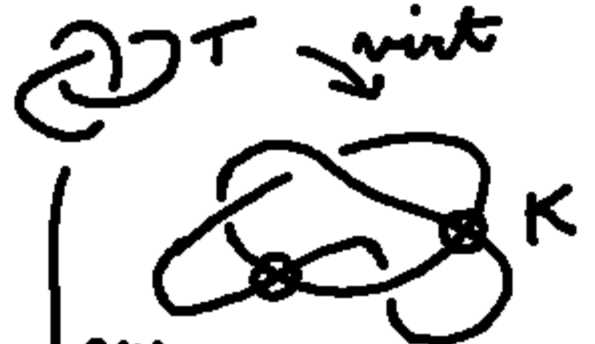
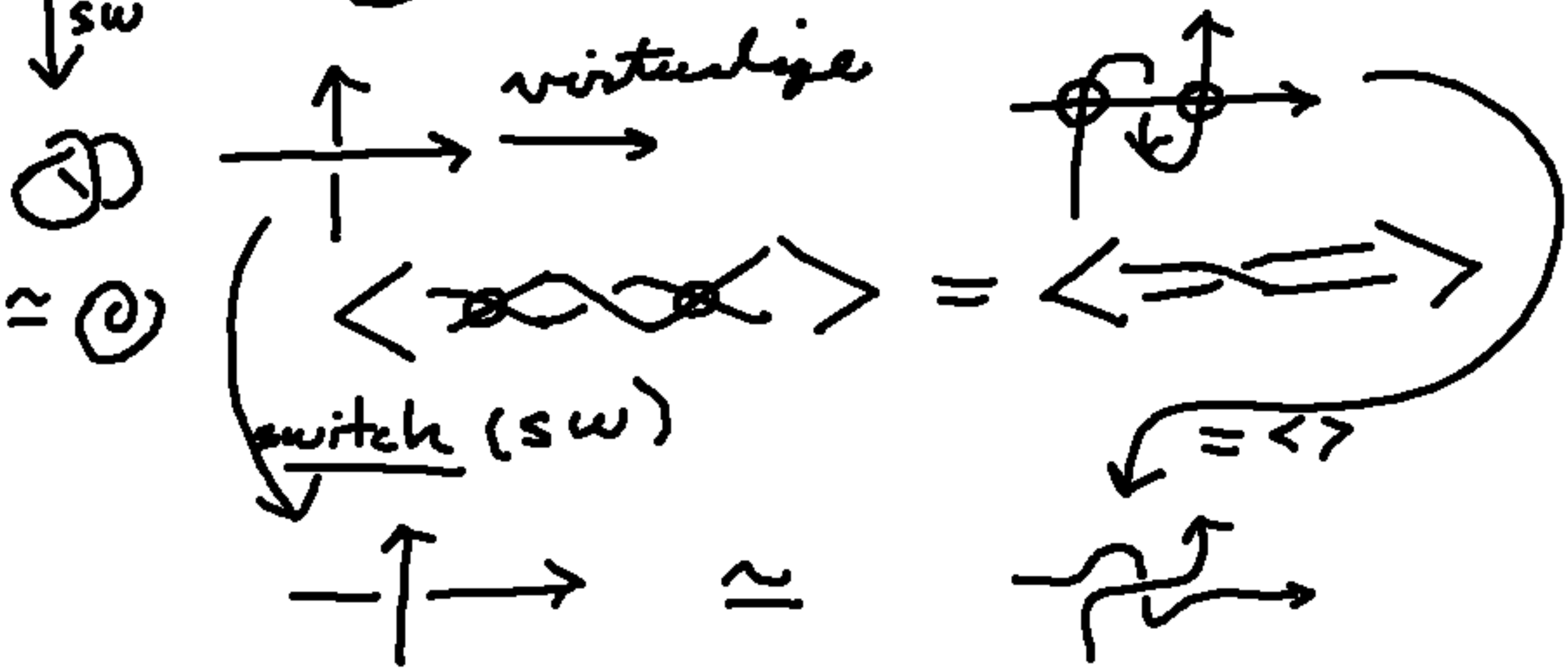
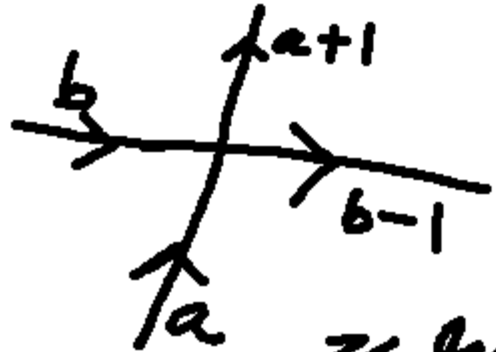
• Last time: reformulated $[K]$. $\xrightarrow{\text{onward}}$ KhoH

given examples of knots K, $[K]$ trivial or $V_K = 1$.



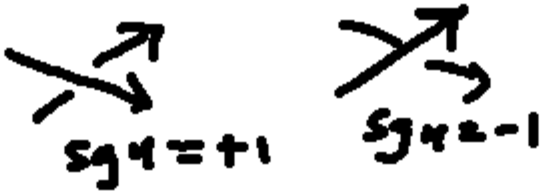


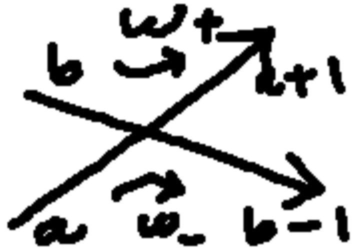
K



$\mathbb{Z}$ labeling

$c \in \mathrm{Crossings}(K)$





$$\left.\begin{aligned} w_+ &= a+1-b \\ w_- &= b-1-a \end{aligned}\right\} \text{note } w_- = -w_+$$

$$P_K = \sum_{c \in Cr(K)} \left(t^{wt(c)} - 1\right) sgn(c) = \sum_c sgn(c)\, t^{wt(c)} - wr(K)$$

$$\boxed{wt(c) = W_{sgn(c)}}$$

invariant under virtuals

<u>$\mathbb{Z}$ labeling</u>

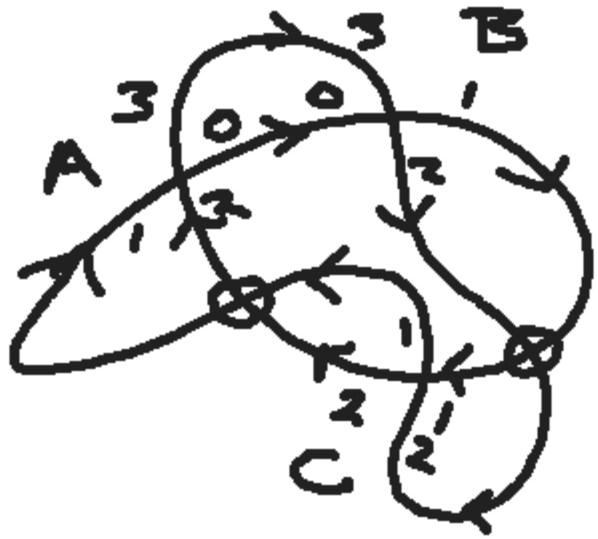
	ω_+	ω_-
A	-1	1
B	1	-1

$$P_K = t^{-1} + t - 2$$

$\Rightarrow$ <u>K is non-trivial & non-classical</u>

- $K \underset{\text{v moves}}{\sim} K' \Rightarrow P_K = P_{K'}$
- K <u>classical</u> $\Rightarrow P_K = \emptyset$.

K



	W_+	W_-
A	-2	2
B	2	-2
C	0	0

$$P_K = t^{-2} + t^{2} - 2$$

$\Longrightarrow$ K non-trivial
non-classical.

Exercise.

(a) Calculate P for another example K' where $V_{K'} = 1$.

(b) Find a K', non triv $V_{K'} = 1$ but $P_{K'} = 0$.

Rewrite as $\langle\ \rangle$ to

[] = [] $- \, q$ []

$$[\bigcirc] = q + q^{-1}$$

$$J_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} [K]$$

↑ invar under all RM's (+ Detour).

$$\left(V_K(t) = J_K(q=\sqrt{t}) \Big/ \left(\sqrt{t} + \tfrac{1}{\sqrt{t}}\right) \right)$$

usual Jones

(so $V_{\bigcirc} = 1$)

Rewrite $\langle\ \rangle$ to

$$[\bigcirc] = q + q^{-1}$$

usual bracket states

$$[K] = \sum_{S} (-1)^{\#B(S)}\, q^{\#B(S)}\, \underline{\underline{(q+q^{-1})^{\#\text{loops}}}}$$

Reformulate to enhanced states: place ± 1 on each loop.

$$\bigcirc : \bigcirc^{+}, \bigcirc^{-} \qquad [\bigcirc^{+}] = q, \quad [\bigcirc^{-}] = q^{-1}.$$

Rewrite $\langle\ \rangle$ to

$$[\bigcirc] = q + q^{-1}$$

usual bracket states

$$[K] = \sum_{S} (-1)^{\#B(S)} q^{\#B(S)} \underline{\underline{(q+q^{-1})^{\#\text{loops}}}}$$

Reformulate to **enhanced states**: place ± 1 on each loop.

$$\bigcirc : \bigcirc^{+}, \bigcirc^{-} \qquad [\bigcirc^{+}] = q, \ [\bigcirc^{-}] = q^{-1}.$$

$$(-1)^3 q^3 (q^3)$$

$$\boxed{(-1)^{\#B} q^{\#B} \quad q^{\#(+)-\#(-)}}$$

$\to q^{-2}$

$\to 1^3 \cdot q q^{-1} = 1$

$$[K] = \sum_{s \in \mathcal{ES}} (-1)^{i(s)} q^{j(s)}$$

Enhanced States

$i(s) = \#B(s)$

$j(s) = i(s) + \lambda(s)$

$\lambda(s) = \#+(s) - \#-(s)$

Think about what kind of cancellations of terms can occur in the sum.

$$[K] = \sum_{s \in \mathcal{ES}} (-1)^{i(s)} q^{j(s)}$$

$i(s) = \# B(s)$

$j(s) = i(s) + \lambda(s)$

$\lambda(s) = \# + (s) - \# - (s)$

Enhanced states

$$= \sum_{i,j} (-1)^i q^j \dim(C^{ij})$$

$/\mathbb{Z}$

C^{ij} = module gen by enhanced states s with $i = i(s)$ & $\underline{j = j(s)}$.

$$= \sum_j q^j \underbrace{\Big(\sum_i (-1)^i \dim(C^{ij})\Big)}_{\text{looks like an Euler Characteristic.}}$$

$$[K] = \sum_j q^j \underbrace{\Big(\sum_i (-1)^i \dim (C^{i,j})\Big)}_{\text{looks like an Euler Characteristic.}} = \sum_j q^j \underline{\underline{\chi(H^{\bullet,j})}}$$

~~If~~ there were a complex

$$C^{0,j} \xrightarrow{\partial} C^{1,j} \xrightarrow{\partial} C^{2,j} \xrightarrow{\partial} \cdots$$

$\partial \circ \partial = 0$

$+$ j not changed by bndry.

$$\Longrightarrow \sum_i (-1)^i \dim C^{i,j} = \chi(C^{\bullet,j})$$

$$= \chi(H(C^{\bullet,j})) = \underline{\underline{\chi(H^{\bullet,j})}}$$

$/\mathbb{Q}$

usually we have

$$H_n(C) \stackrel{def}{=} \frac{\mathrm{Ker}(\partial : C_n \to C_{n-1})}{\mathrm{Im}(\partial : C_{n+1} \to C_n)}$$

$$\partial : \underset{n \text{ cells}}{C_n} \longrightarrow \underset{(n-1) \text{ cells}}{C_{n-1}}$$

$$C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2}$$

$$\partial \circ \partial = \emptyset$$

$$\partial(\partial\langle 012\rangle) = \partial\big(\langle 12\rangle - \langle 02\rangle + \langle 01\rangle\big) = \langle 2\rangle - \langle 1\rangle - \langle 2\rangle + \langle 0\rangle + \langle 1\rangle - \langle 0\rangle = \emptyset$$

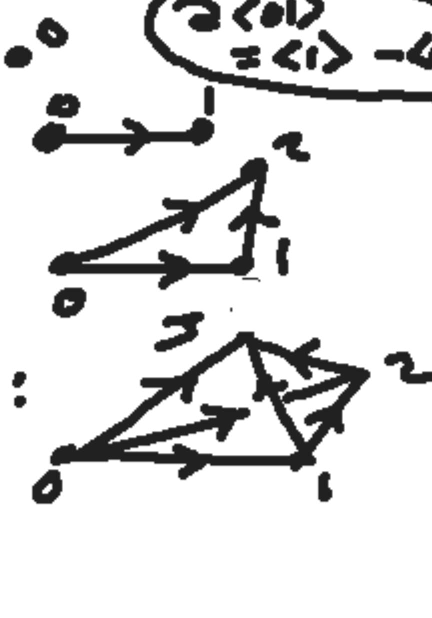
Abstract simplex $\langle 012\cdots n\rangle$

$\langle 0\rangle$:

$\langle 01\rangle$:

$\langle 012\rangle$:

$\langle 0123\rangle$:



$$\partial\langle 01\rangle = \langle 1\rangle - \langle 0\rangle$$

$$\partial\langle 012\cdots n\rangle = \langle \hat{0}12\cdots n\rangle - \langle 0\hat{1}2\cdots n\rangle + \langle 01\hat{2}\cdots n\rangle \pm \cdots + (-1)^n\langle 012\cdots n{-}1\,\hat{n}\rangle$$

$\hat{i}$ = remove i

$$\partial\circ\partial\langle 012\cdots n\rangle = \emptyset$$

<u>Exercise</u>: Prove that $\partial\partial\langle 012\cdots n\rangle = \emptyset$.

Category

A category $\mathcal{C}$ consists in a collection of objects $\mathcal{O}$ and a collection $\mathcal{M}$ of arrows (morphisms). To each arrow is associated two objects

$$A \xrightarrow{f} B \leftarrow \text{target}(f) = t(f)$$

$\uparrow$ arrow f

source$(f) = s(f)$

1) $A \xrightarrow{f} B \xrightarrow{g} C$

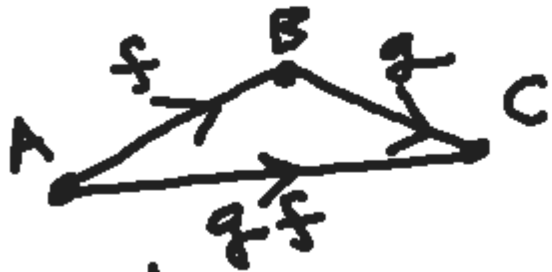
$\Rightarrow \exists \; A \xrightarrow{gf} C$

2) $A \circlearrowleft 1_A \quad (\exists)$

$$\underbrace{A \xrightarrow{1_A} A \xrightarrow{f} B}_{f \circ 1_A = f} \quad \underbrace{A \xrightarrow{f} B \xrightarrow{1_B} B}_{1_B f = f}$$

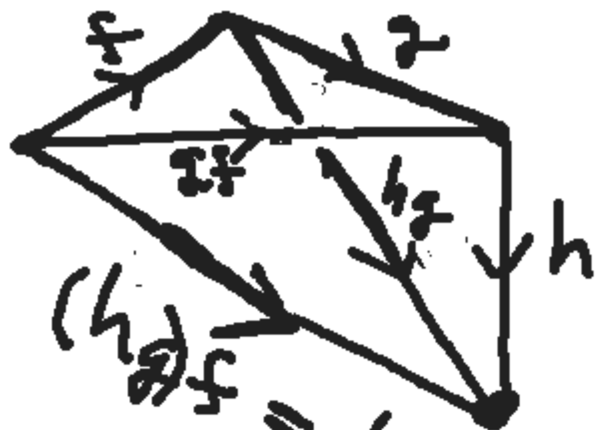
3) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

$\Rightarrow h(gf) = (hg)f$



$$\Big\downarrow \partial$$

$$\partial[f,g] = f + g - gf$$



$$(hg)f = h(gf)$$

$1_A, f, f^2, f^3, \dots$

$[f]$

$[f,f]$

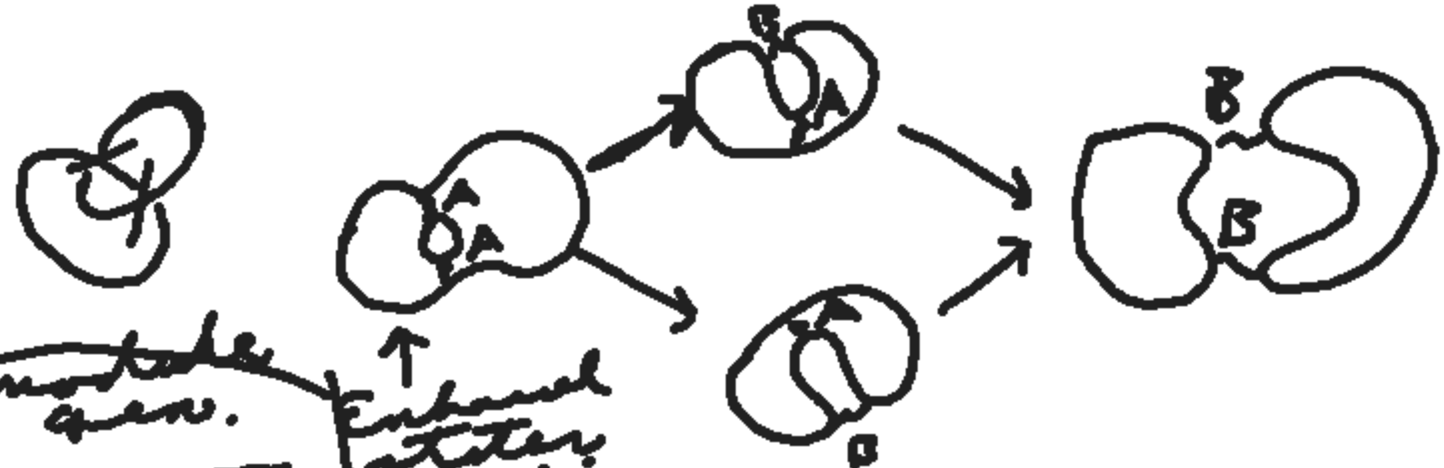
$[f,f,f]$

$\vdots$

$$\partial[f,g,h]$$
$$= [g,h] - [gf,h] + [f,hg]$$
$$- [f,g]$$

$$\partial[f,g,h,k]$$
$$\shortparallel$$
$$[g,h,k]$$
$$-[gf,h,k]$$
$$+[f,hg,k]$$
$$-[f,g,kh]$$
$$+[f,g,h]$$

Categories are directly associated with simplices and homology.



smoothing gen.

Enhanced states to this

AA	AB
BA	BB

$$
\begin{tikzcd}
 & {[BA]} \arrow[dr] & & \\
{[AA]} \arrow[ur] \arrow[dr] & \oplus & {[BB]} \\
 & {[AB]} \arrow[ur] &
\end{tikzcd}
$$

cube category

$$C^{0\bullet} \longrightarrow C^{1\bullet} \longrightarrow C^{2\bullet}$$

Module assoc to each object

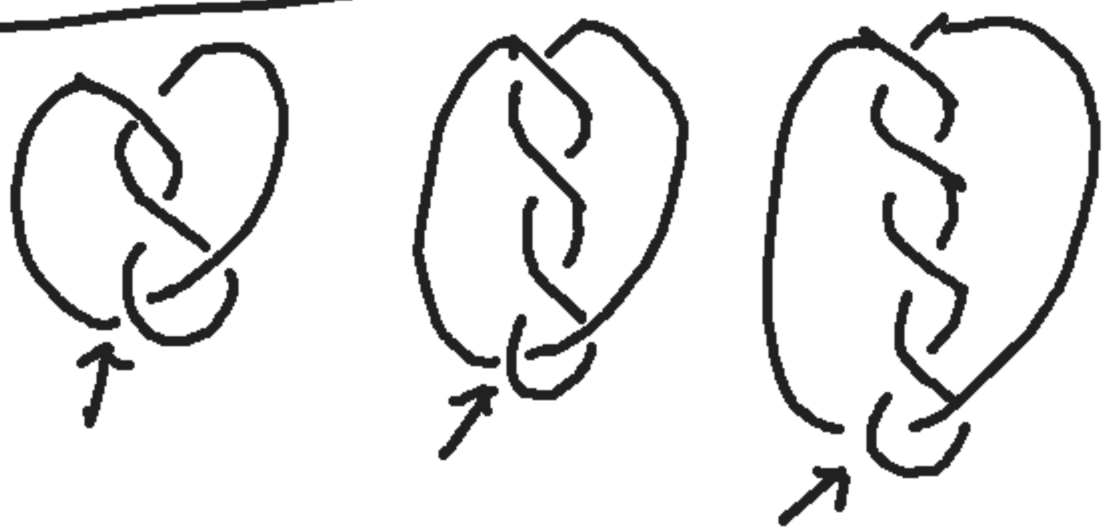
$[AA], [BA], \ldots$

$\boxed{C^{i\bullet}}$

all states i B's

$C^{ij} = \bigoplus$ Modules for all states with i, B's + given j

Agenda:
- Finish defn of Kho H for classical knots.
- examples & ideas & facts.
- virtuals & how to extend the defn



Exercise:
Virtualize the ↗ crossings … & find $P_{K_1}, P_{K_2}, \ldots$ for the series of virtual knots.