# Lecture 13


## Doubled Lee | See Rushworth Paper

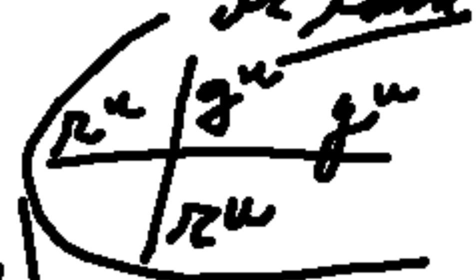
r g labelling

alternate r/g sol


often not Seifert


$rg = 0$

color upper or lower




$\mathrm{rank}(DKh'(L)) = 2\,|\{\text{alt sol smoothings of } L\}|$

[illegible]

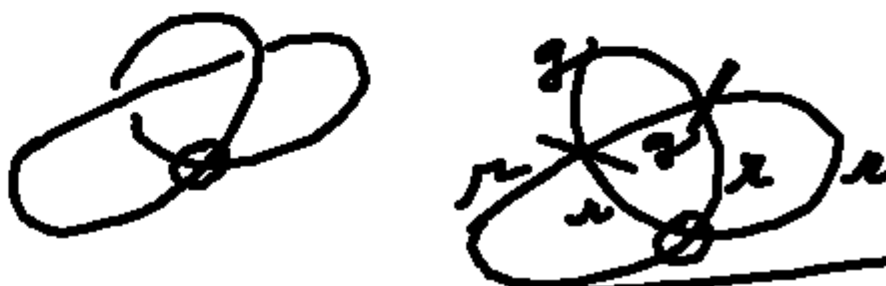
$DKho'(L)$ categorifies the binary bracket.

$$\{K\} = \mathrm{Bin}\langle K \rangle = \sum_{\substack{r,g \\ \text{roles}}} \langle L | \sigma \rangle$$

$\left.\begin{matrix} \text{roles} \\ L \end{matrix}\right\} \sigma$

$$\langle L | \sigma \rangle = \prod A \text{ or } A^{-1}$$

according to the smoothing.



$2A^{-2}$

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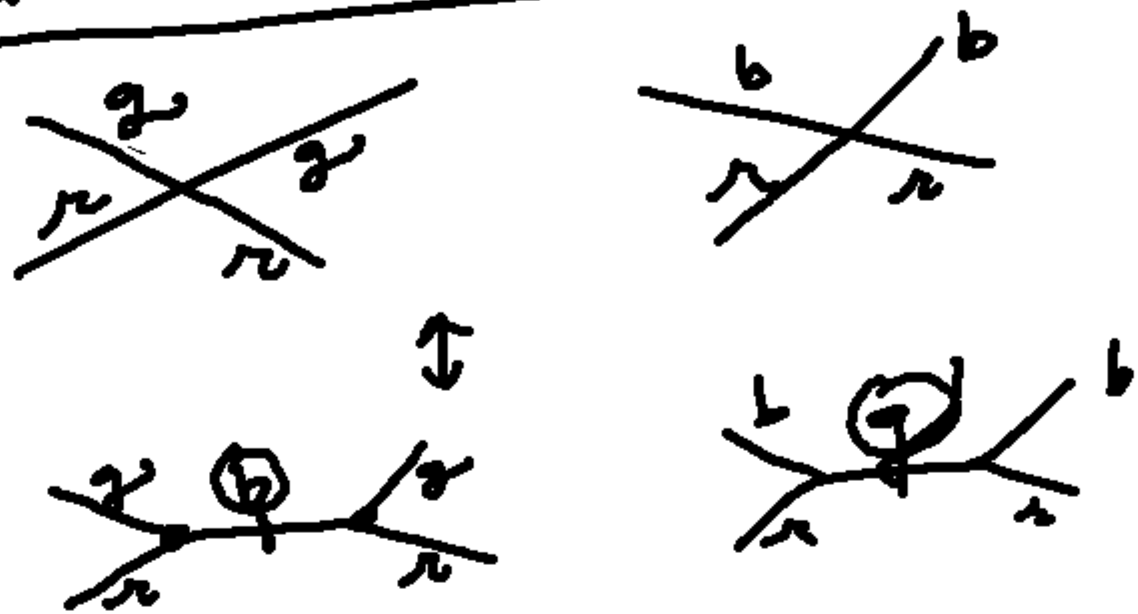
<u>Question</u>: What about Homology in relation to non-topological generalizations of $\{L\}$?

Cons consider $r, g, b$ 3 colrs

$$\left\{ \begin{array}{l} \{\times\} = A\{\asymp\} + B\{)(\} \\ \{O\} = 3 \end{array} \right.$$

colored diff

not invar und RM's



To have an appropriate 3 color algebra we would want

$$\left.\begin{matrix} rg = 0 \\ rb = 0 \\ bg = 0 \end{matrix}\right\}$$ so that diagram colorings would correspond to cycles in the homology.

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In Lee alg. with 2 colors $r, g$ we have

$rg = 0$, $r + g = 1$

$\Delta(r) = 2\, r \otimes r$

$\Delta(g) = -2\, g \otimes g$

$g^2 = g$, $r^2 = r$

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So we could try $rg = rb = bg = 0$

$r^2 = r$, $g^2 = g$, $b^2 = b$

$\Delta(r) = 2\, r \otimes r$

$\Delta(g) = -2\, g \otimes g$

$\Delta(b) = 2\, b \otimes b$

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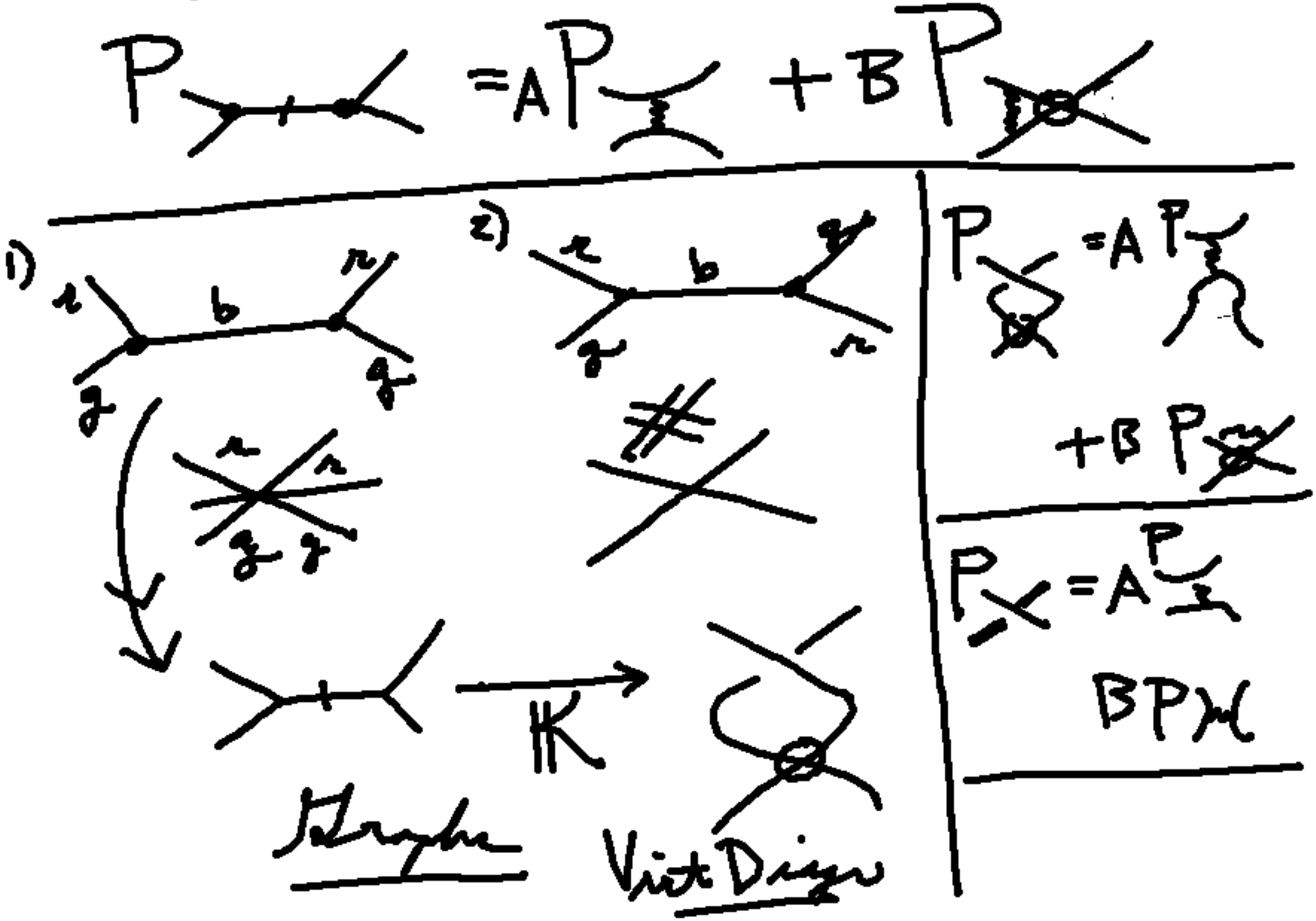
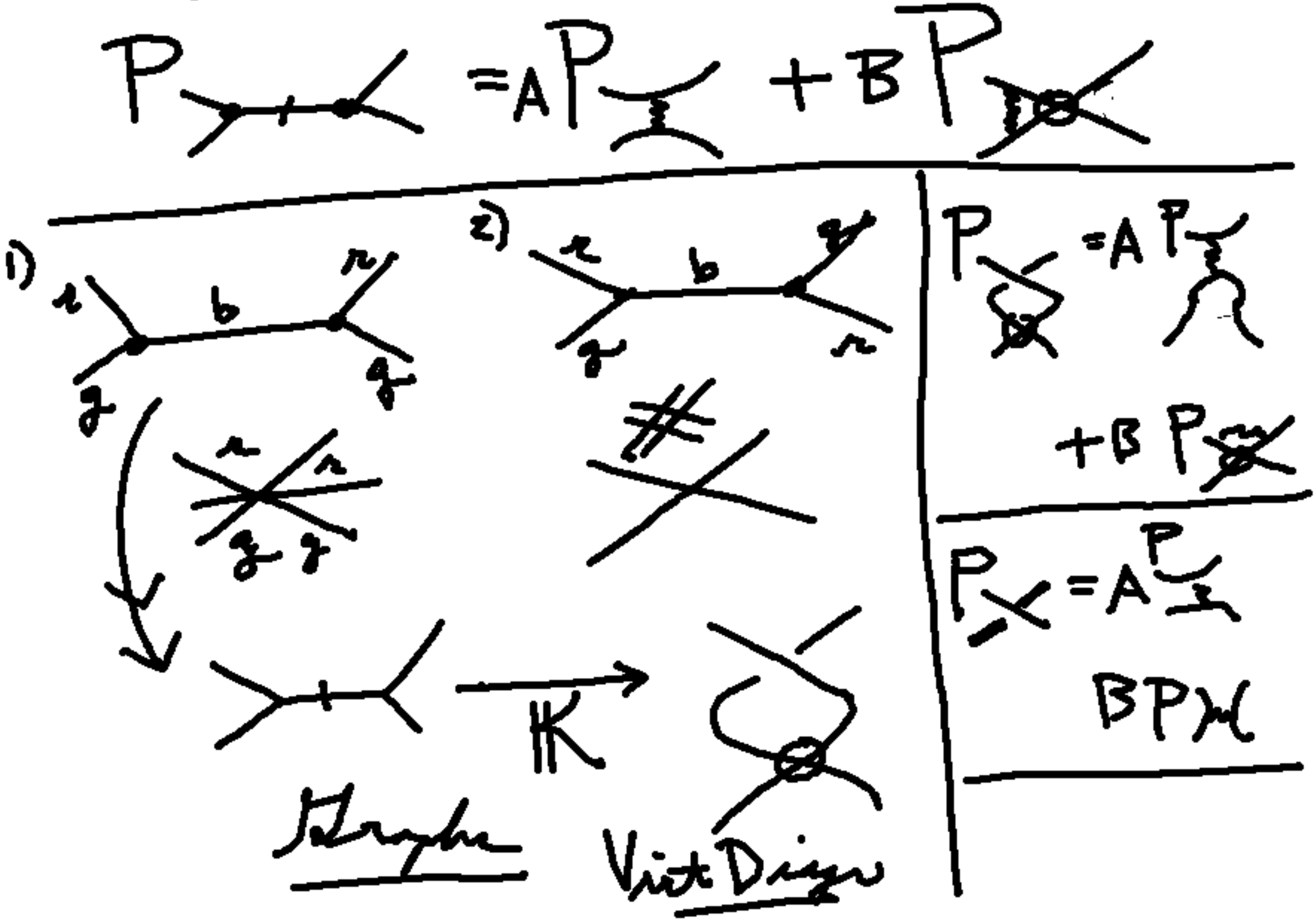
But what will happen? If you can experiment.

At graph levels (3 cols)



At graph level (3 rule)



$\{r, g, b\}$

**Question**

Consider $\boxed{\begin{array}{l} P_{\times} = A P_{\asymp} + B P_{)(} \\ P_{O} = 3 \end{array}}$

**Invariable**

Is there a natural categorification of it?
An $r, g, b$ algebra.

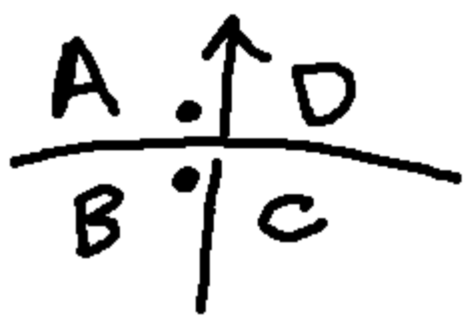
And if we had it,
would it be used
to study graph colorings?

# FKT States / Kauffman States

## Related to Knot Floer Homology

J.W. Alexander, 1928, TAMS

$\hookrightarrow$ Alexander Polynomial $\Delta_K(t)$.

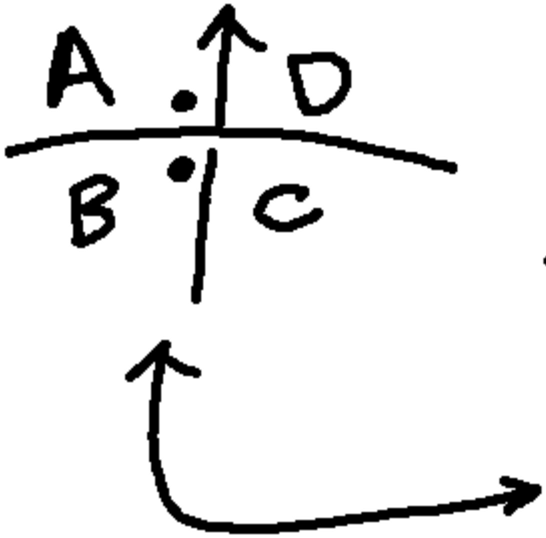



(A, B, C, D, ... region labels)

Construct module with region generators, and one relation per crossing. Module over the ring $\mathbb{Z}[t, t^{-1}]$.

$$(tA - tB + C - D)$$

$$\mathbb{Z}[A, B, C, D, \dots ; t, t^{-1}] / (\text{relns})$$

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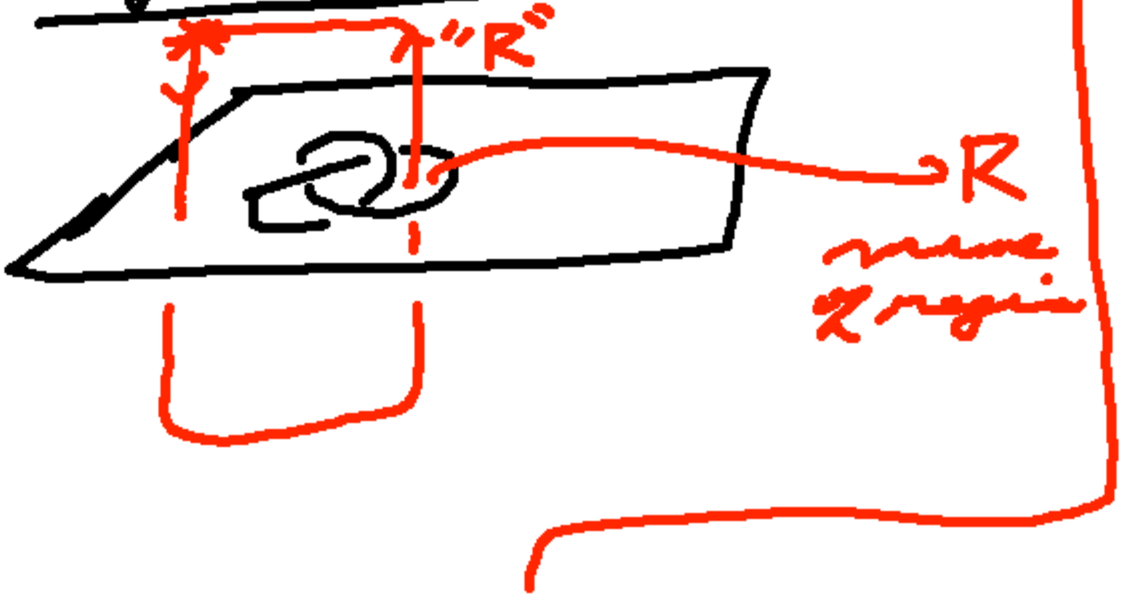
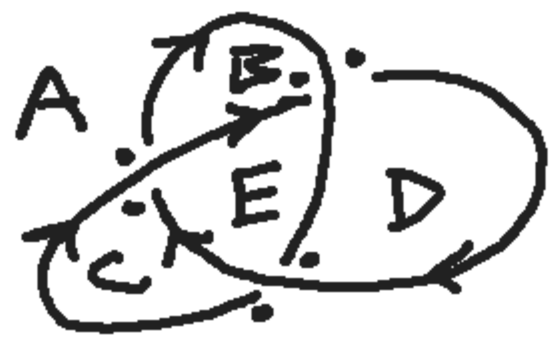
Construct module with region generators, and one relation per crossing. Module over the ring $\mathbb{Z}[t,t^{-1}]$.

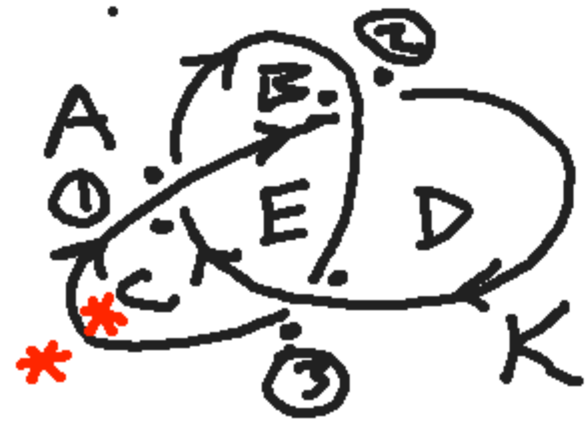
$$(tA - tB + C - D)$$

$$\mathbb{Z}[A,B,C,D,\ldots;\ t,t^{-1}]/(\text{relns})$$

Regions corresp. to using Dehn Pres of $\pi_1(\mathbb{R}^3 - K)$






1. $tA - tC + E - B$
2. $tA - tB + E - D$
3. $tA - tD + E - C$


$$\tilde{M} = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline ① & t & -1 & -t & 0 & 1 \\ ② & t & -t & 0 & -1 & 1 \\ ③ & t & 0 & -1 & -t & 1 \end{array}$$

$M$

Regions = Nodes + 2

$\doteq$ (up to $\pm t^N$ factor)

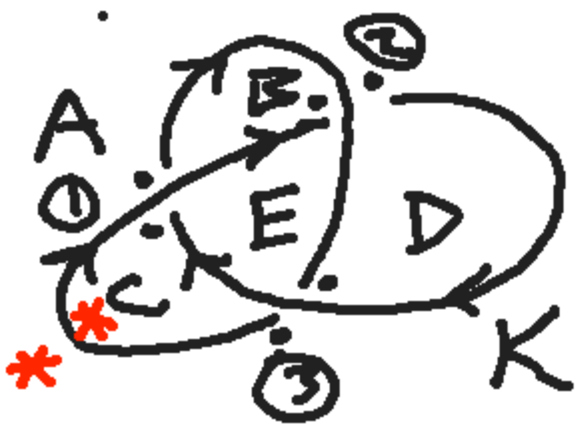
$$\Delta_K(t) \doteq \mathrm{Det}(M)$$

$$\Delta_K = t^2 - t + 1$$

$$\mathrm{Det}(M) = \begin{vmatrix} -1 & 0 & 1 \\ -t & -1 & 1 \\ 0 & -t & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ -t & 1 \end{vmatrix} + \begin{vmatrix} -t & -1 \\ 0 & -t \end{vmatrix}$$

$$= -(-1+t) + (t^2)$$

$$= 1 - t + t^2 = t^2 - t + 1$$




1. $tA - tC + E - B$
2. $tA - tB + E - D$
3. $tA - tD + E - C$

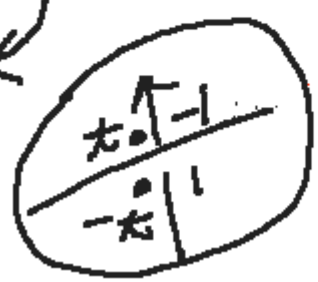

sgn(S) = sgn(S's perm)

$\tilde{M} =$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| ① | $t$ | $-1$ | $-t$ | $0$ | $1$ |
| ② | $t$ | $-t$ | $0$ | $-1$ | $1$ |
| ③ | $t$ | $0$ | $-1$ | $-t$ | $1$ |

$M$

$\Delta_K = t^2 - t + 1$


$\Delta_K(t) = \sum_S \prod$ (algebra bits marked by markers) $\times \boxed{\mathrm{sgn}(S)}$

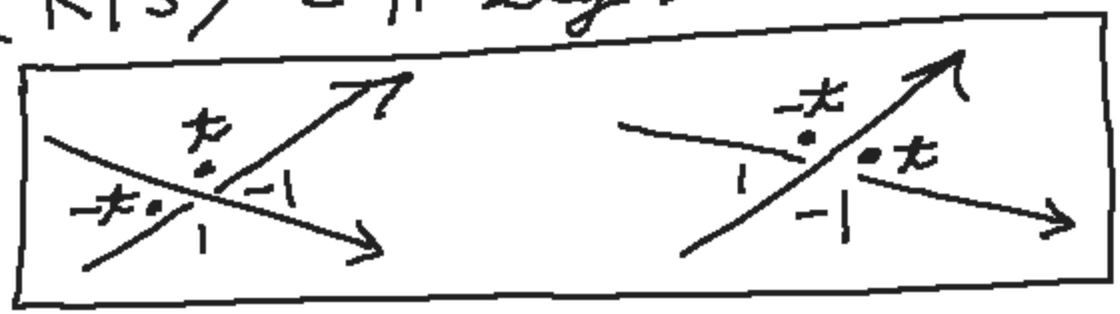
S a marker state

It turns out that $\mathrm{sgn}(S)$ can be replaced by

$$(-1)^{\#} = (-1)^{\#(\text{Black Holes in } S)}$$

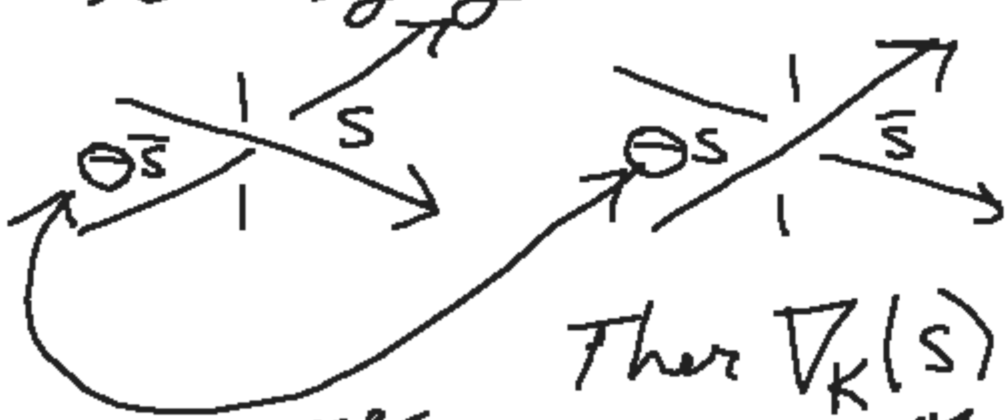
$$\Delta_K(t) = \sum_{S \in \text{Marker States}} \langle K|S\rangle (-1)^{\#}$$

$\langle K|S\rangle = \prod$ alg. site



$$(1 - t + t^2) = \Delta_K(t)$$

Can get the Conway Alex Polynomial just by changing the weights.




$\bar{s} = \frac{1}{s}$

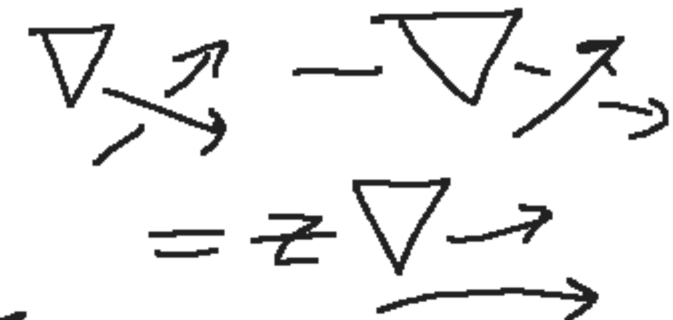
$z = s - \bar{s}$

Then $\nabla_K(s) = \sum_S \langle K|S\rangle$

or $z$

take care of $(-1)^{\#}$ absorbed in $\langle K|S\rangle$

Classical Knots



$\nabla_{O} = 1$

invar all RM's

Marker States $\rightsquigarrow \nabla_K(z)$.

$\Big\downarrow$ ?

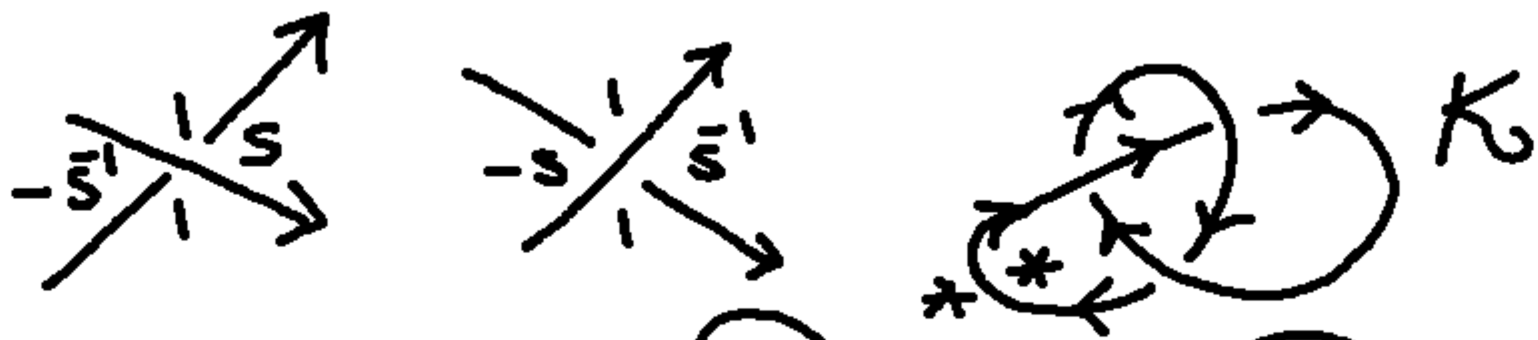
Complex Related to $\rightsquigarrow$ Categorif of $\nabla_K(z)$.

ans: Yes! Oszvath + Szabo

In their theory
the {Marker States} = $\mathcal{S}$
generates a chain complex
+ (originally) $\partial$ : Module($\mathcal{S}$) $\circlearrowleft$
comes from high dim'l
diff'l topology.

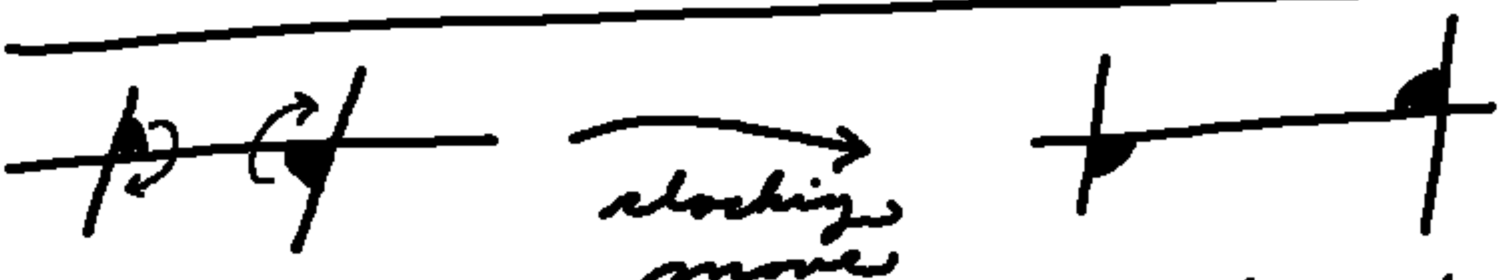
$\boxed{\text{?? Comb Version of } \partial \text{ ??}}$

Partial Soln
in Border Floer Homology
Osvath etc.





$$\nabla_K = s^2 - 1 - s^{-2} = (s - s^{-1})^2 + 1 = z^2 + 1.$$




It is possible that clocking moves could generate chain complex maps. More ...