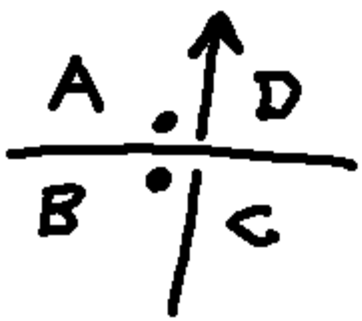
Lecture 14

("Formal Knot Theory")

FKT Model for Alexander

Alexander 1928, TAMS



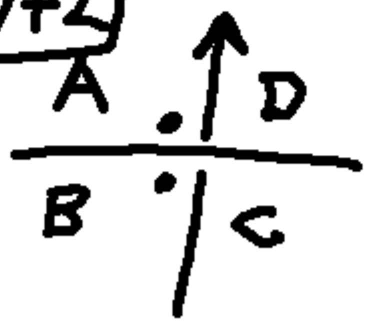
Module gen by symbols for regions and relns for each crossing).

$tA - tB + C - D$

relns for module over $\mathbb{Z}[t,t^{-1}]$.

(comes from Dehn Pres)

$\boxed{R_0 = V+2}$



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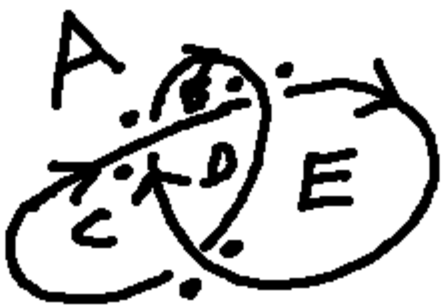


K

$tA - tC + D - B$
$tA - tB + D - E$
$tA - tE + D - C$

A	B	C	D	E
t	-1	$-t$	1	0
t	$-t$	0	1	-1
t	0	-1	1	$-t$

(M = the C, D, E columns)

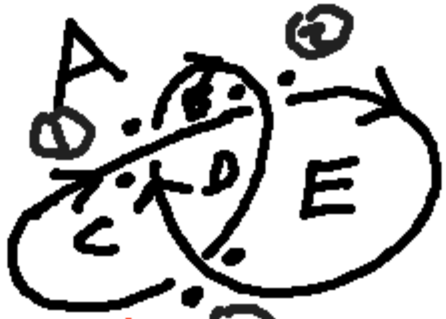
$\Delta_K(t) \doteq \mathrm{Det}(M)$

$$\mathrm{Det} = -t\begin{vmatrix} 1 & -1 \\ 1 & -t \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -1 & -t \end{vmatrix}$$

$$= -t(-t+1) - (-1)$$

$$= t^2 - t + 1 \doteq \Delta_K(t)$$

($\doteq$ means $=$ up to factor of $\pm t^N$)



$\begin{pmatrix} t & -1 \\ -t & 1 \end{pmatrix}$ alg bits

$tA - tC + D - B$

$tA - tB + D - E$

$tA - tE + D - C$

	A	B	C	D	E
1.	t	-1	$-t$	1	0
2.	t	$-t$	0	1	-1
3.	t	0	-1	1	$-t$

$\Delta_K(t) \doteq \mathrm{Det}(\tilde{A})$

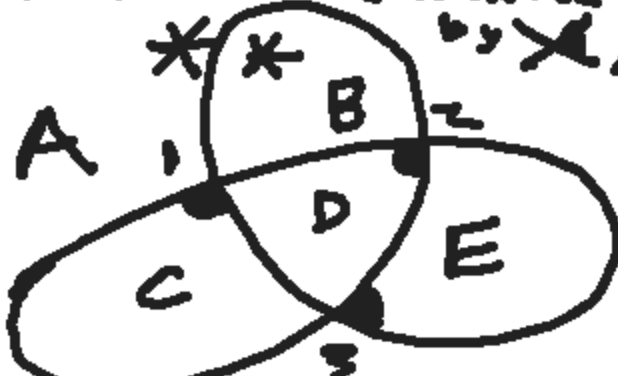
($\doteq$ means = up to factor of $\pm t^N$)

$$\mathrm{Det} = -t\begin{vmatrix} 1 & -1 \\ 1 & -t \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -1 & -t \end{vmatrix}$$

$$= -t(-t+1) - (-1)$$

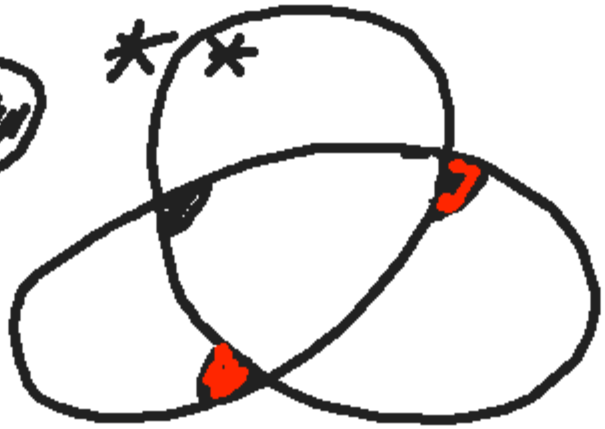
$$= t^2 - t + 1 \doteq \Delta_K(t)$$

($\langle K|S\rangle = \pi$ bits of alg marked by S) $\neq 0$ terms in $\mathrm{Det}(\tilde{A})$ $\leftrightarrow$ region to node choices.



$$\Delta_K(t) \doteq \sum_{\text{marker states } S} \langle K|S\rangle \, \mathrm{sgn}(S)$$

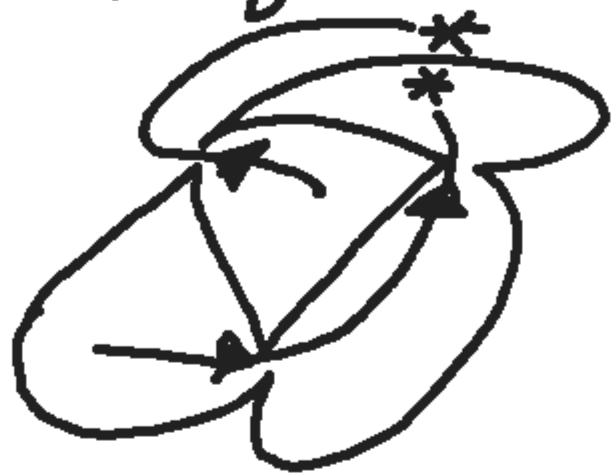
($\mathrm{sgn}(S)$ = sgn of perm)

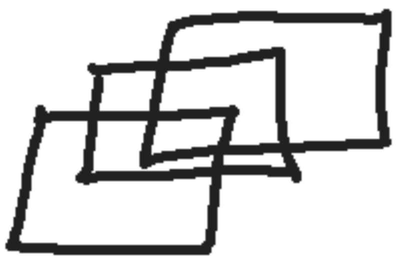


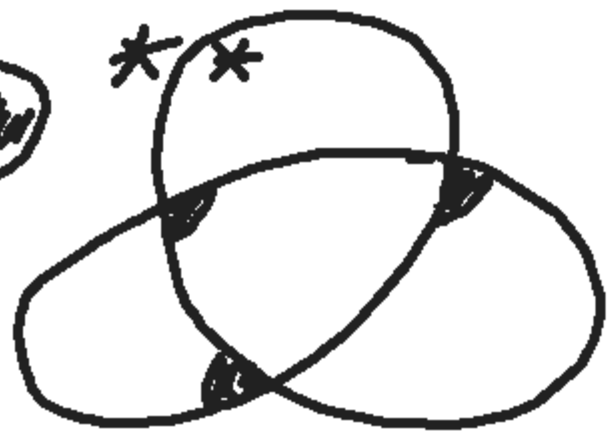
marker states

⟺ Euler ~~Jordan~~ Trails on Diagram.

(finding such trails in a puzzle articulated by Lewis Carroll)







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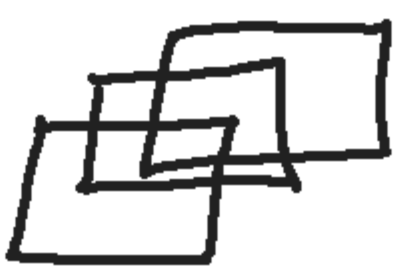
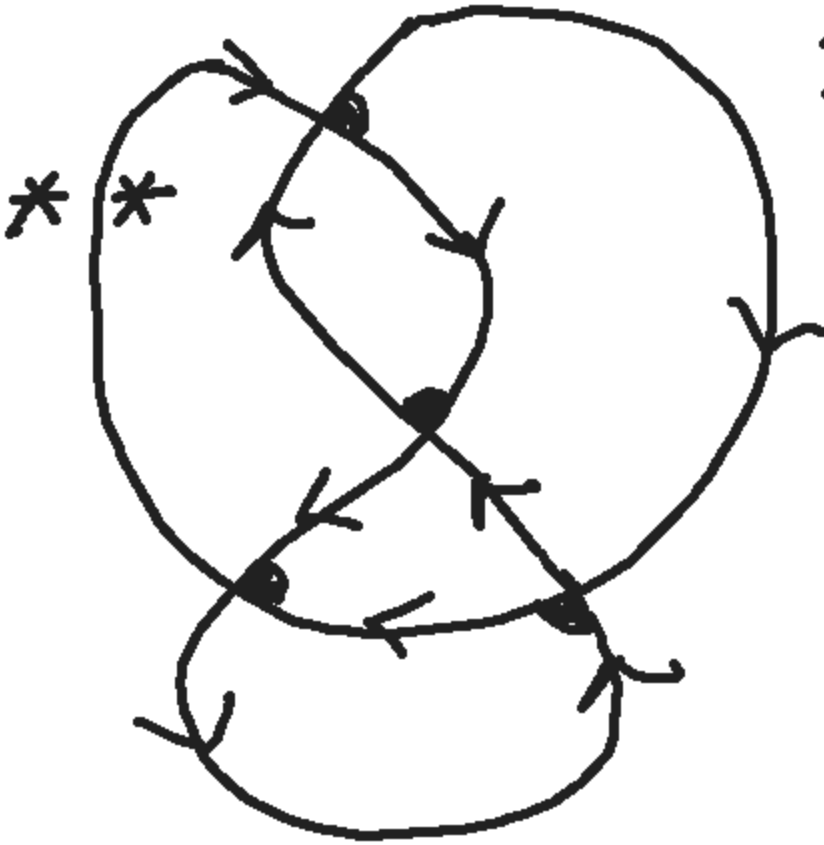
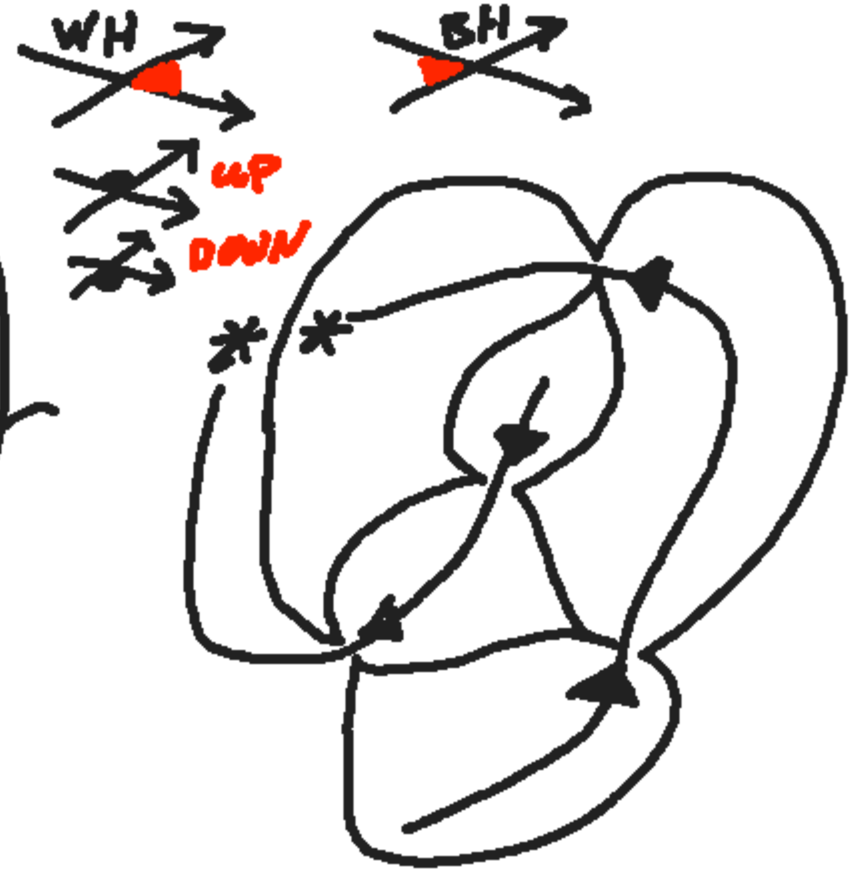


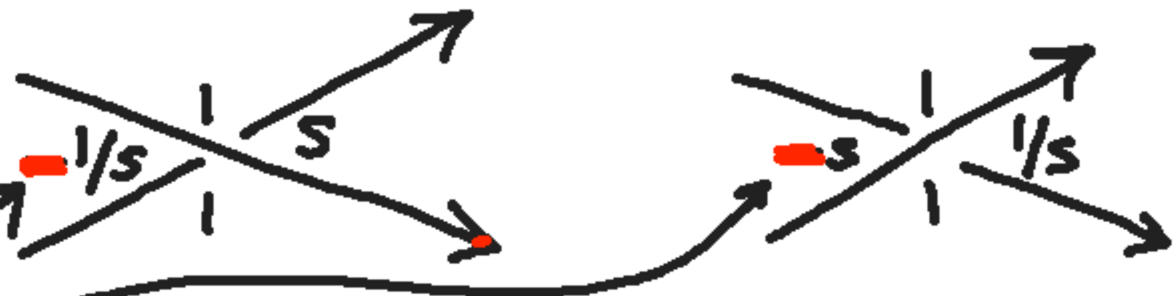




$$\Delta_K(t) \doteq \sum_S \langle K|S\rangle (-1)^{\#(BH)}$$

To get Conway Alex

change with:



$z = \left(s - \frac{1}{s}\right)$

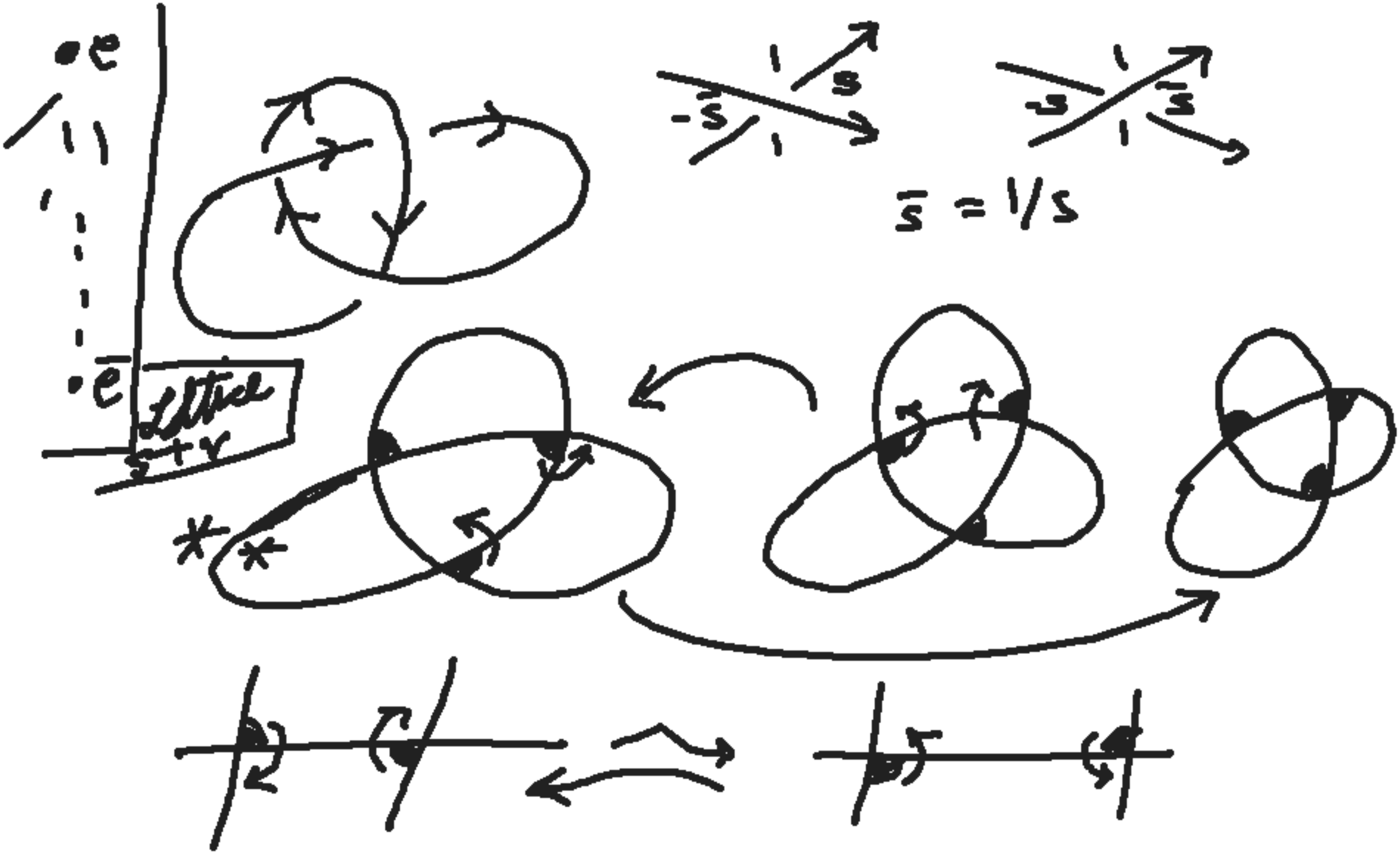
(for the perm)

$$\nabla_K(s) = \sum_S \langle K|S\rangle$$

(z) for classical knots + links

$$\nabla_{\text{positive crossing}} - \nabla_{\text{negative crossing}} = \left(s - \frac{1}{s}\right)\nabla_{\text{smoothing}}$$

$$\nabla_{O} = 1$$


e
ē
Lattice
s
-s
s̄
s̄ = 1/s

$z = (s - \bar{s}^{1})$

$$\nabla_K = s^2 - 1 + \bar{s}^2 = z^2 + 1$$

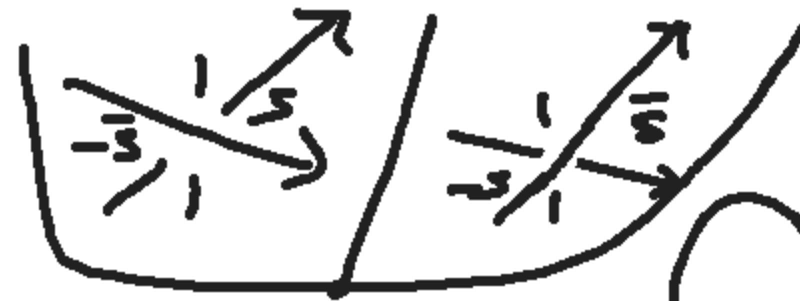
Heegaard Floer Knot Homology

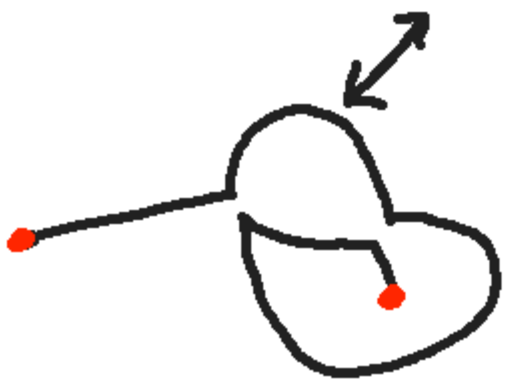
{marker states} gen complex.

$$\nabla_{K^*}(s) = \nabla_K(s^{-1})$$

$$\nabla_{\overleftarrow{K}}(s) = \nabla_{\overrightarrow{K}}(-s^{-1})$$

[illegible] Sequence



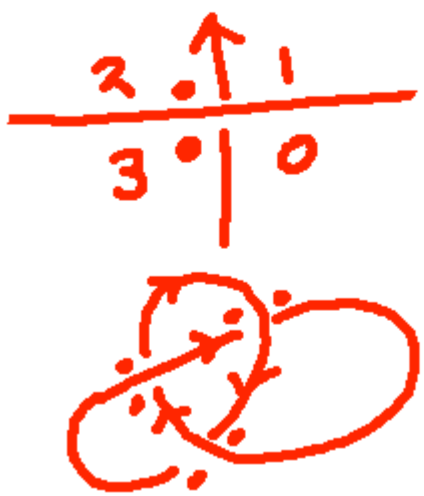


$$\nabla_K(s) = s^2 + s - s^{-1}$$

$$\nabla_{K^*}(s) = s^{-2} + s^{-1} - s$$

$$\nabla_{\overrightarrow{K}} \neq \nabla_{\overleftarrow{K}}$$

We will examine a diagram homology due to Luse + Rong, which (it turns out) is not invar under RM'S (verif from Marithania Silvero via conversations with LK + in her thesis. This work is also a project AS + LK plan to continue ...).



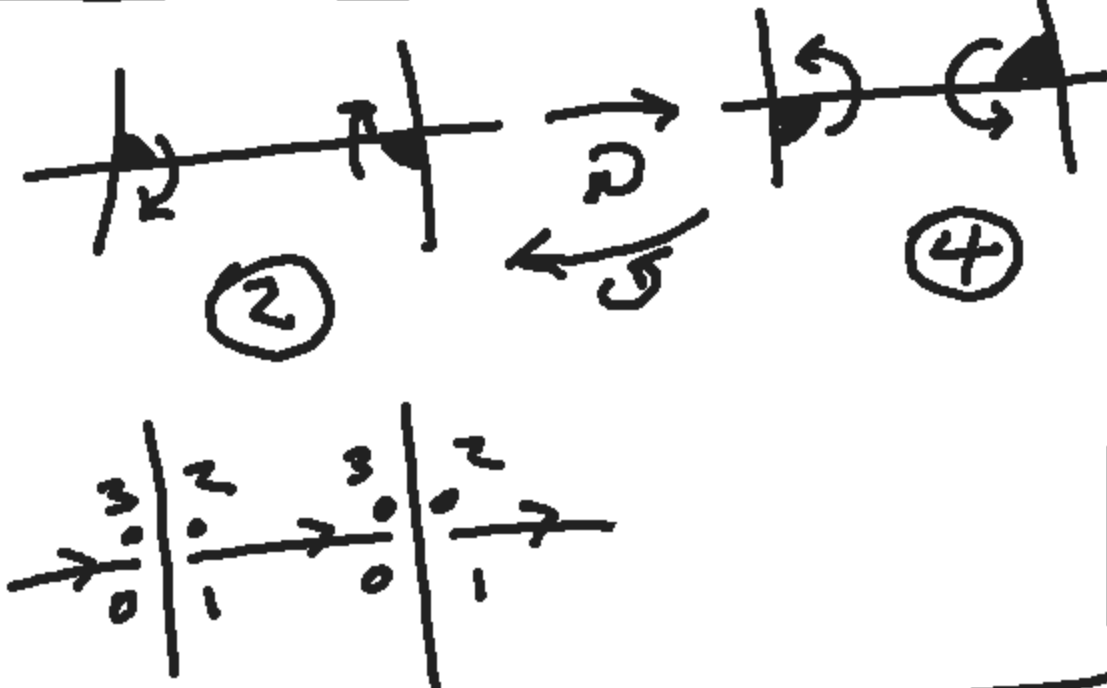
Given a marker state S

$gr(S) = \sum$ local marker numbers.

$gr(S) = 1 + 1 + 0 = 2$

Claim: $S \rightleftarrows S' \Rightarrow gr(S') = gr(S) \pm 2$

clocking move



Clock Theorem (FKT)

Any two states are connected by a sequence of clocking moves.

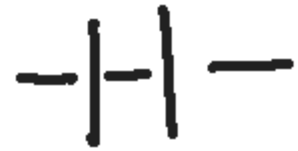
$gr(S)$

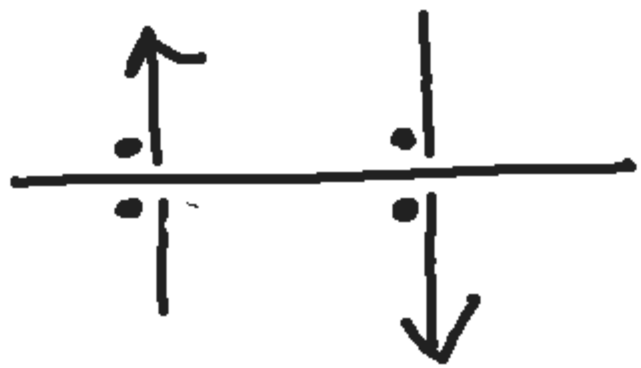
"quantum grading"

Alexander Grading

$a(S) = \#$ of $\bullet$'s marked by markers.

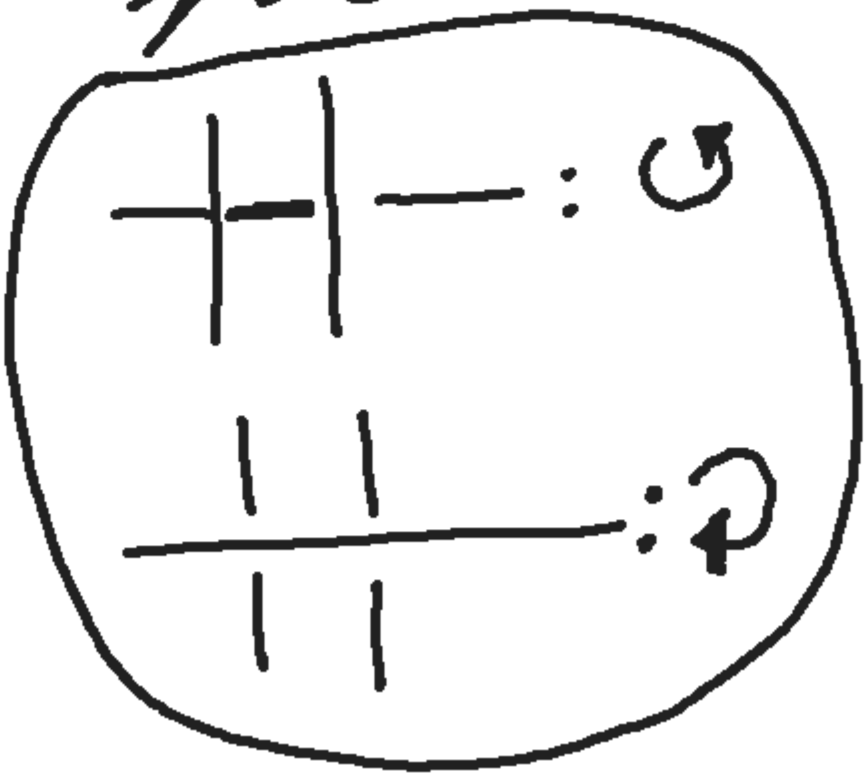
Alex: $|\langle K|S\rangle| = t^{a(S)}$

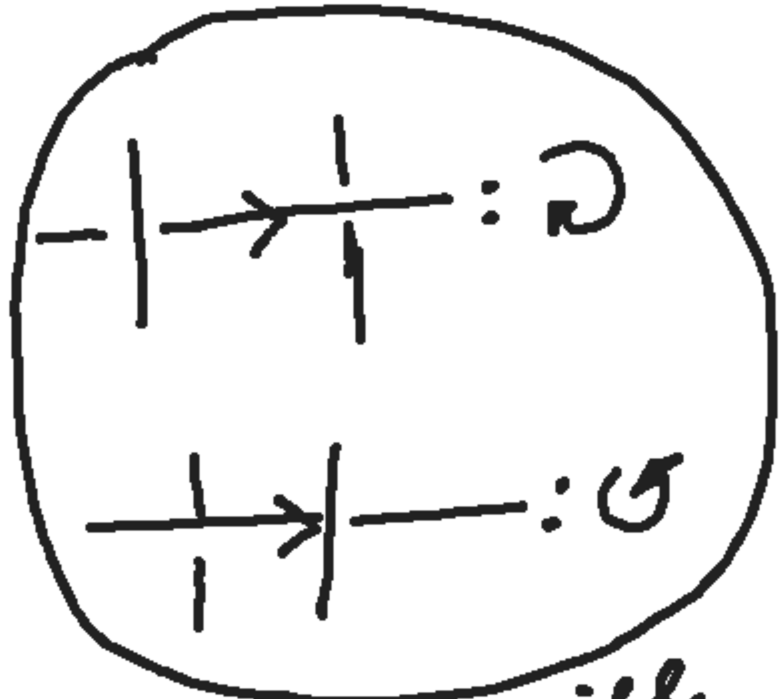
N.B. $A(S)$ invariant under moves (closely) on  & .



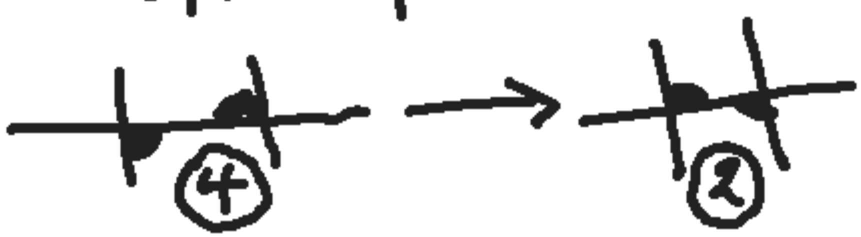
etc.

Grading gr(S) decreases for:





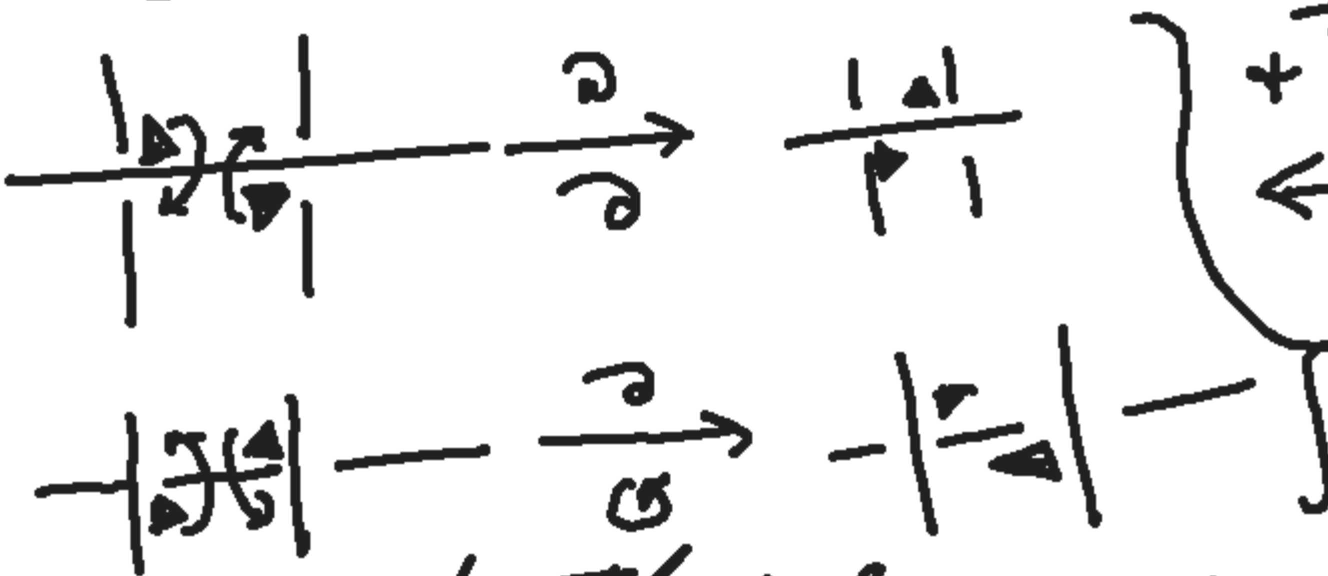
there will be defined to produce Φ boundary maps.



C_K = module gen by the states S with $g(S) = K$.

$\partial : C_K \longrightarrow C_{K-2}$

This has homology i.e. $\partial \circ \partial = \phi$ + categorifies $\Delta_K(t)$.

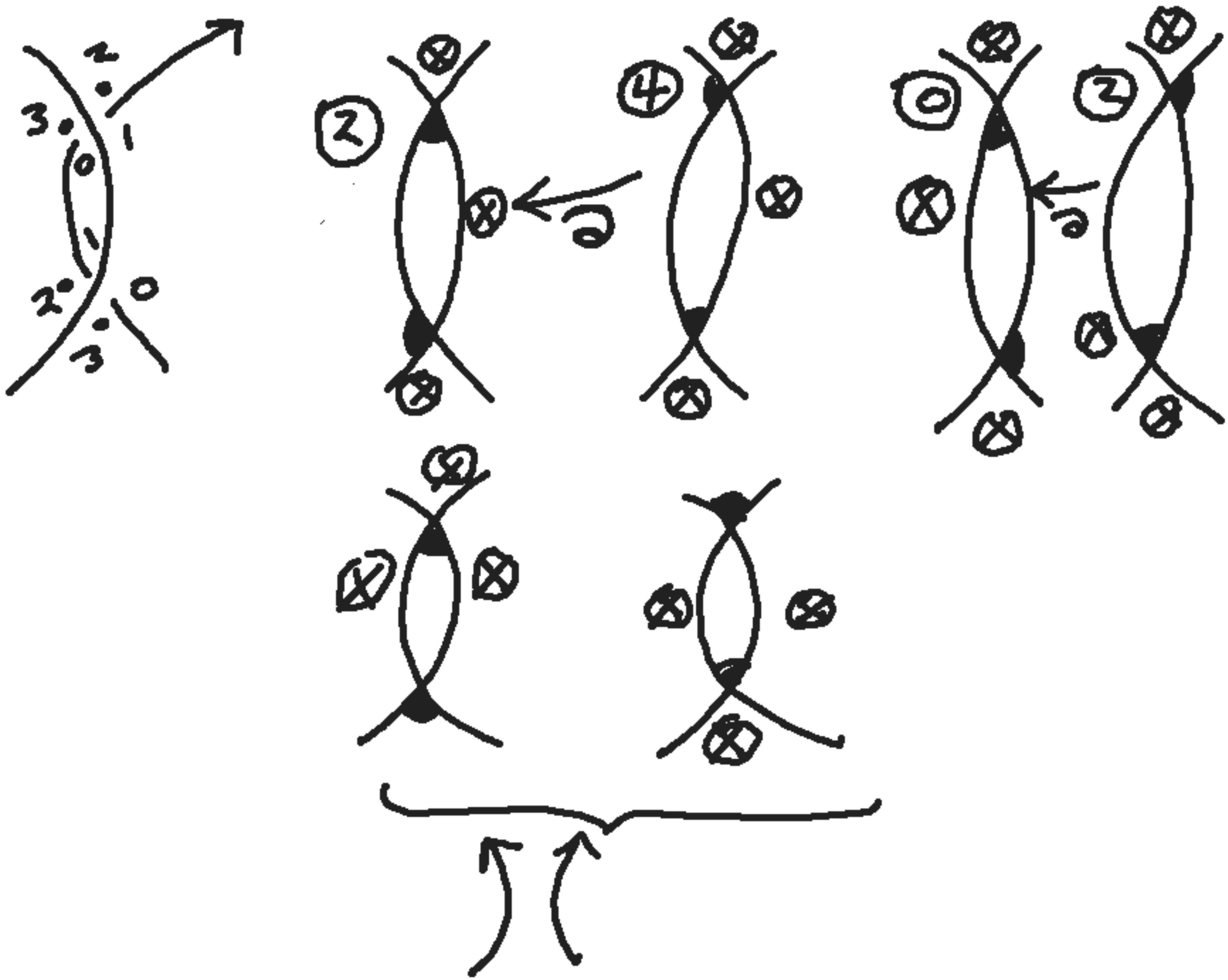


$\partial = \phi$ otherwise

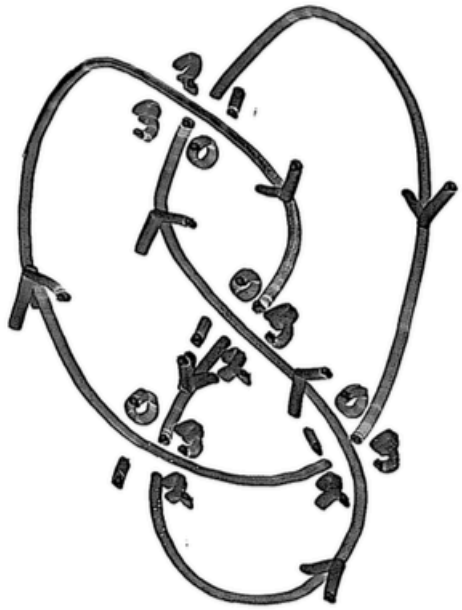
$\partial S = \sum \pm S'$, S' obtained from S as above

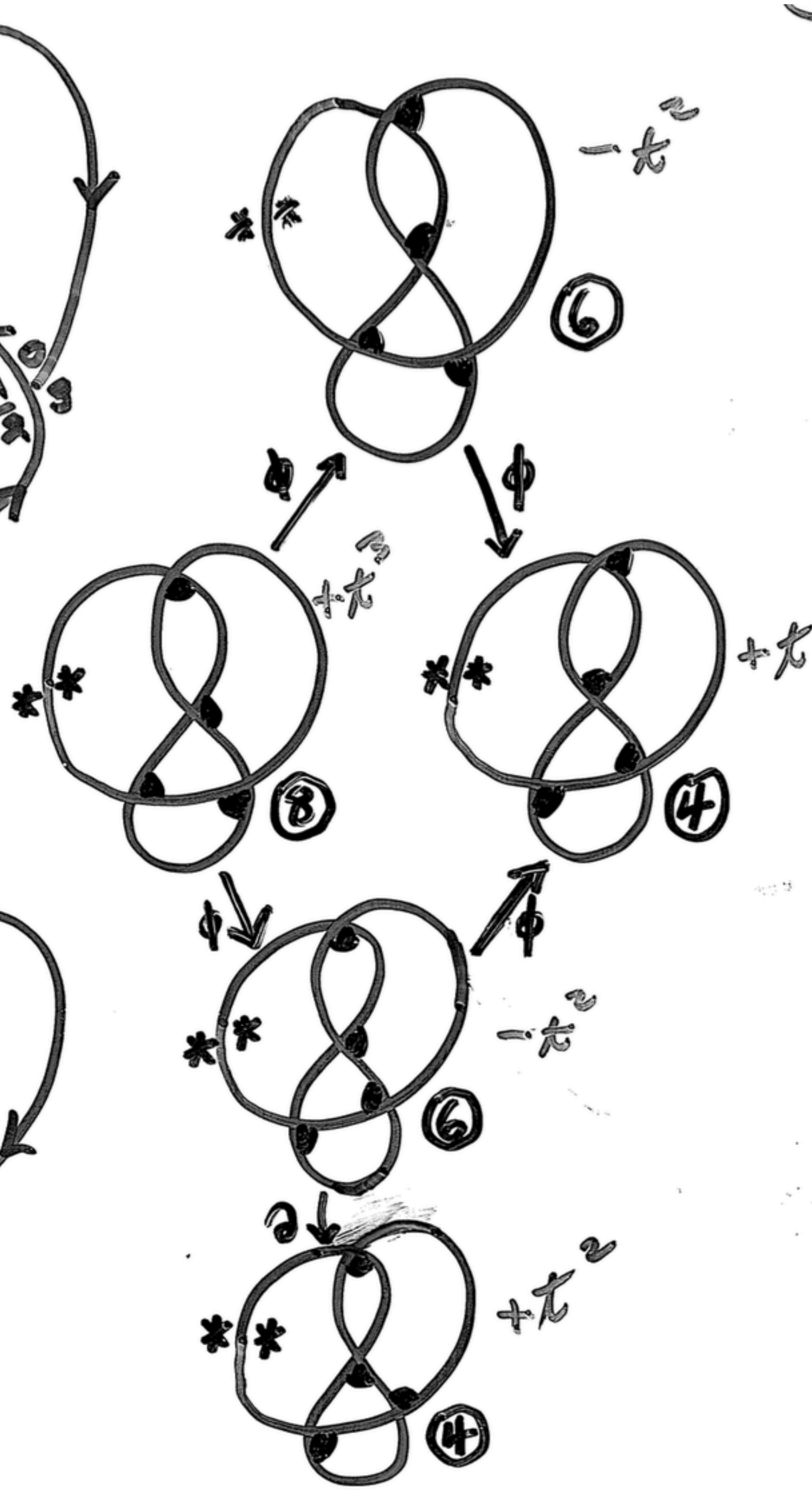
$\pm$ can be set$/\mathbb{Z}$ in "usual" way.

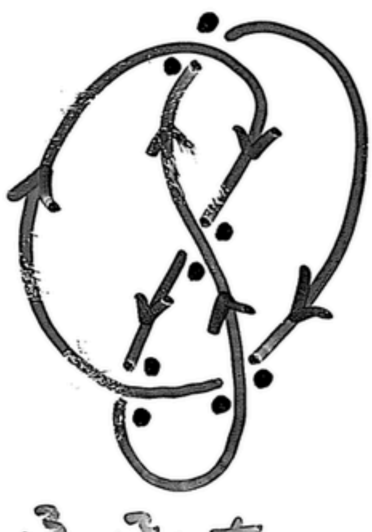
or we can work mod Z.



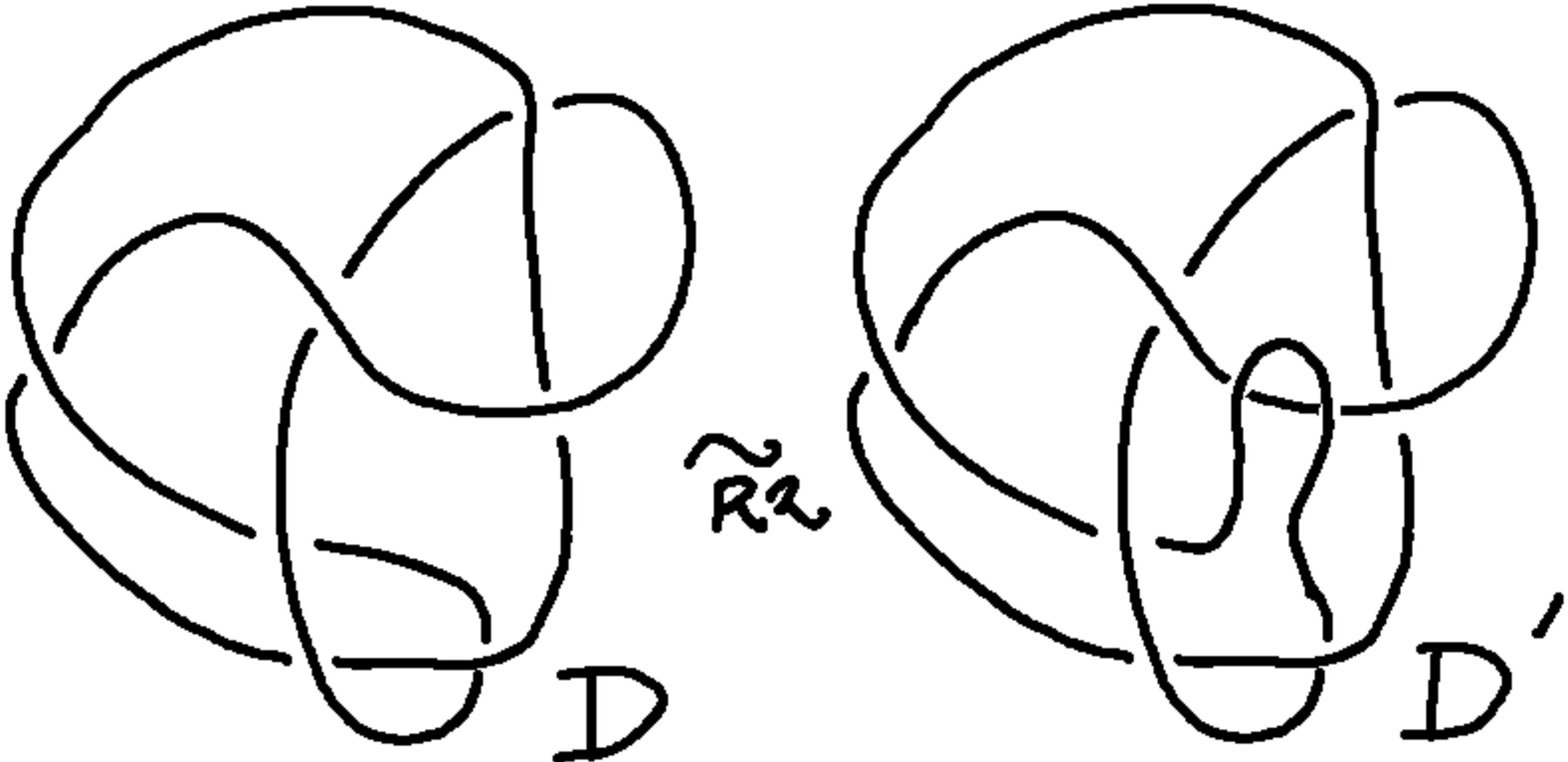
2
3
1
0
1
2
0
3
2
4
0
2







$t^3 - t^2 + t$
$\doteq t^2 - t + 1$



Marathania Silvero's Example

D and D' have different slack homology. They differ by one R2 move. More work is needed here!