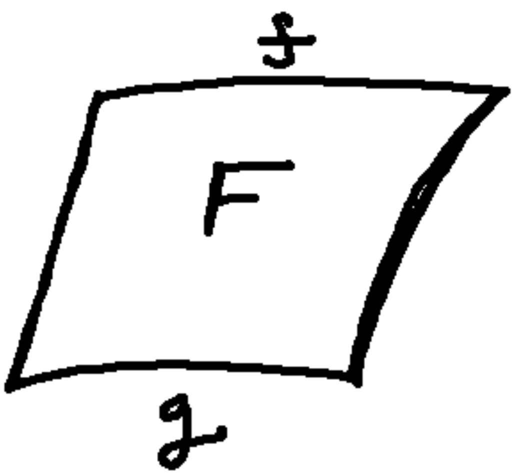
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

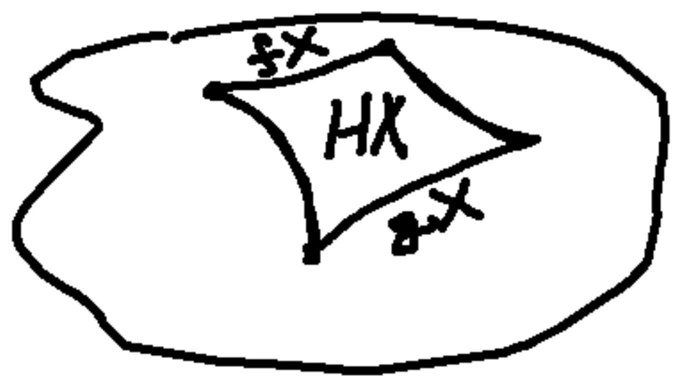


$$\boxed{\partial HX = fX + gX + H\partial X}$$

$$X \times I \xrightarrow{F} Y$$

$f \sim g \iff \exists F$ s.t. $F(x,0) = f$
$F(x,1) = g$





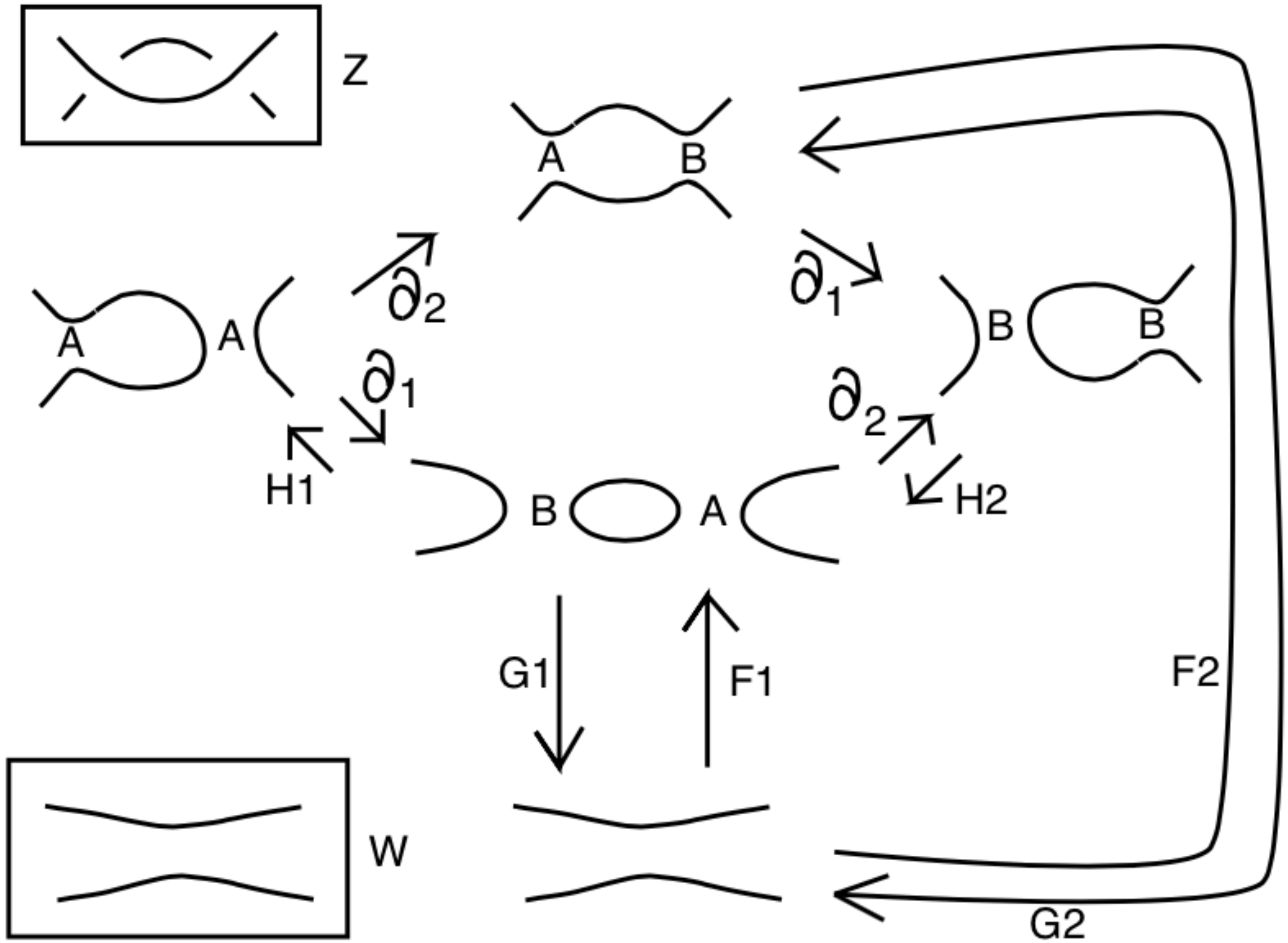


Figure 11: **Complexes for Second Reidemeister Move**

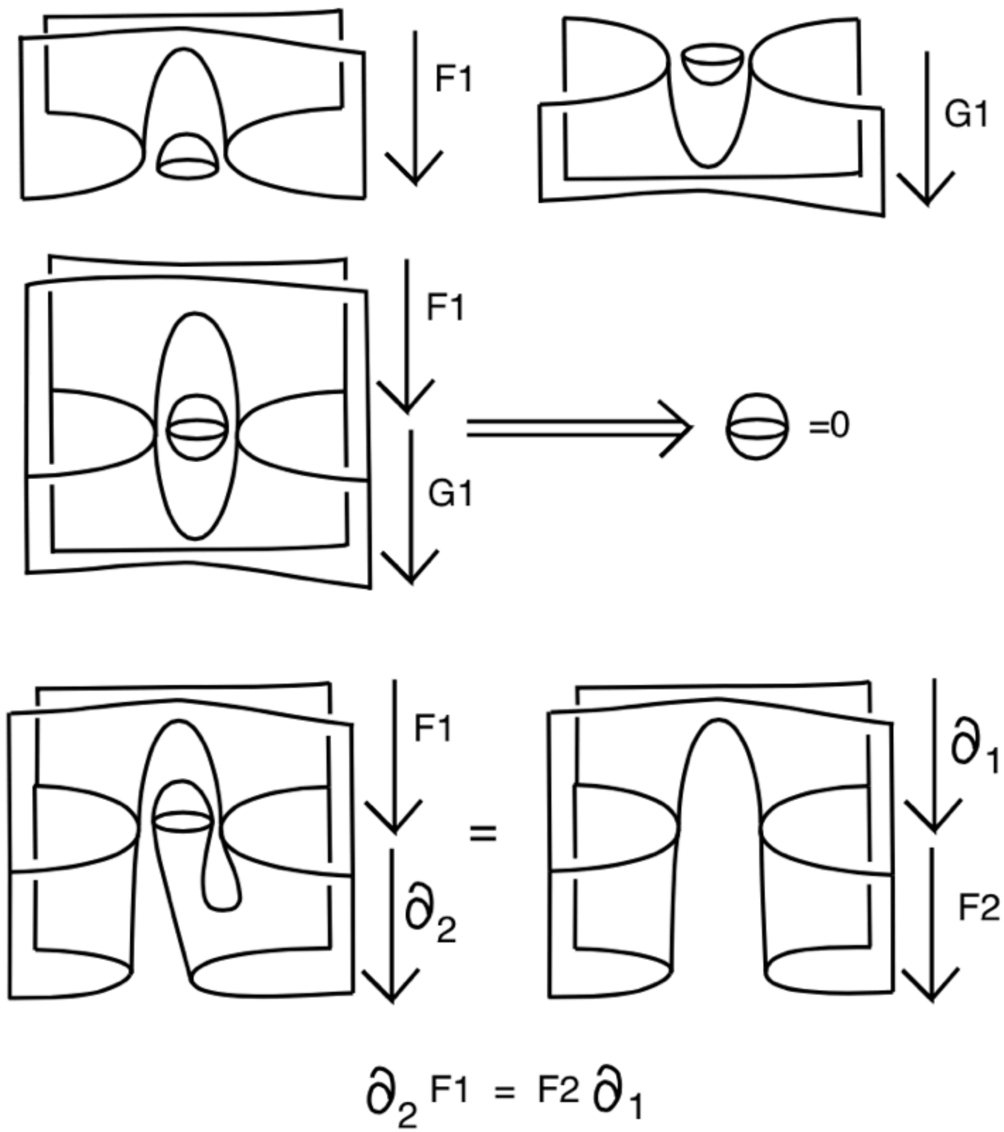


Figure 12: **Cobordism Compositions for Second Reidemeister Move**

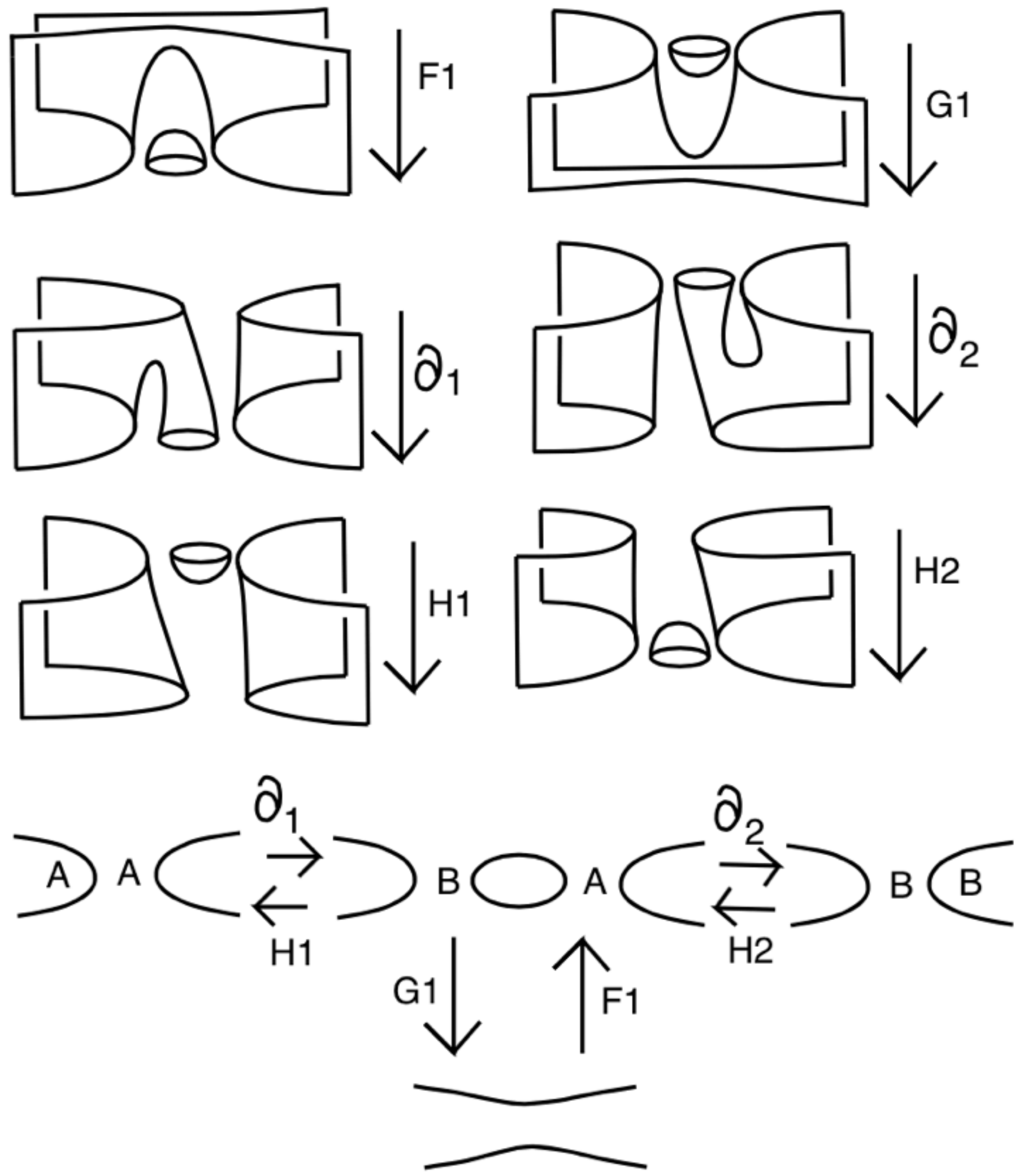


Figure 13: **Preparation for Homotopy for Second Reidemeister Move**

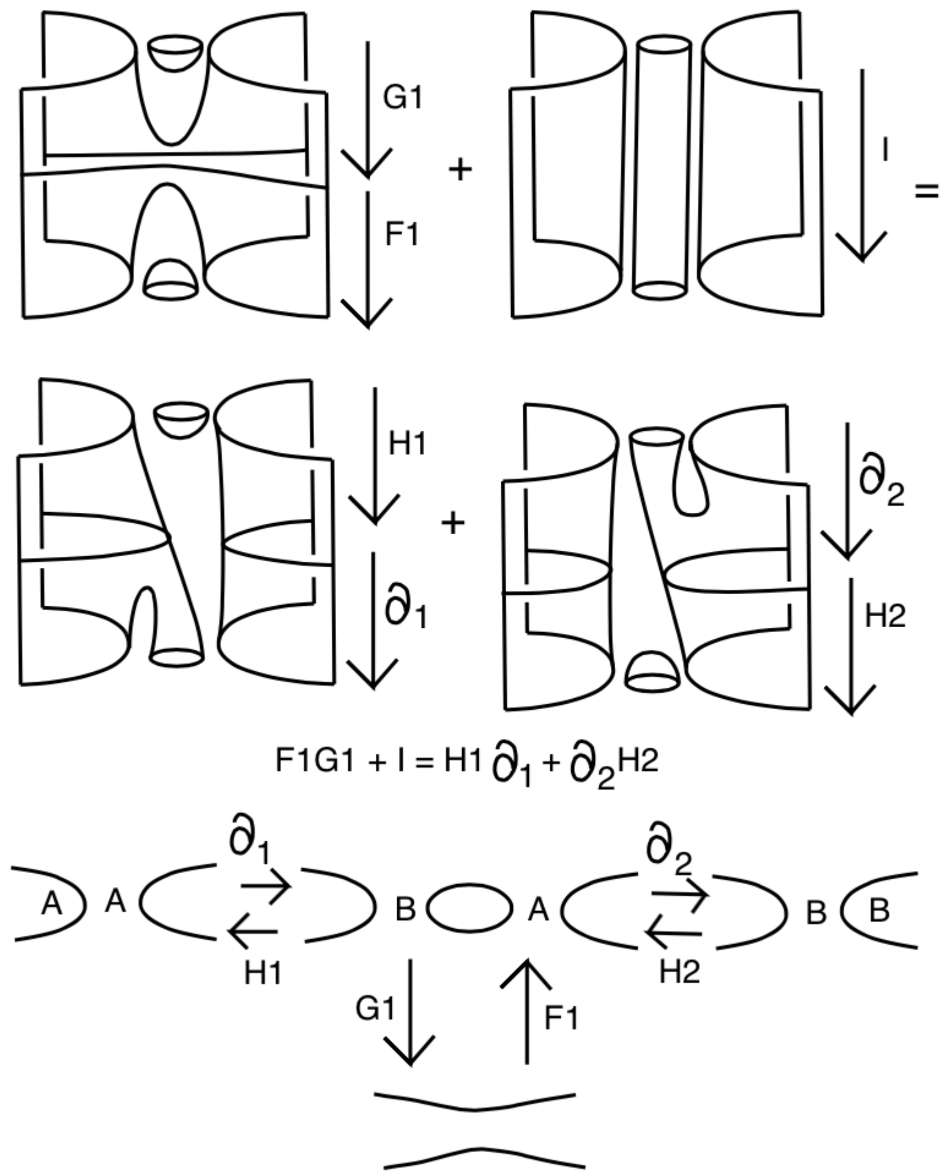


Figure 14: **Homotopy for Second Reidemeister Move**

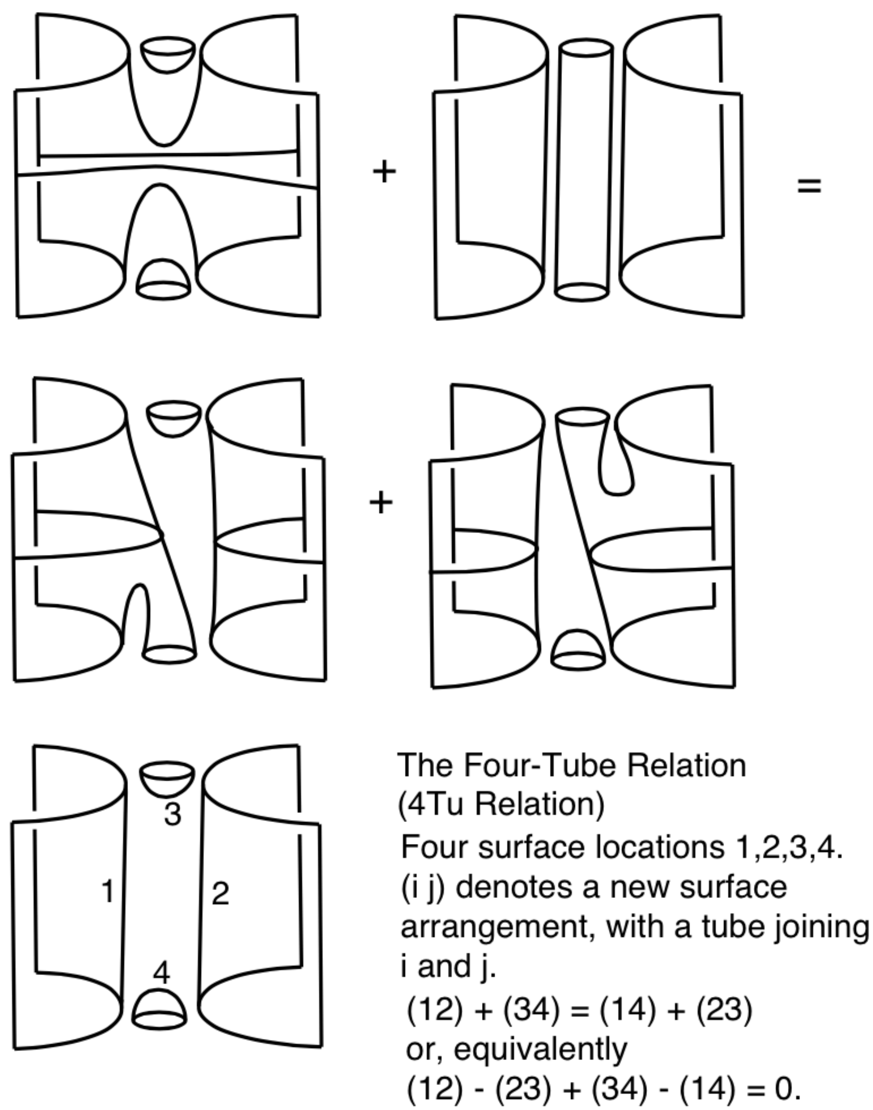
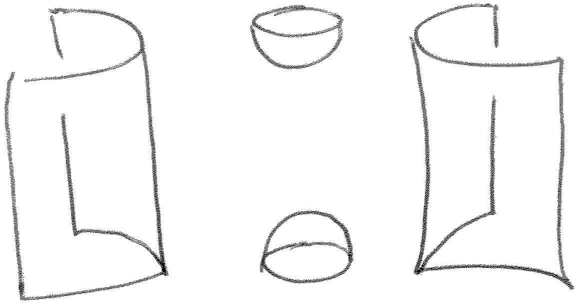
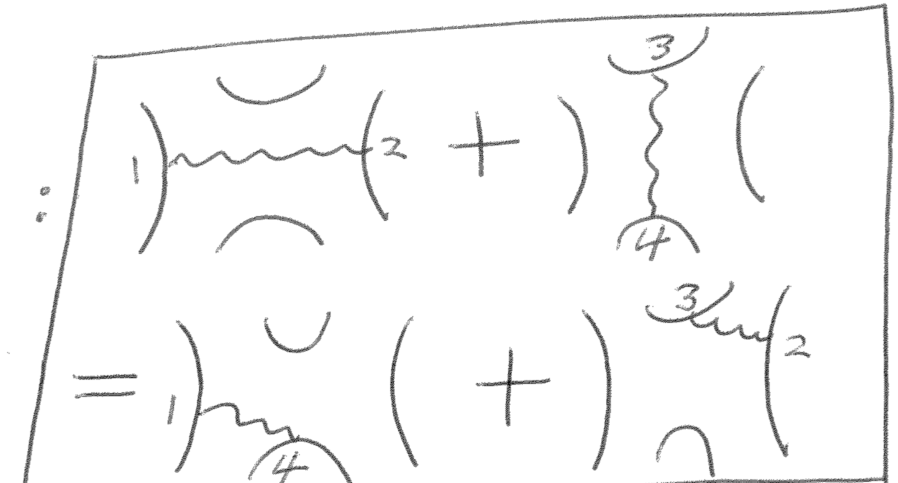


Figure 15: **Four-Tube Relation From Homotopy**



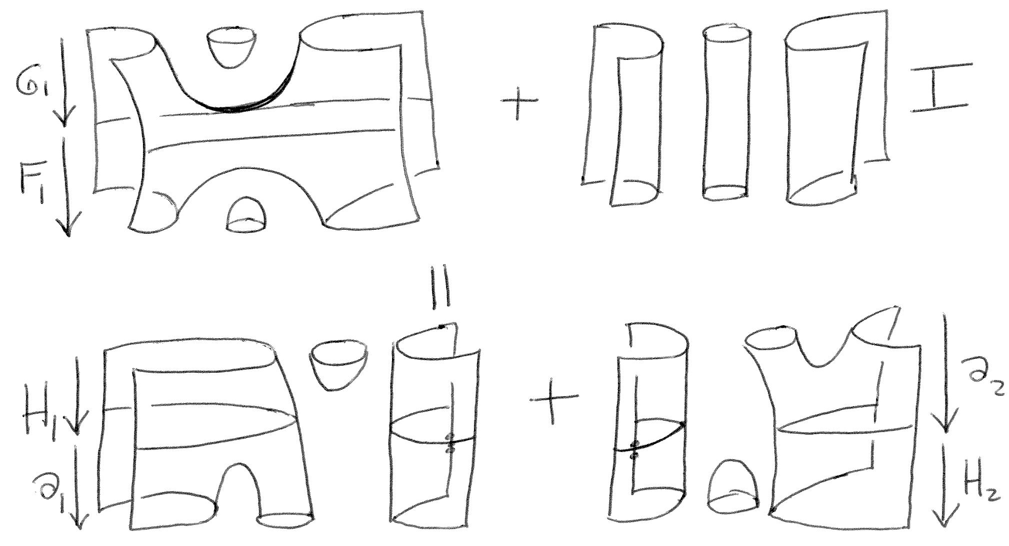
$4Tu + \{\ominus = \phi\}$

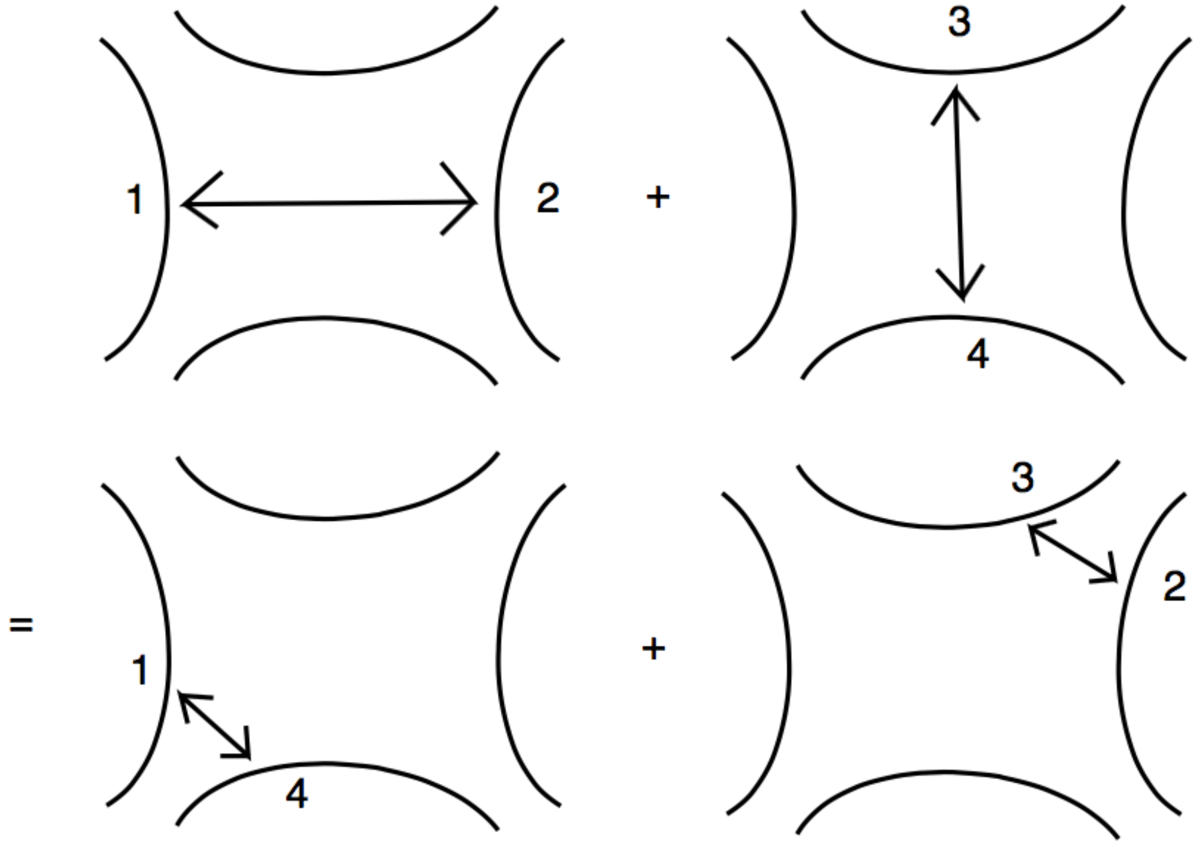
$\Longrightarrow$ Invariance



$\widehat{12}34 - 1\widehat{23}4 + 12\widehat{34} - \widehat{1234} = \phi$

4Tu Relation

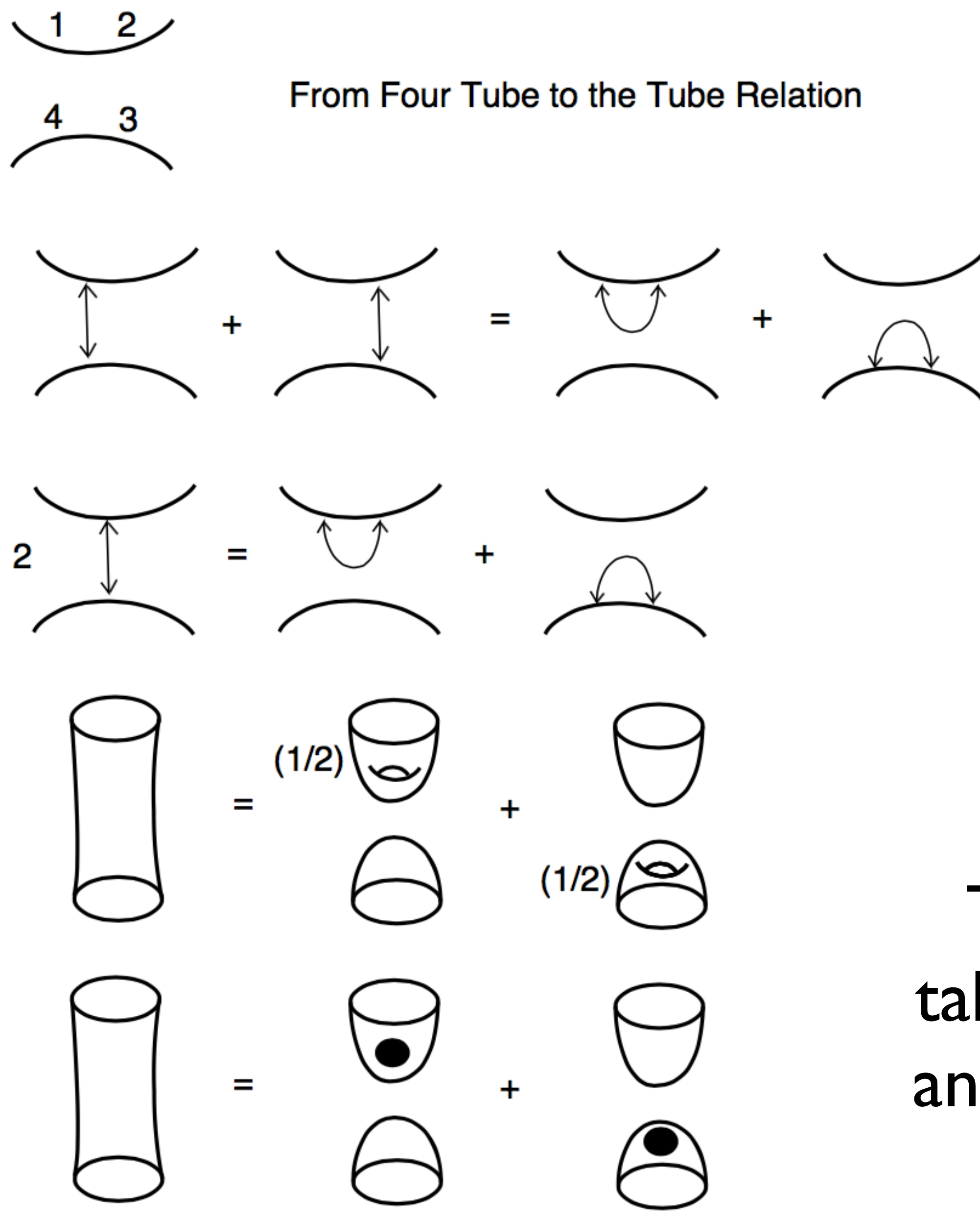




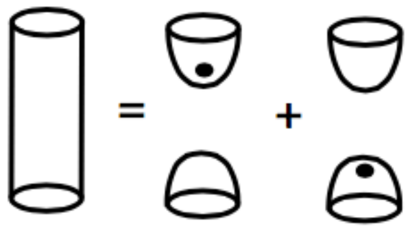
$$\overset{\frown}{12}34 - 1\overset{\frown}{23}4 + 12\overset{\frown}{34} - \overset{\frown}{1234} = 0$$

Schematic Four-Tube Relation

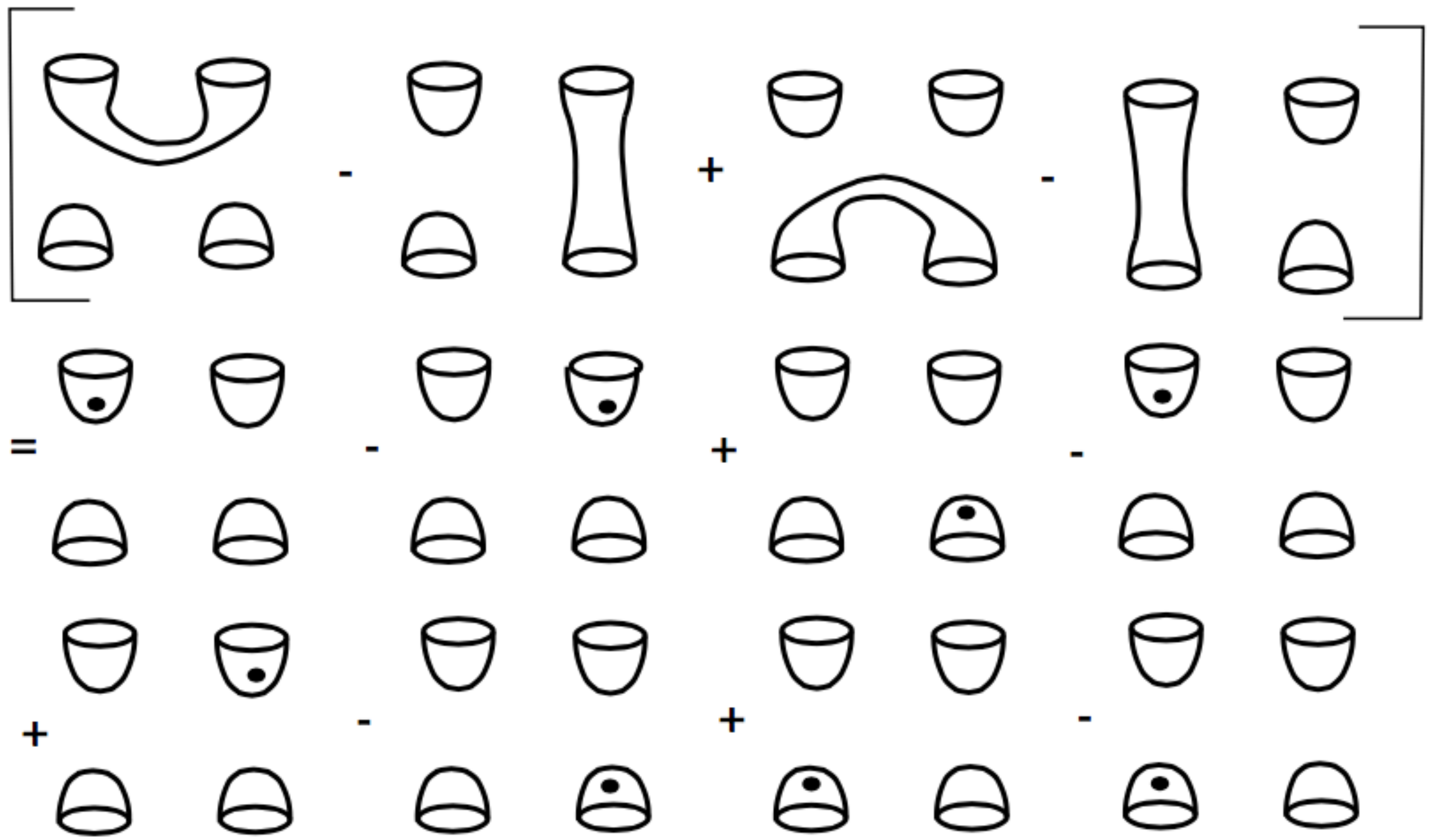
From Four Tube to the Tube Relation



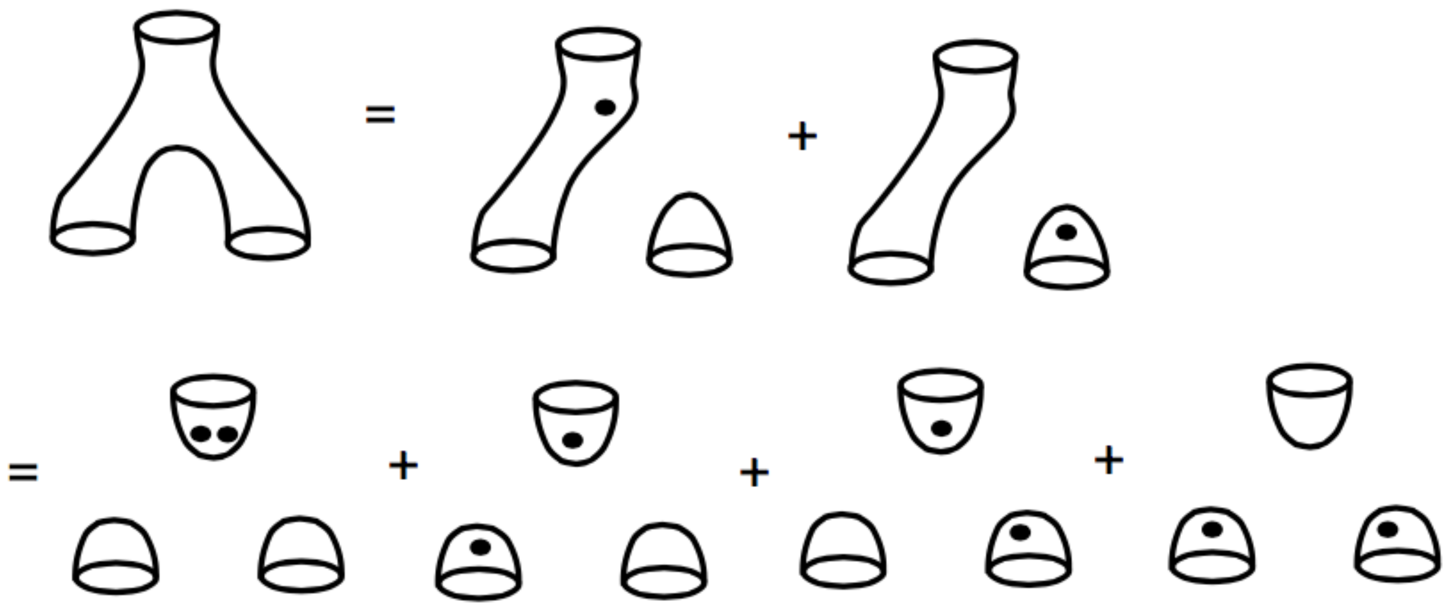
The dot can be taken to represent an algebra element x.



The Tube Relation implies the Four Tube Relation.

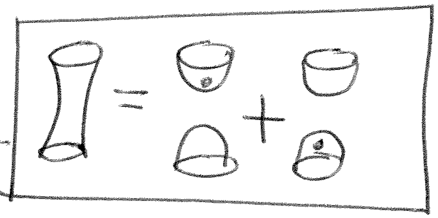


= 0.



Coproduct via the Tube-Relation

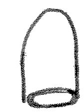
From 4Tu to Frobenius Algebra



Tube Relation

Now find algebra $\mathcal{A}$

 $\mathcal{A} \xrightarrow{\varepsilon} \mathbb{Z}$ counit

 $\mathbb{Z} \xrightarrow{\eta} \mathcal{A}$ unit

$\bullet \equiv x \in \mathcal{A}, \quad \alpha \in \mathcal{A}$

$$\Rightarrow \boxed{\alpha = \varepsilon(\alpha x)1 + \varepsilon(\alpha)x}$$

$$\Longrightarrow \left.\begin{array}{l} x = \varepsilon(x^2)1 + \varepsilon(x)x \\ 1 = \varepsilon(x)1 + \varepsilon(1)x \end{array}\right\} \quad \boxed{\begin{array}{l} \varepsilon(x) = 1 \\ \varepsilon(x^2) = 0 \\ \varepsilon(1) = 0 \end{array}}$$

$$x^2 = \varepsilon(x^3)1 + \varepsilon(x^2)x$$

$$\Rightarrow x^2 = k1, \quad k \in \mathbb{Z}$$

$$\boxed{x^2 = k}$$

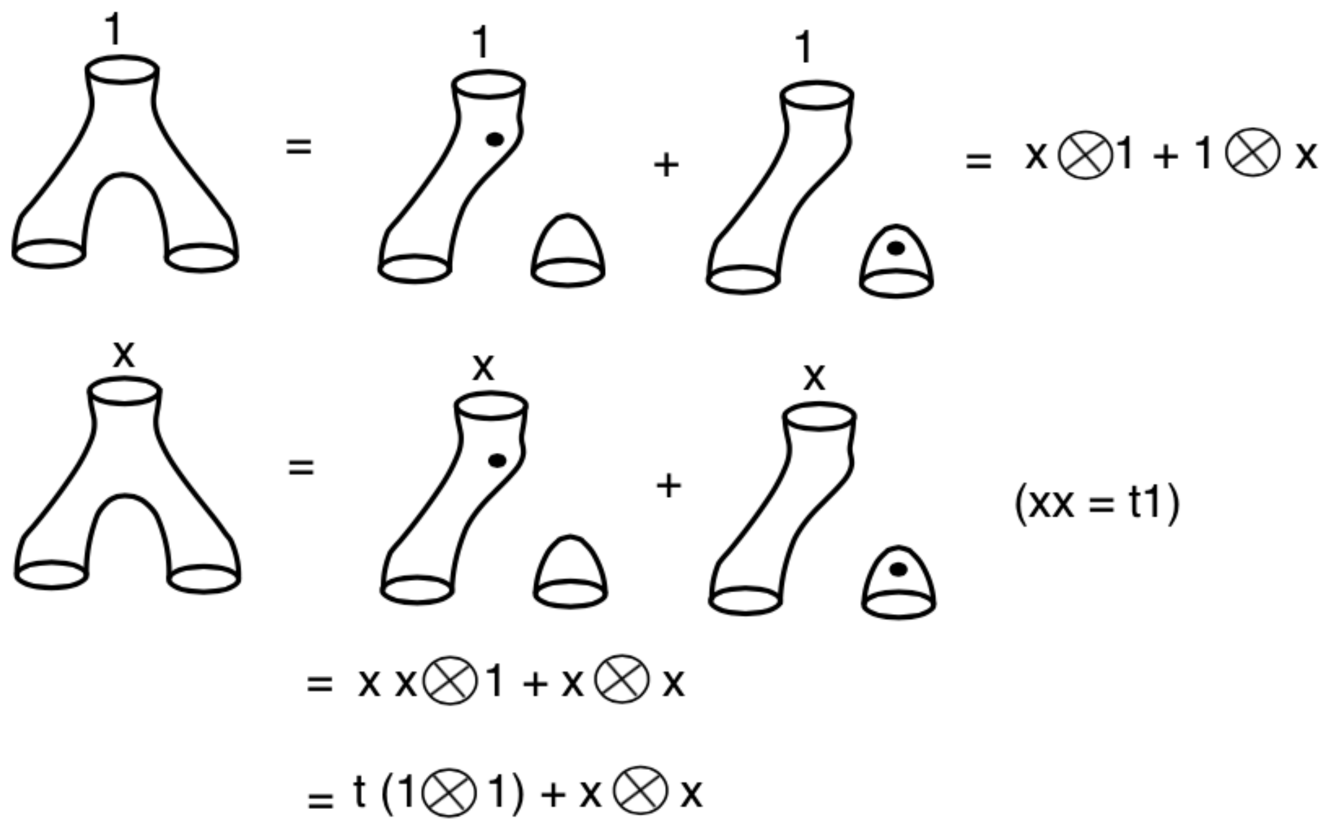
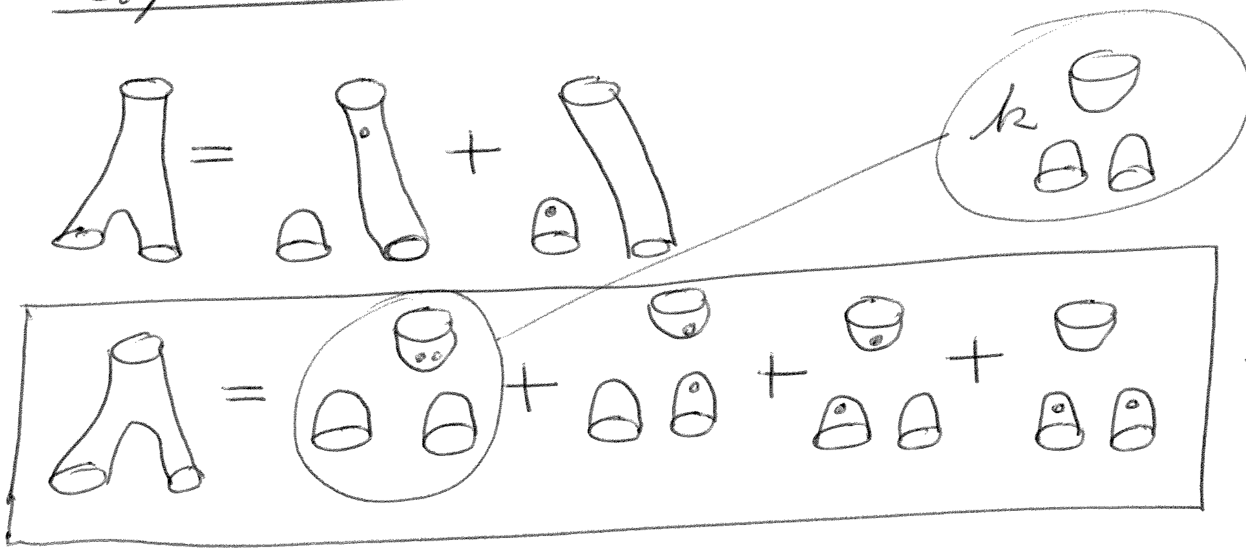


Figure 20: **Coproducts of** 1 **and** x **Via Tube-Cutting Relation**

Coproduct



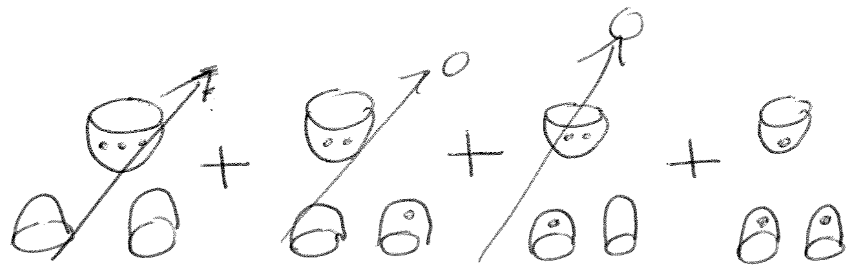
$$\boxed{\begin{aligned} \varepsilon(x) &= 1 \\ \varepsilon(x^2) &= 0 \\ \varepsilon(1) &= 0 \end{aligned}}$$

$x^2 = \varepsilon(x^3)1 + \varepsilon(x^2)x$

$x^2 = k1$, $k \in \mathbb{Z}$

$\boxed{x^2 = k}$

$\Delta(x) =$



$$\boxed{\Delta(x) = k(1 \otimes 1) + x \otimes x}$$

$$\boxed{\Delta(1) = 1 \otimes x + x \otimes 1}$$

Algebra from 4Tu - Guaranteed to Produce Link Homology

$\mathcal{A} = \mathbb{Z}[x]/(x^2 - k)$

$\varepsilon(x) = 1, \ \varepsilon(1) = 0$

$\Delta(1) = 1 \otimes x + x \otimes 1$

$\Delta(x) = k(1 \otimes 1) + x \otimes x$

$k = 0$: Khovanov

$k = 1$: Lee

Lee's Algebra

$$x^2 = 1,$$

$$\Delta(1) = 1 \otimes x + x \otimes 1,$$

$$\Delta(x) = x \otimes x + 1 \otimes 1,$$

$$\epsilon(x) = 1,$$

$$\epsilon(1) = 0.$$

This gives a link homology theory that is distinct from Khovanov homology. In this theory, the quantum grading j is not preseved, but we do have that

$$j(\partial(\alpha)) \geq j(\alpha)$$

for each chain α in the complex. This means that *one can use j to filter the chain complex for the Lee homology*. The result is a spectral sequence that starts from Khovanov homology and converges to Lee homology.

Lee's Algebra

$$\mathcal{A} = \mathbb{Q}[x]/(x^2-1)$$

$$\varepsilon(x) = 1, \quad \varepsilon(1) = 0$$

$$\Delta(1) = 1 \otimes x + x \otimes 1$$

$$\Delta(x) = (1 \otimes 1) + x \otimes x$$

Let $r = \frac{1+x}{2}$, $g = \frac{1-x}{2}$

$$\varepsilon(r) = 1/2, \quad \varepsilon(g) = -1/2$$

$$r + g = 1$$

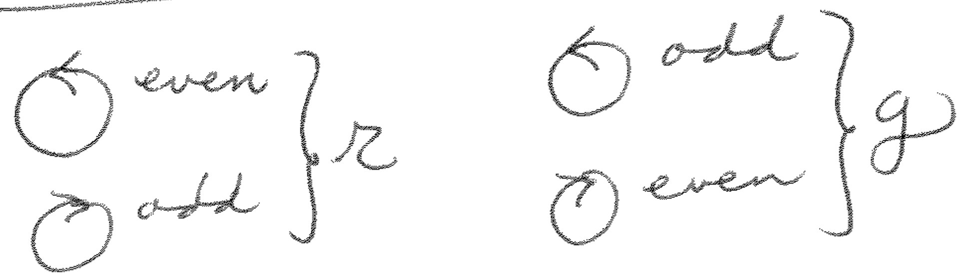
$$r^2 = r, \quad g^2 = g$$

$$rg = 0$$

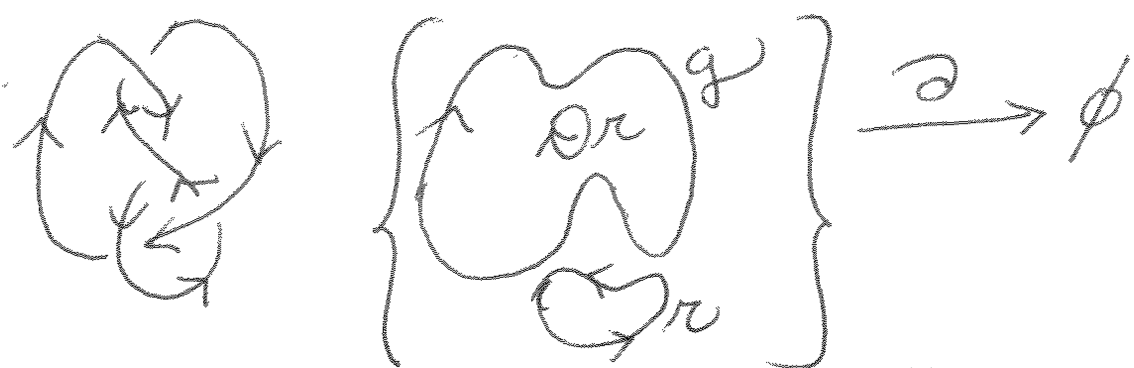
$$\Delta(r) = 2\, r \otimes r$$

$$\Delta(g) = -2\, g \otimes g$$

Lee Homology is gen by Seifert smoothing states for all choices of orientation of link.



e.g.



Lee homology is simple. One has that the dimension of the Lee homology is equal to $2^{comp(L)}$ where $comp(L)$ denotes the number of components of the link L. Up to homotopy, Lee's homology has a vanishing differential, and the complex behaves well under link concondance. In his paper [4] Dror BarNatan remarks "In a beautiful article Eun Soo Lee introduced a second differential on the Khovanov complex of a knot (or link) and showed that the resulting (double) complex has non-interesting homology. This is a very interesting result." Rasmussen [49] uses Lee's result to define invariants of links that give lower bounds for the four-ball genus, and determine it for torus knots. This gives an (elementary) proof of a conjecture of Milnor that had been previously shown using gauge theory by Kronheimer and Mrowka [29].

Rasmussen's result uses the Lee spectral sequence. We have the quantum (j) grading for a diagram K and the fact that for Lee's algebra $j(\partial(s)) \geq j(s)$. Rasmussen uses a normalized version of this grading denoted by $g(s)$. Then one makes a filtration $F^kC^*(K) = \{v \in C^*(K) | g(v) \geq k\}$ and given $\alpha \in Lee^*(K)$ define

$$S(\alpha) := max\{g(v) | [v] = \alpha\}$$

$$s_{min}(K) := min\{S(\alpha) | \alpha \in Lee^*(K), \alpha \neq 0\}$$
$$s_{max}(K) := max\{S(\alpha) | \alpha \in Lee^*(K), \alpha \neq 0\}$$

and

$$s(K) := (1/2)(s_{min}(K) + s_{max}(K)).$$

This last average of s_{min} and s_{max} is the Rasmussen invariant.

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$$\begin{cases} n_+ = \#\text{ of } + \text{ crossings in } K. \\ n_- = \#\text{ of } - \text{ crossings in } K. \end{cases}$$

$$\boxed{\begin{aligned} j(s) = \#(\text{B smoothings}) \\ + \#(1\text{'s}) - \#(x\text{'s}) \end{aligned}}$$

We now enter the following sequence of facts:

1. $s(K) \in Z$.
2. $s(K)$ is additive under connected sum.
3. If K^* denotes the mirror image of the diagram K, then
$$s(K^*) = -s(K).$$
4. If K is a positive knot diagram (all positive crossings), then
$$s(K) = -r + n + 1$$
where r denotes the number of loops in the canonical oriented smoothing (this is the same as the number of Seifert circuits in the diagram K) and n denotes the number of crossings in K.
5. For a torus knot $K_{a,b}$ of type (a, b), $s(K_{a,b}) = (a-1)(b-1)$.
6. $|s(K)| \leq 2g^*(K)$ where $g^*(K)$ is the least genus spanning surface for K in the four ball.
7. $g^*(K_{a,b}) = (a-1)(b-1)/2$. This is Milnor's conjecture.

This completes a very skeletal sketch of the construction and use of Rasmussen's invariant.

Grading

$$q(s) = j(s) + (n_+ - 2n_-)$$

$$\begin{cases} n_+ = \# \text{ of } + \text{ crossings in } K. \\ n_- = \# \text{ of } - \text{ crossings in } K. \end{cases}$$

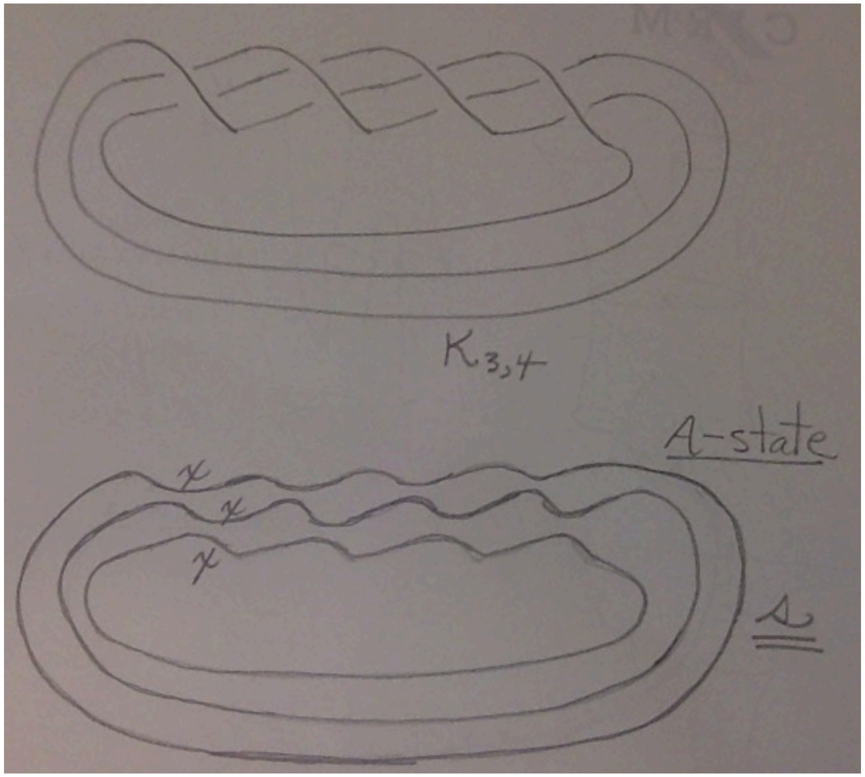
$$\boxed{\begin{aligned} j(s) = \#(\text{B smoothings}) \\ + \#(1\text{'s}) - \#(x\text{'s}) \end{aligned}}$$

Facts: $s_{max}(K) = s_{min}(K) + 2$

$s(K) = s_{min}(K) + 1$

A-State: $s(K) = 1 - (\#\text{loops}) + (\#\text{ crossings}) =$
$2\text{genus}(\text{Seifert}(K))$

For positive knot all loops labelled x.



For A-state of a $K_{p,q}$ torus knot have (with all x's) :

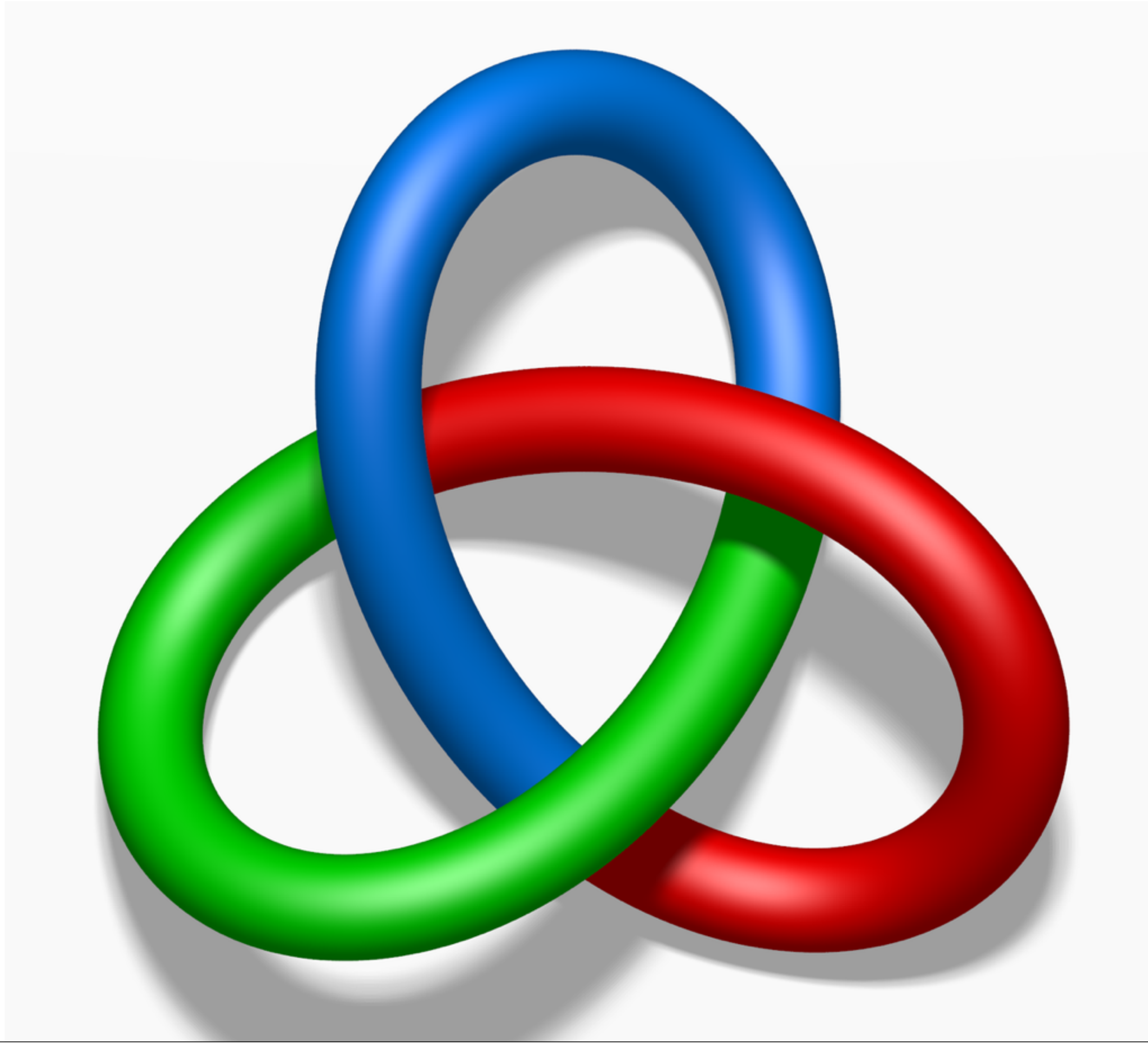
- p loops
- $(p-1)q$ crossings

So $g(s) = (0-p) + (p-1)q$

$= pq - q - p$

$g(s) = (p-1)(q-1) - 1$

$$\Longrightarrow \boxed{s(K_{pq}) = (p-1)(q-1)}$$



For Virtual Knots we need to add a single cycle arrow. More on this next lecture.