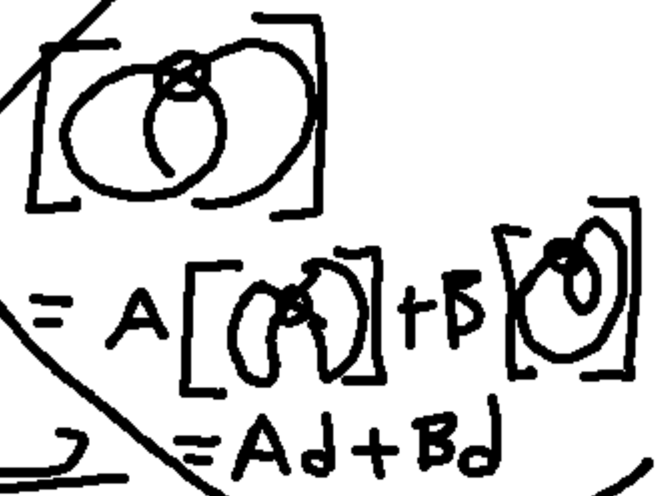
1. defn of VKT
2. interp via $S^1 \hookrightarrow S_g \times I$.
3. parity + odd writhe
4. knotoids
5. ~



Bracket Model for Jones Poly

D

$$[\times] = A[\asymp] + B[)(]$$

$$[\text{loop}] = d$$

loop:  no classical crossings

well-defined comb. poly assoc to a given virt D.

A <u>state</u> S of a diagram D is obtained by choosing a smoothing for each crossing.

mnemonic

Each S is a collection of loops + their labels at the smoothings.

$\|S\| = \#$ loops in S

$$[D] \underset{def}{=} \sum_S [K|S]\, d^{\|S\|}$$

$[K|S] =$ product of labels of S.

$$[K] = \sum_{S} [K|S]\, d^{\|S\|}$$

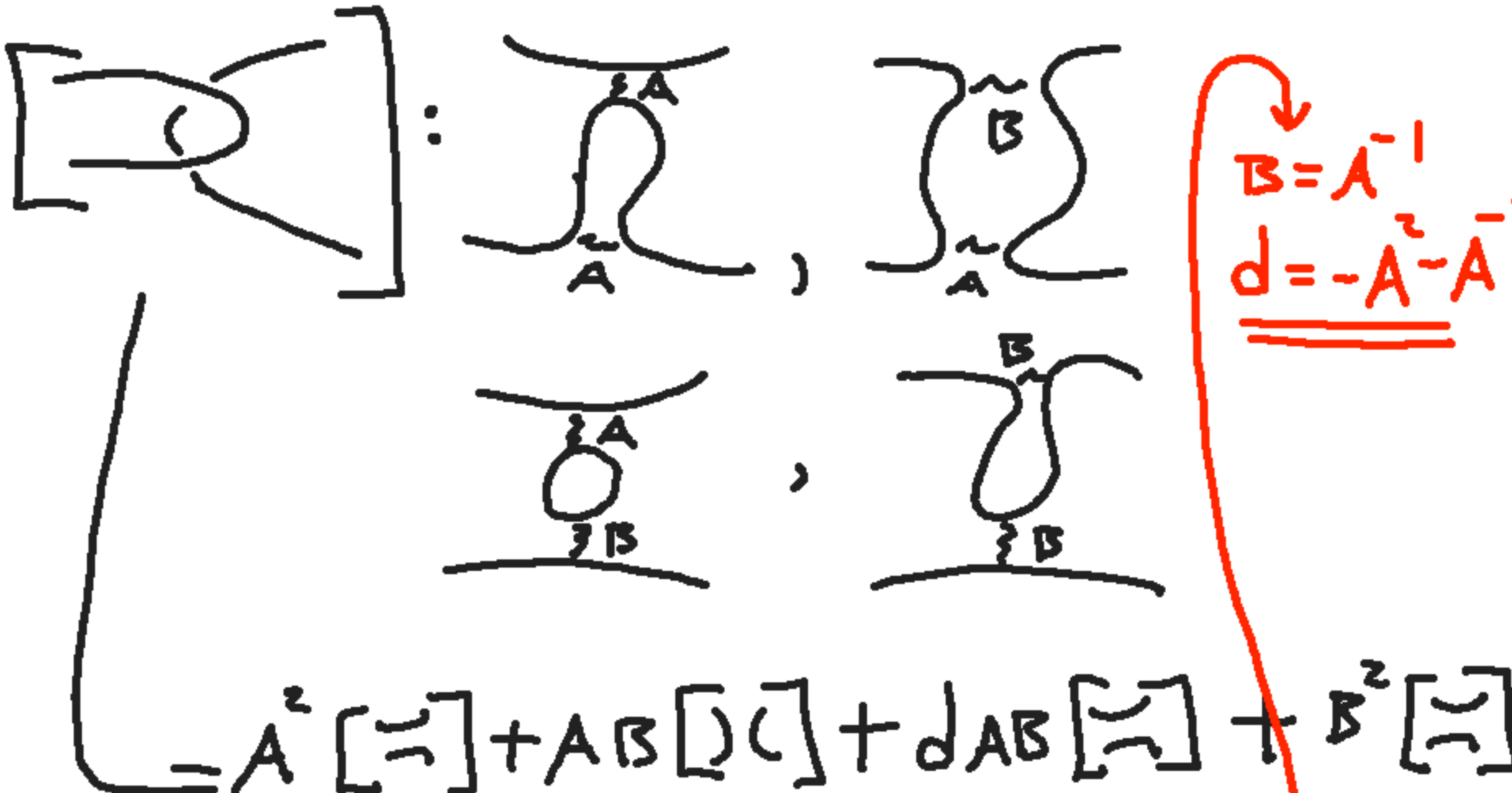
$$\Rightarrow \left[\diagup\!\!\!\!\diagdown\right] = A\left[\asymp\right] + B\left[\,)(\,\right]$$

$$[O \sqcup K] = d\,[K]$$

$$[O] = d$$

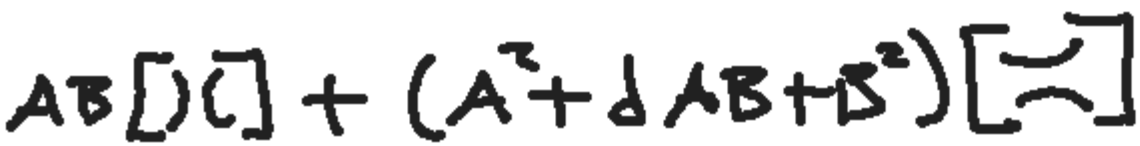
Note $[K]$ invariant under detour moves.
(detour moves do not change $\|S\|$.)

Examine  .



$B = A^{-1}$

$\underline{\underline{d = -A^2 - A^{-2}}}$

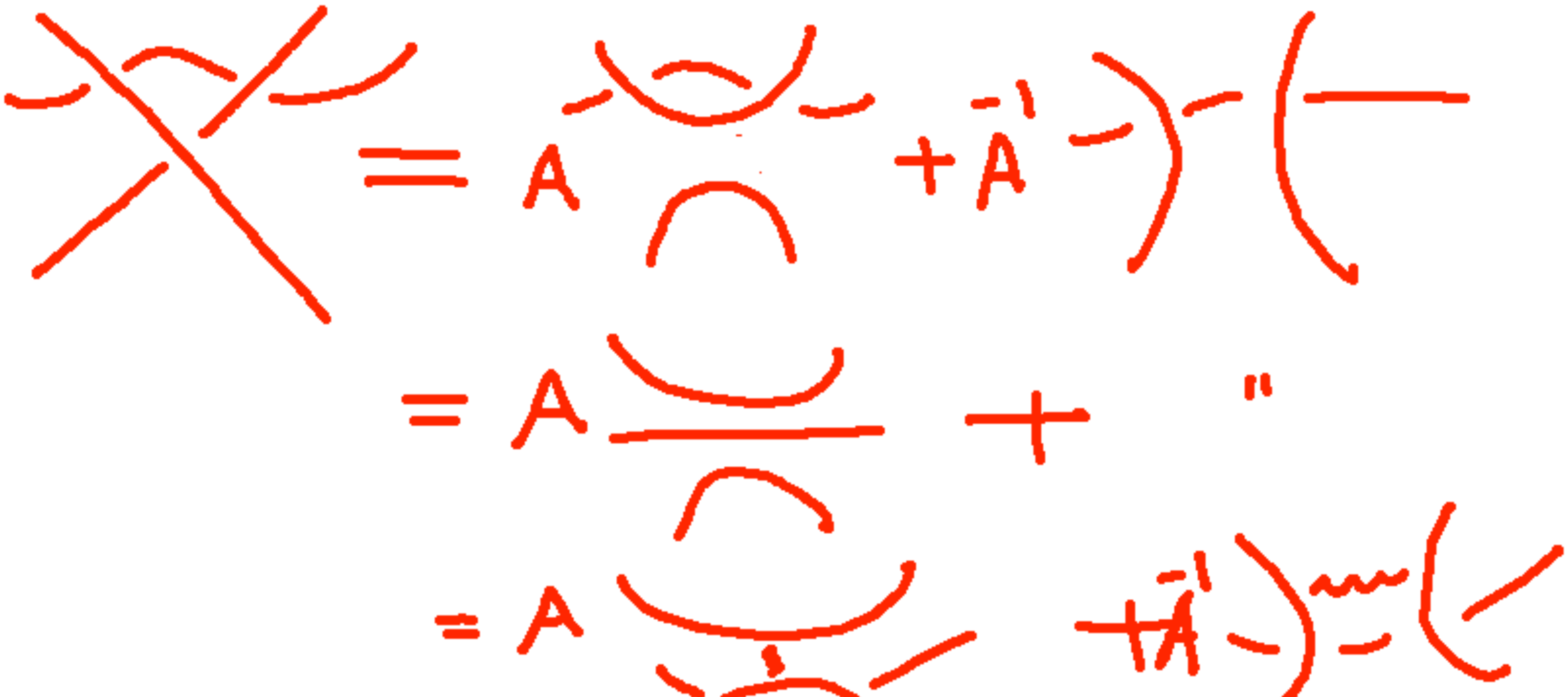


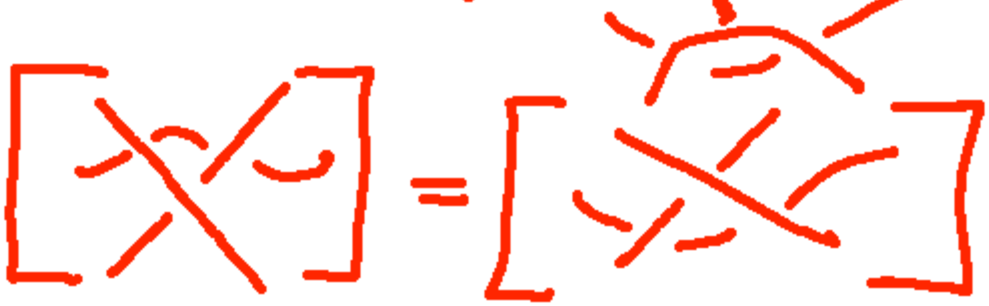
$= AB[\,)(\,] + (A^2 + dAB + B^2)[\asymp]$.

Set
$AB = 1$
$A^2 + dAB + B^2 = 0$

$d = -A^{2} - A^{-2}$, $B = A^{-1}$

$[\,\bigcirc\,] = -A^{2} - A^{-2}$





✓

$\Big[$  $\Big] = A\,\Big[$  $\Big] + B\,\Big[$  $\Big]$

pos curl

$= (Ad+B)\,\Big[$  $\Big]$

$$Ad+B = A(-A^{2}-A^{-2})+A^{-1} = -A^{3}$$

$\Big[$  $\Big] = A\,\Big[$  $\Big] + B\,\Big[$  $\Big]$

neg curl

$= (A+Bd)\,\Big[$  $\Big]$

$$A+Bd = A+A^{-1}(-A^{2}-A^{-2}) = -A^{-3}$$

$[K]$ is invar under R2, R3 $\underbrace{\text{+ Detour.}}_{\text{regular isotopy.}}$

Let $f_K(A) = (-A^3)^{-wr(K)} [K]/\delta$

$\delta = -A^2 - A^{-2}$, $\underline{wr(K)} = \sum_{c \in Cr(K)} sgn(c)$

invar under R2, R3, D

$\Rightarrow$ f_K invar under R1, R2, R3, Detour

& $f_{\bigcirc} = 1$

We call $V_K(t) = f_K(t^{-1/4})$ the Jones Polynomial (for virtuals)

⊛ $K^* =$ mirror image of K
(switch all crossings)

then $\boxed{f_{K^*}(A) = f_K(A^{-1})}$.

$\Bigg[$  $\Bigg]_K = A\Bigg[$  $\Bigg] + A^{-1}\Bigg[$  $\Bigg]$

$= A\Big(A\Big[$  $\Big] + A^{-1}\Big[$  $\Big]\Big) + A^{-1}\big[-A^{-3}\delta\big]$

$= A^2\delta + \delta + (-A^{-4})\delta$

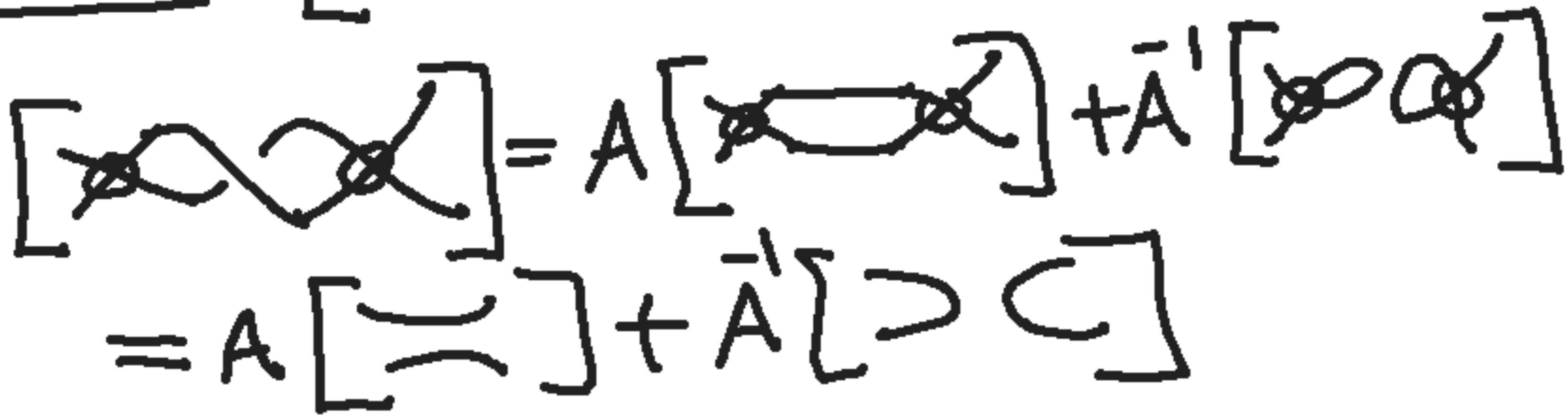
$= (A^2 + 1 - A^{-4})\delta$

$K \not\cong K^*$

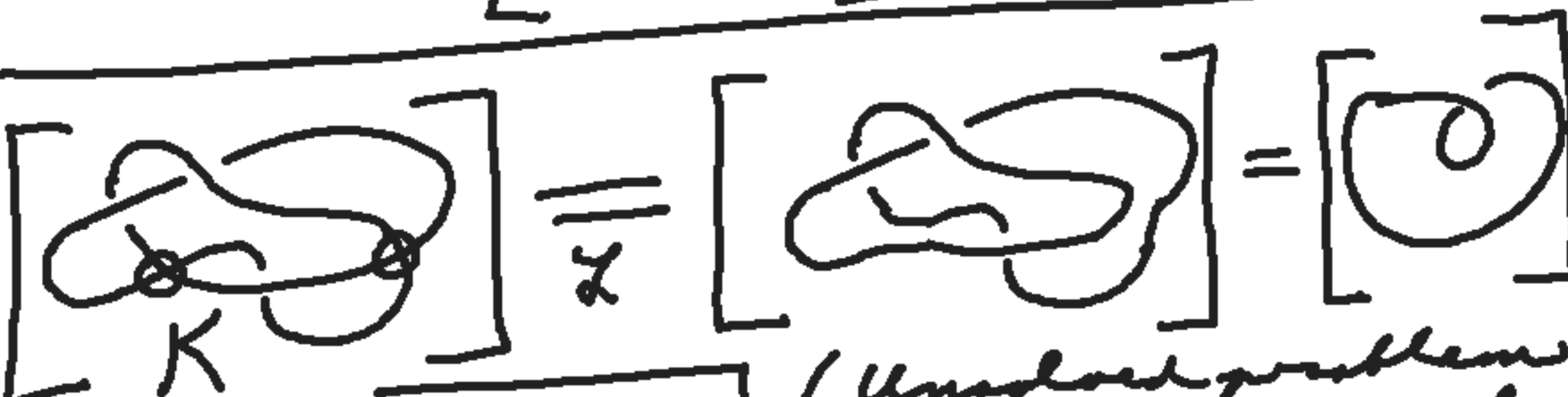
$\Uparrow$

$f_K(A) = (-A^3)^{-2}(A^2+1-A^{-4}) = \underline{-A^{-4} - A^{-6} + A^{-10}}$

Lemma. 

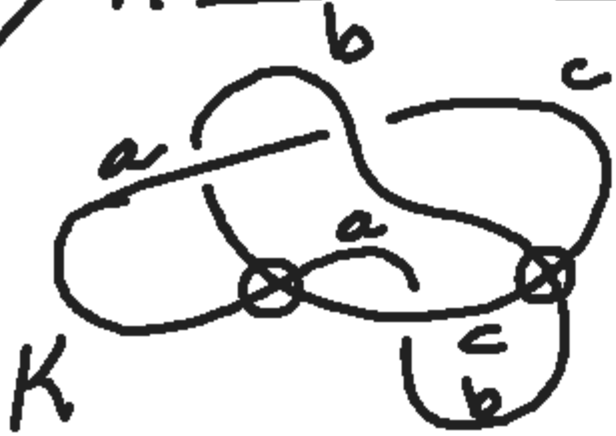
Pf. 

 QED



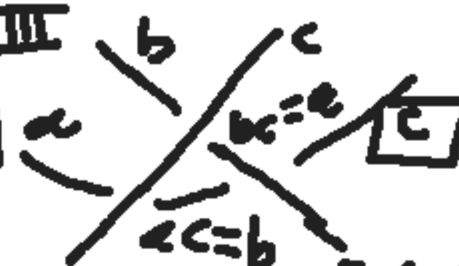
$\Longrightarrow$ $\boxed{f_K = 1}$ (Unsolved problem to find classical knots K with $f_K = 1$.)

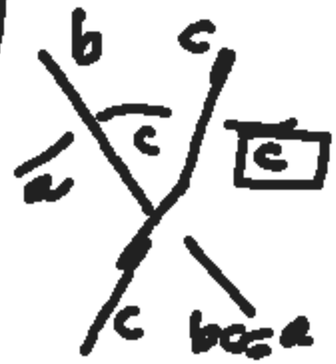
✓ K non-trivial



$$\boxed{\begin{aligned}&(xy)z\\&=(xz)(yz)\end{aligned}}$$

III



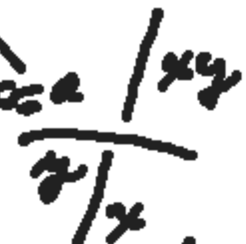


recall 3 coloring

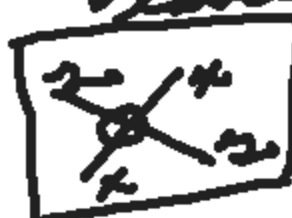
$\mathcal{Q} = \{a, b, c\}$

$$\left\{\begin{aligned}
&ab = ba = c\\
&ac = ca = b\\
&bc = cb = a\\
&aa = a\\
&bb = b\\
&cc = c
\end{aligned}\right.$$

quandle



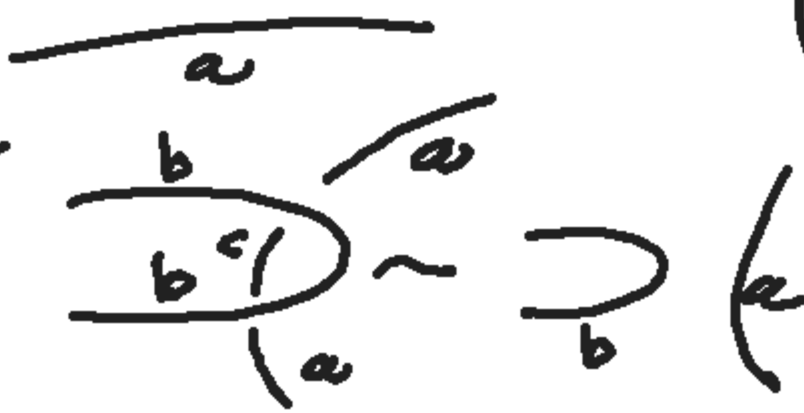
quandle labeling



I

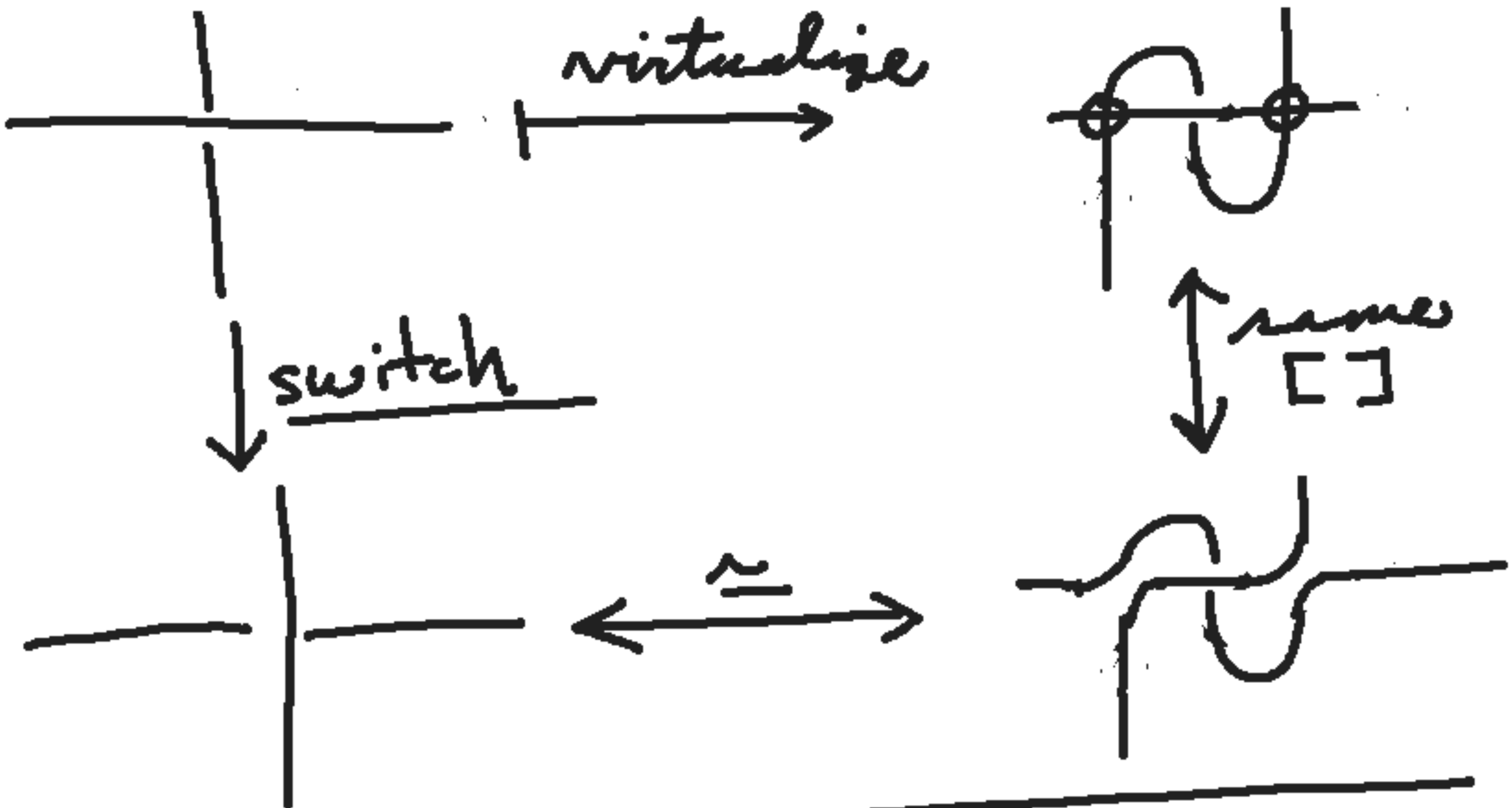
$$\boxed{xx = x}$$

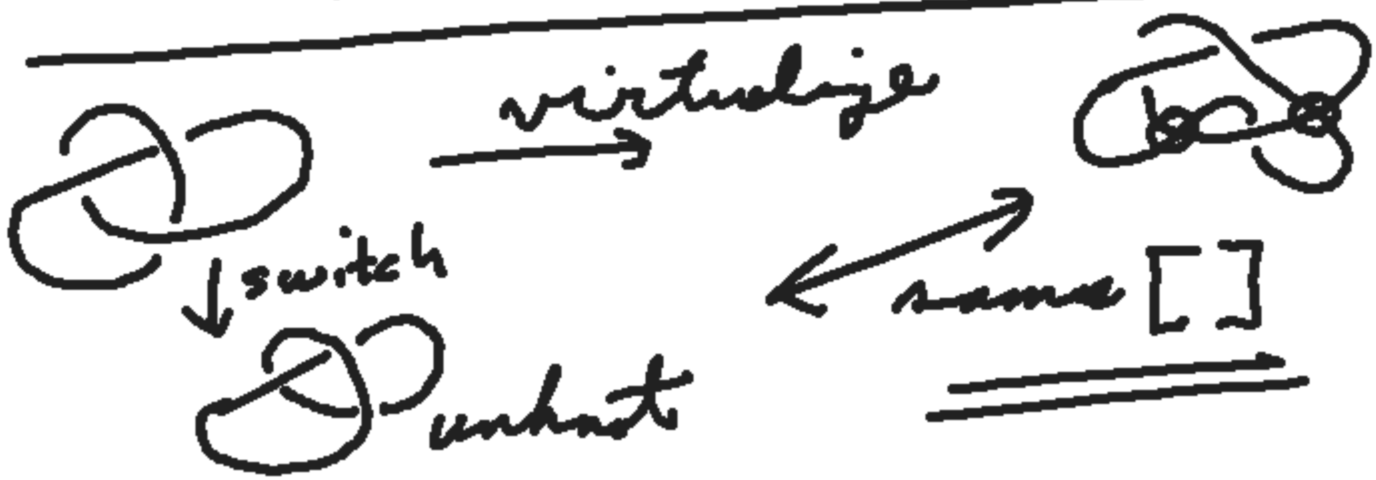
✓

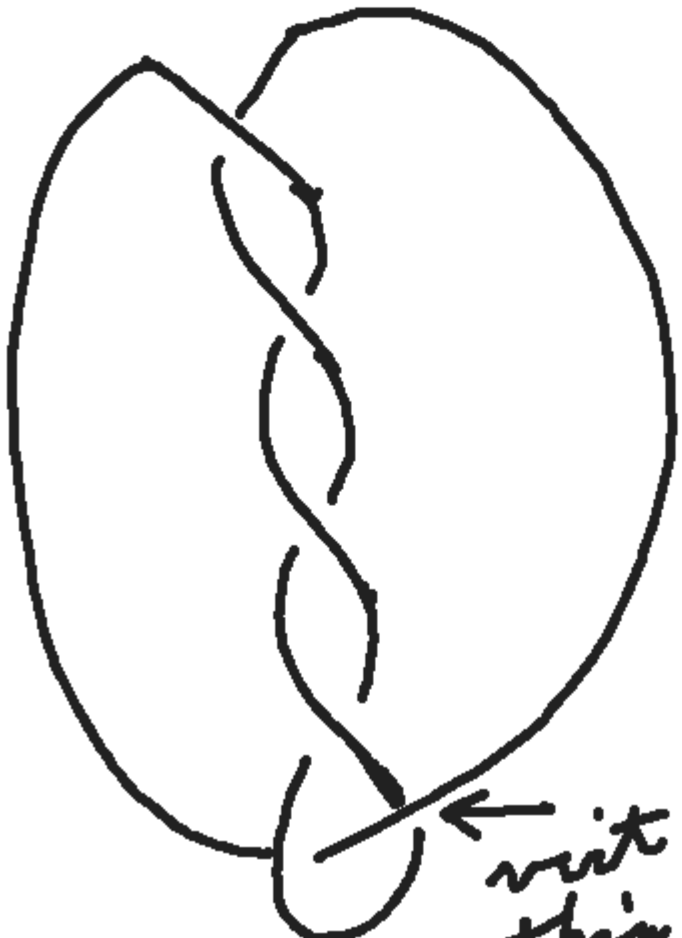


II

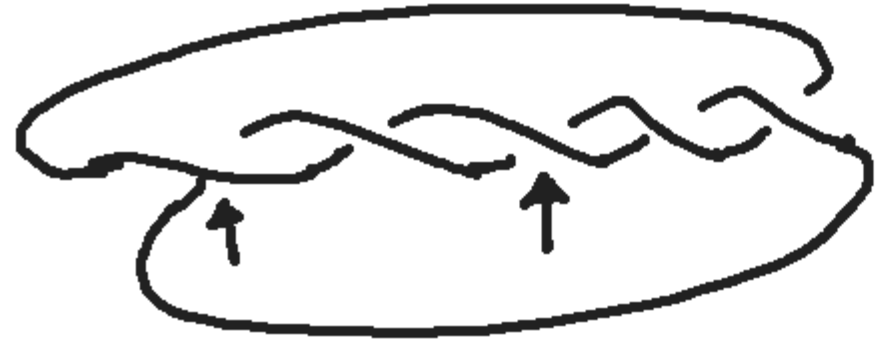
$$\boxed{(xy)y = x}$$


virtualize
switch
≃
same [ ]


virtualize
switch
unknot
same [ ]



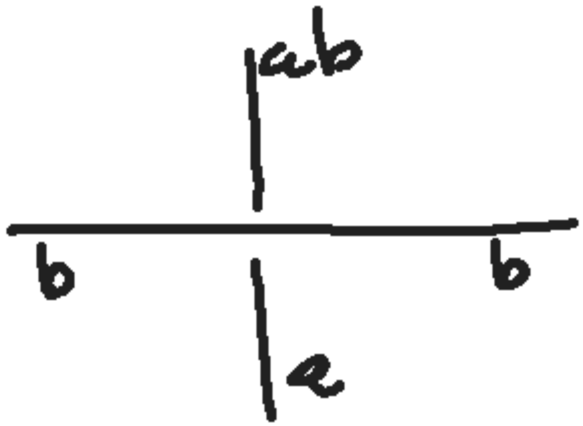
virt this



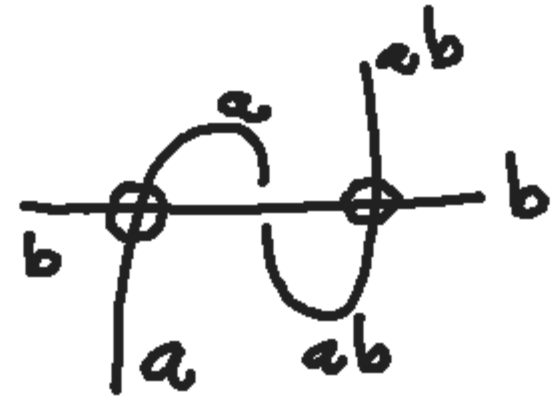
Infinitely many examples of v-knots K non-trivial with $f_K(A) = 1$.

---

virtualization does not change the quandle



$\xrightarrow{\text{virt}}$



$\Rightarrow$ Quandle$(K) \cong$ Quandle$(\text{Virt}(K))$

Claim Theorem. Quandle(K) detects the unknot for classical knots.

$\Rightarrow$ K classical not unknot (knotted) $\Rightarrow$ virt(K) knotted

One aim of this course is to prove that all such examples are non-trivial non-classical virtual knots.

Use Khovanov Homology

We can use e.g. affine index poly to show that

or Sawollek poly

is not classical.

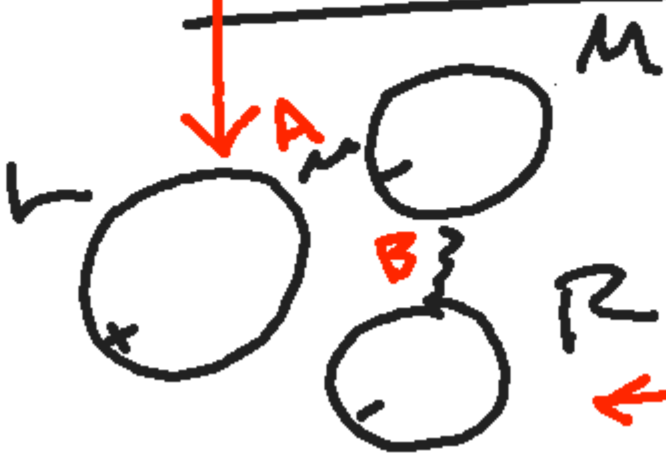
Now reformulate $[K]$ in direction of Kho Homology.

$$[K] = \sum_{S} [K|S]\, d^{\|S\|} \qquad \underbrace{\sum_{\sigma \in ES} A^{\#(A)} (A^{-1})^{\#(B)} (-A^{2})^{\#+} (-A^{-2})^{\#-}}$$

$$\Downarrow$$

$$= \sum_{S} A^{\#A(S)} (A^{-1})^{\#B(S)} (-A^{2} - A^{-2})^{\|S\|}$$

enhanced state



Binomial Expansion means to choose +1 or -1 for each loop in S.

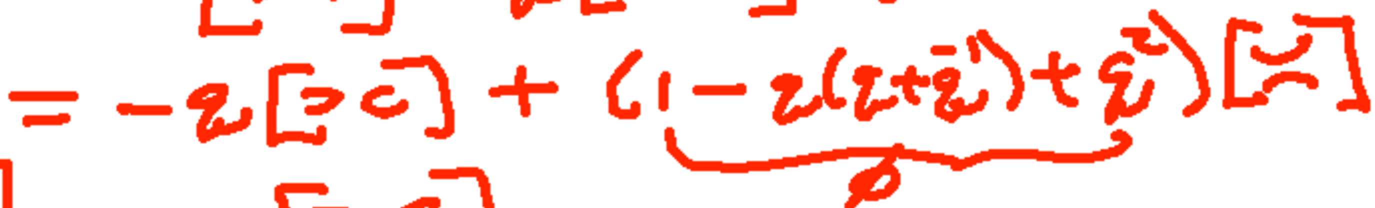
$$\underbrace{A(A^{-1})(-A^{2})(-A^{-2})(-A^{-2})}$$

contributes to labeled ($\pm$) state.

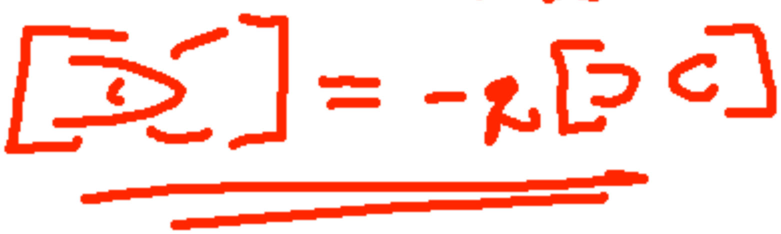
$$\underset{L}{(-A^{2} - A^{-2})}\ \underset{M}{(-A^{2} - A^{-2})}\ \underset{R}{(-A^{2} - A^{-2})}$$

Useful algebraic changes:

$$= \underbrace{q+q^{-1}}=\delta \qquad (q = A^{-2}) \quad \text{etc.}$$



$$\underbrace{1-q(q+q^{-1})+q^{2}}_{0}$$



$$\boxed{[\,\times\,] = [\,\asymp\,] - q\,[\,)(\,]\,, \quad [\bigcirc] = q + q^{-1}}$$

$$= (q + q^{-1} - q) = q^{-1}$$

$n_- = 0, \; n_+ = 1$

$n_- = 1, \; n_+ = 0$

$$= (1 - q(q+q^{-1})) = (-q^2)$$

$wr = n_+ - n_-$

positive negative

$$J_K(q) = (-1)^{n_-}\, q^{n_+ - 2n_-}\, [K]$$

check that J_K is normalized of invar under all RM.

$$\boxed{V_K(t) = J_K(\underbrace{t^{-1/2}}_{=q})}$$