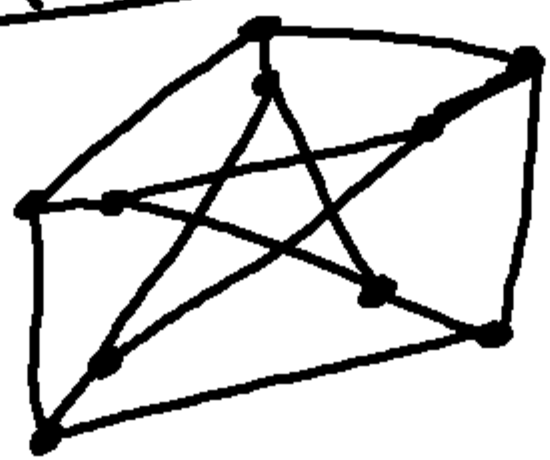
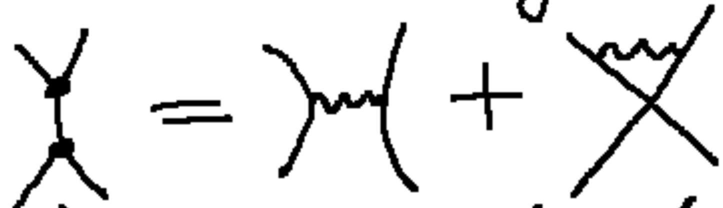
Lecture 11

An uncolorable graph.
(uncol trivalent (non-planar) graphs are called "snarks". This terminology is due to Blanche Descartes (William Tutte)).
(Lewis Carroll "The Hunting of the Snark")



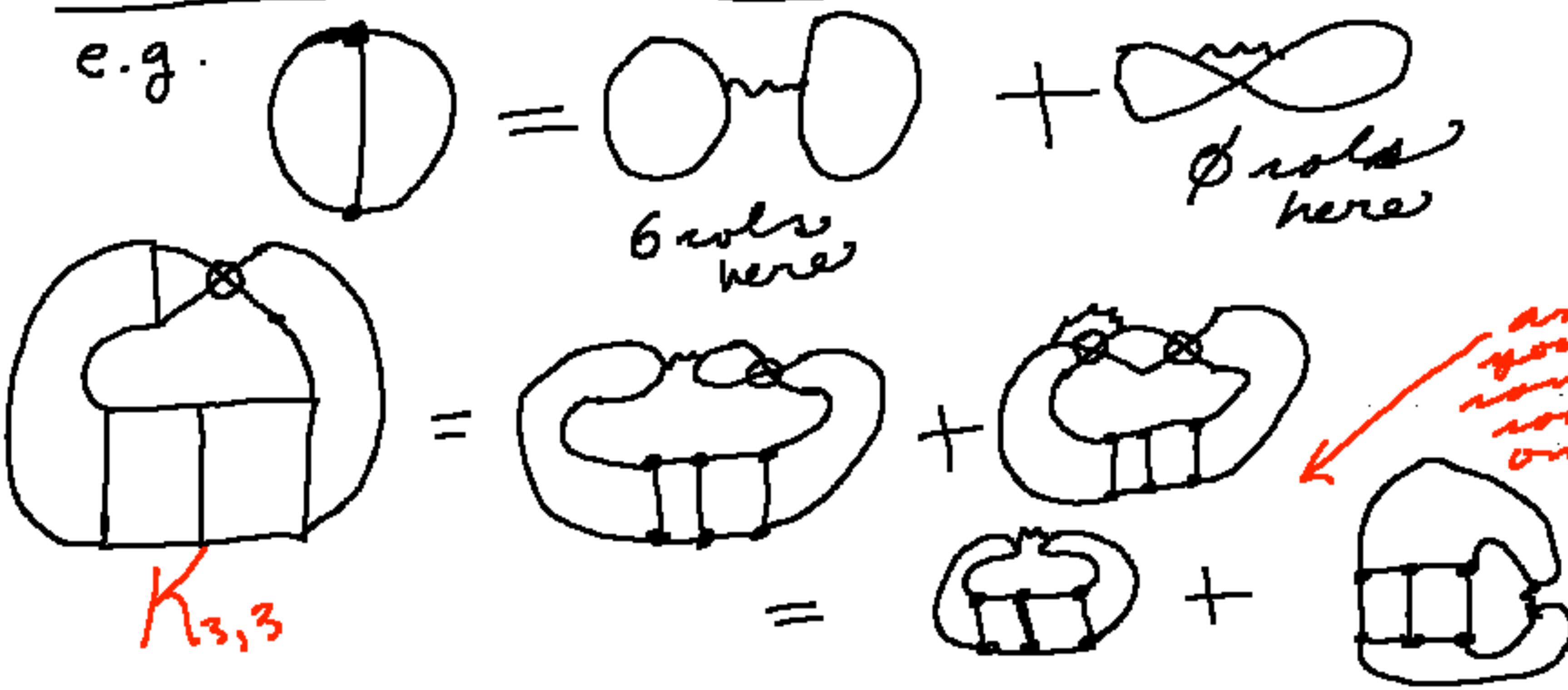
P (Petersen Graph)
$\nexists$ edge 3-coloring of P.
(See book "The Petersen Graph")

Recall the tautological expansion

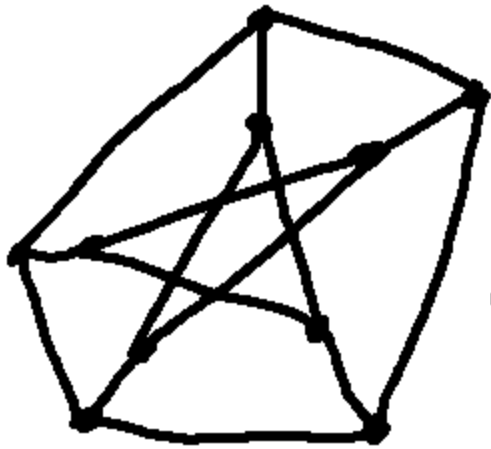


any coloring of [diagram] is obtained either from a coloring of ${}^{a}\!\!\rangle\!\!\sim\!\!\langle^{b}$ where $a \neq b$ or a coloring of $a \times b$ where $a \neq b$.

e.g.

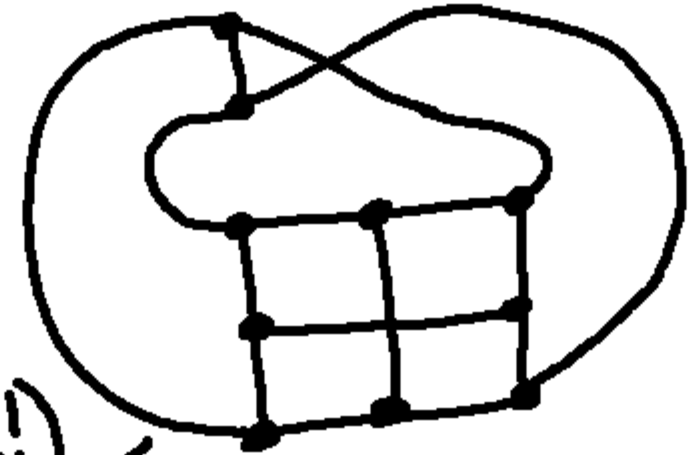


and you now continue on to find 12 colors

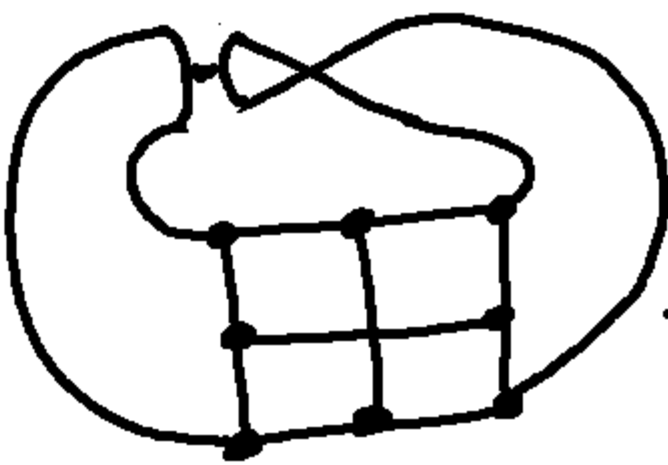


$\cong$

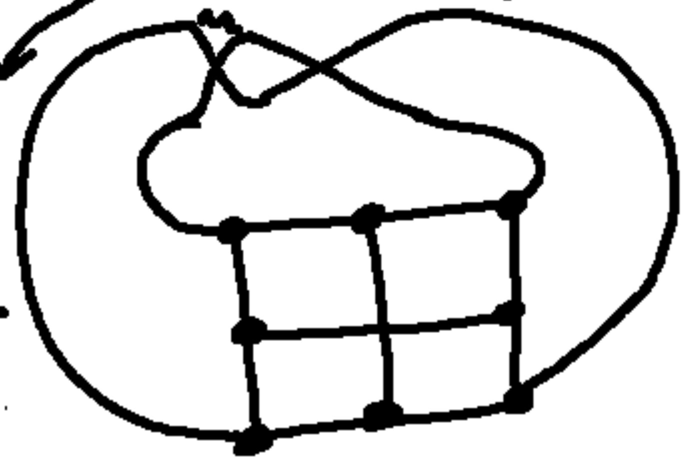
Graph
Isom
(Exercise!)

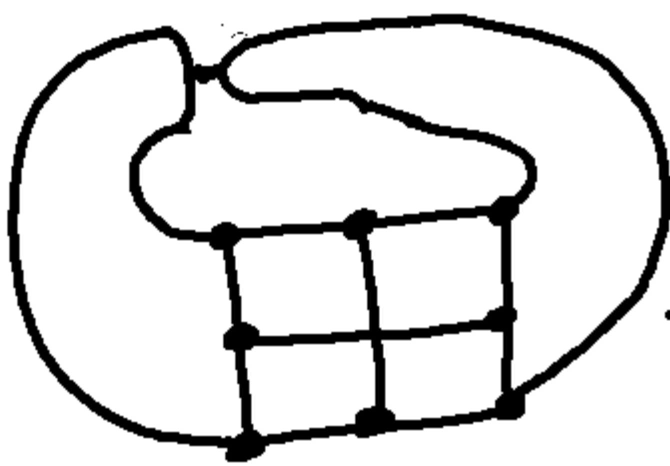


Tautology



+





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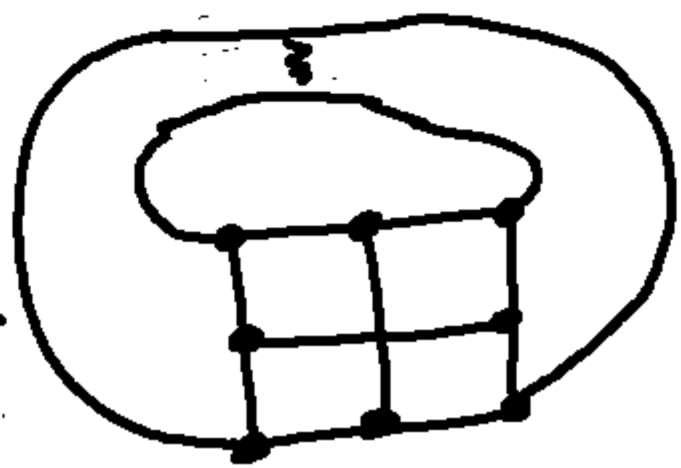
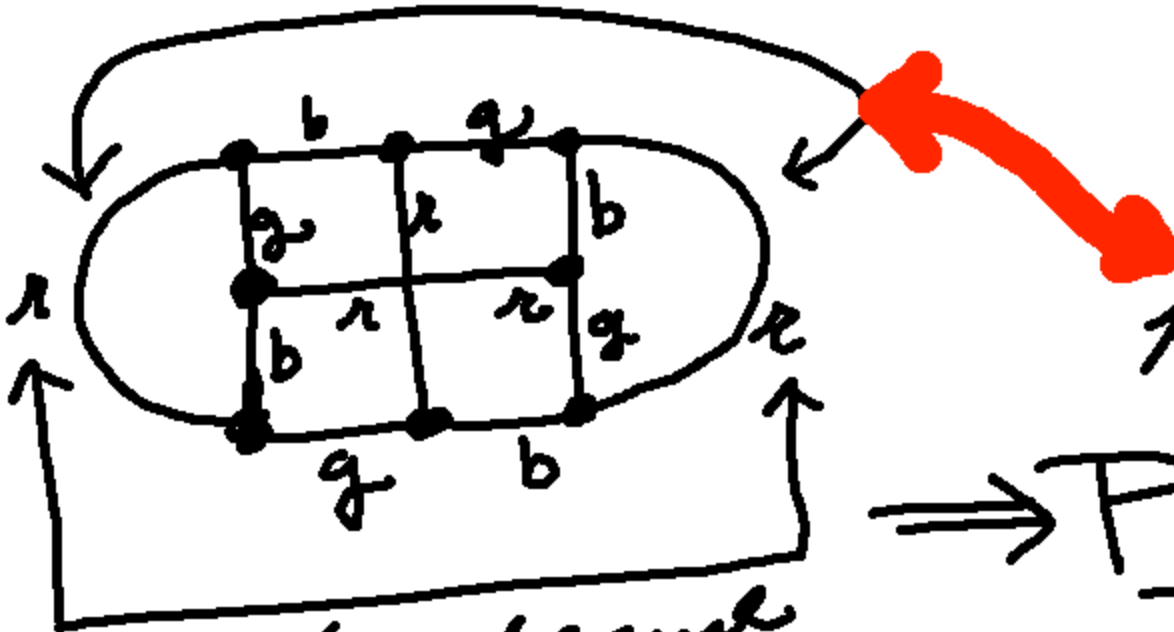


both
same
cond



forced same

forced equal

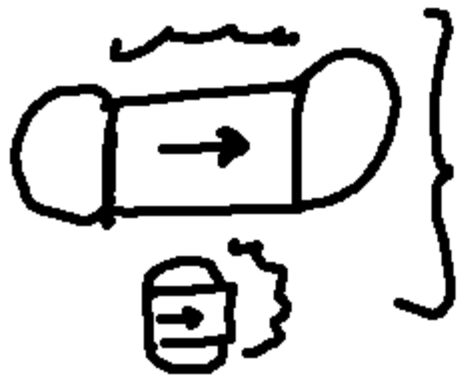
$\Longrightarrow$ Punch.

Note:

Some graph

Suppose this is a critical unavoidable. (any smaller is col.)

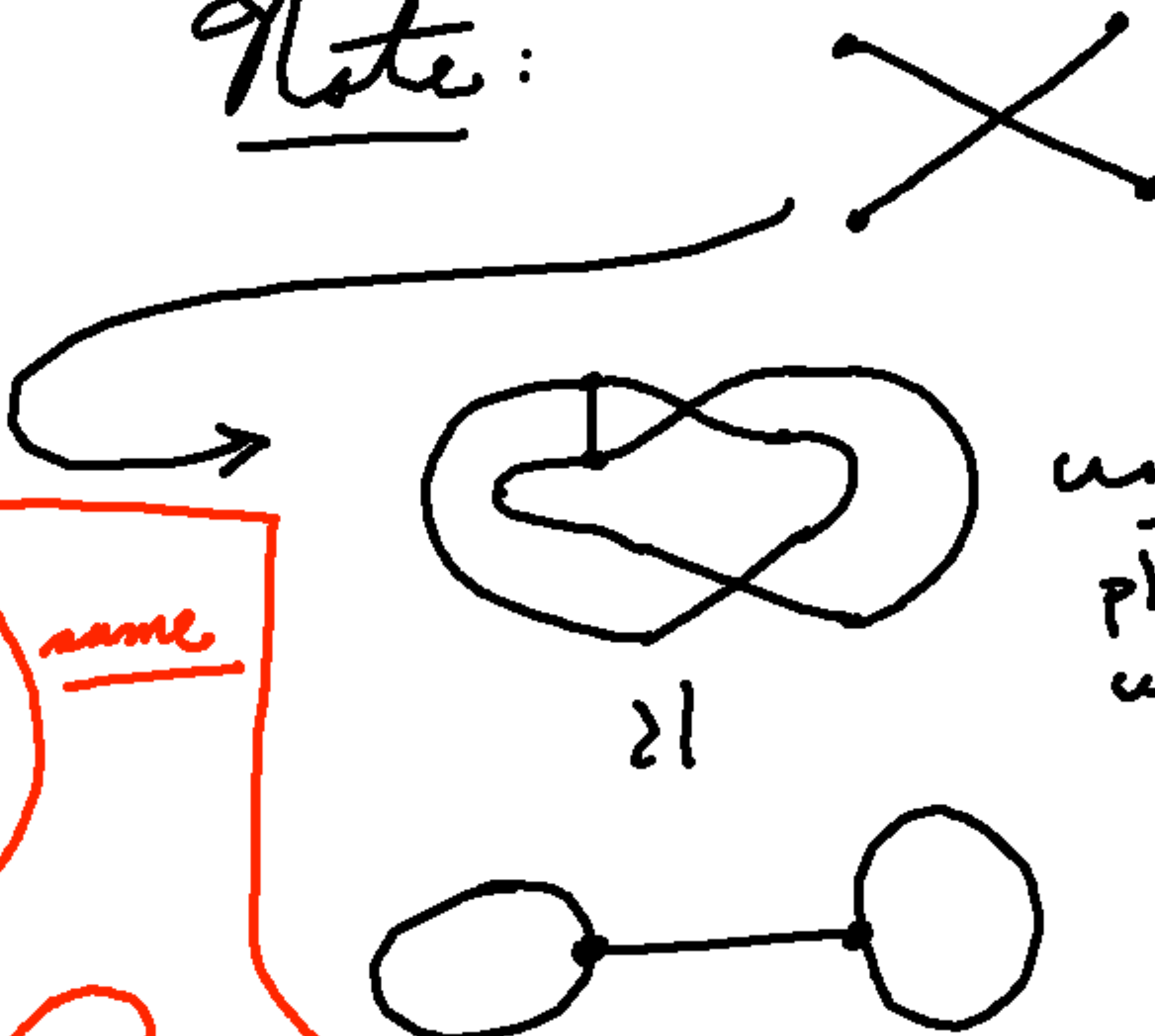
Same arg. shows that



both force same

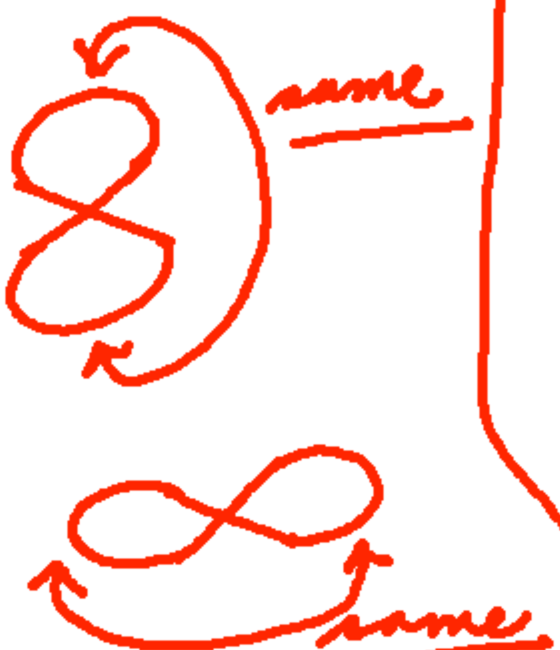
Note:

most elem forcing box



uncolorable

planar with isthmus



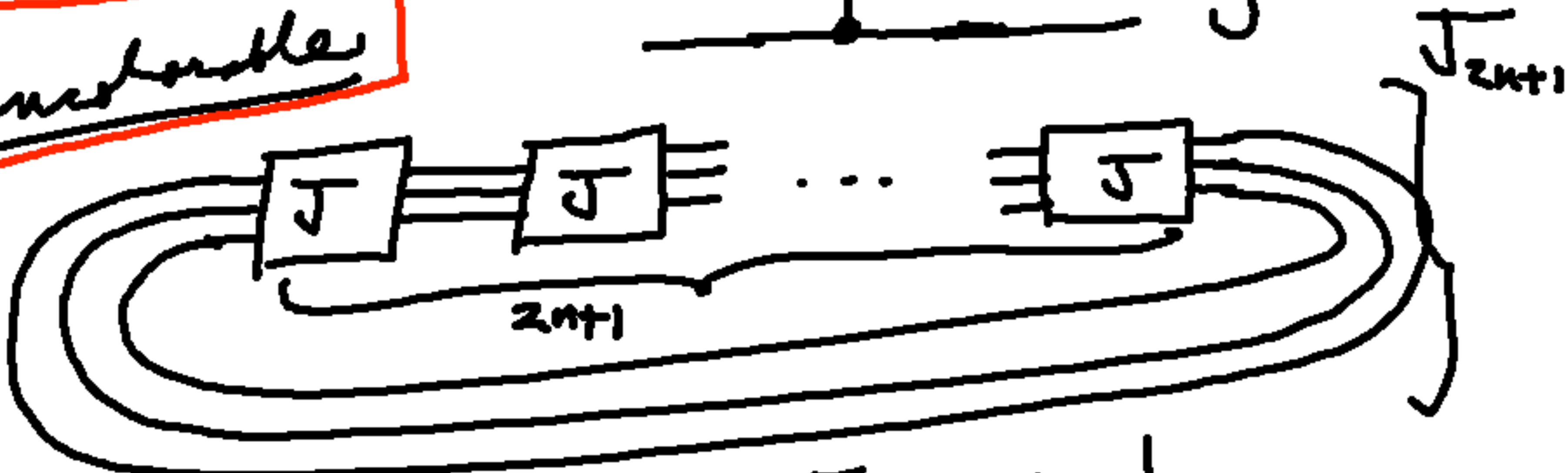
The simplest uncolorable.

Rufus Isaacs (Amer Math Monthly)

Showed how to construct infinitely many non-planar snarks.

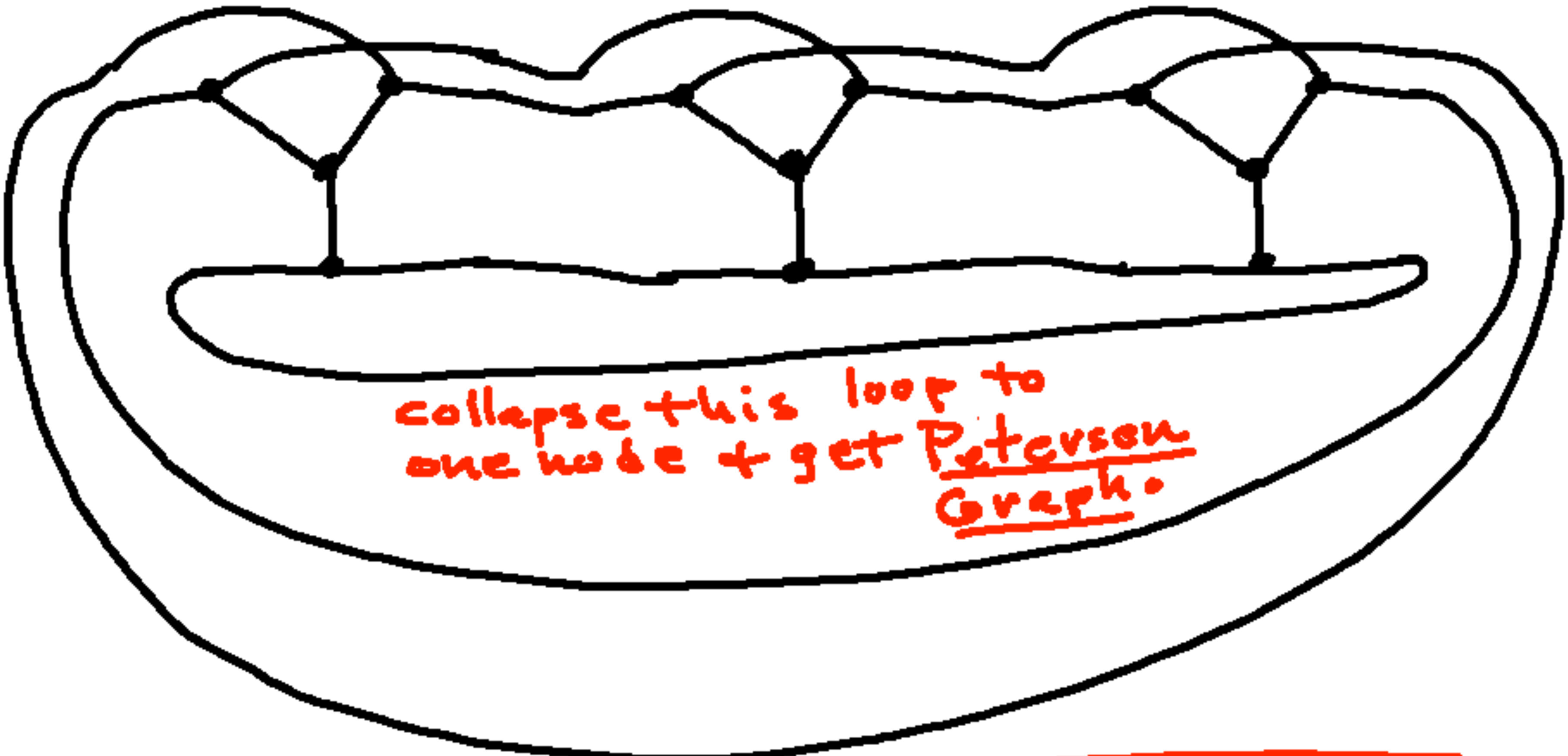
Gadget:

J_{2n+1} is **Uncolorable**

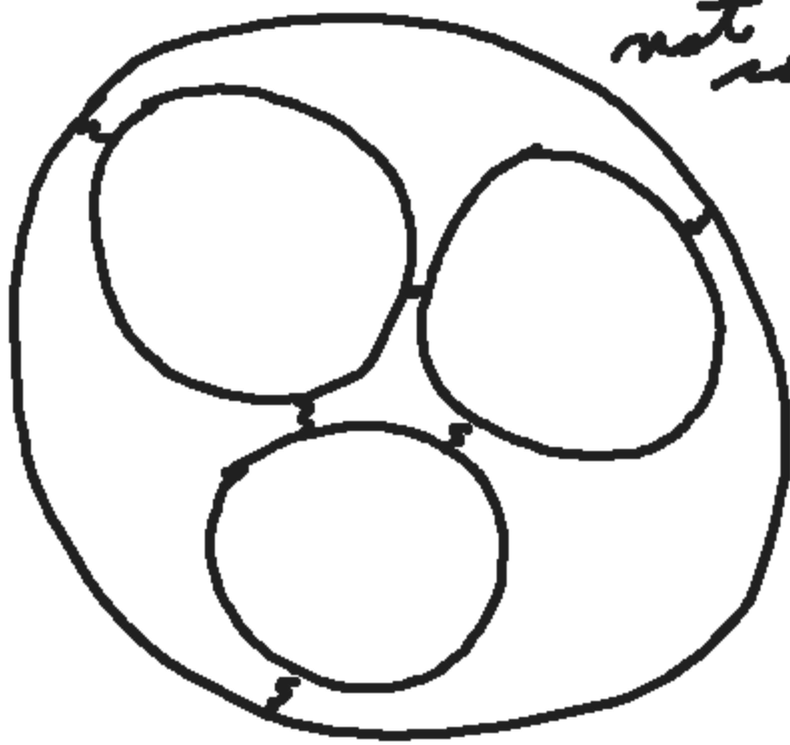


Exercise!

Isaac's J_3



Tutte Conj + Robertson, Seymour + Thomas pvd: Every non planar snark has a Petersen minor (obtained by deleting or contracting edges)



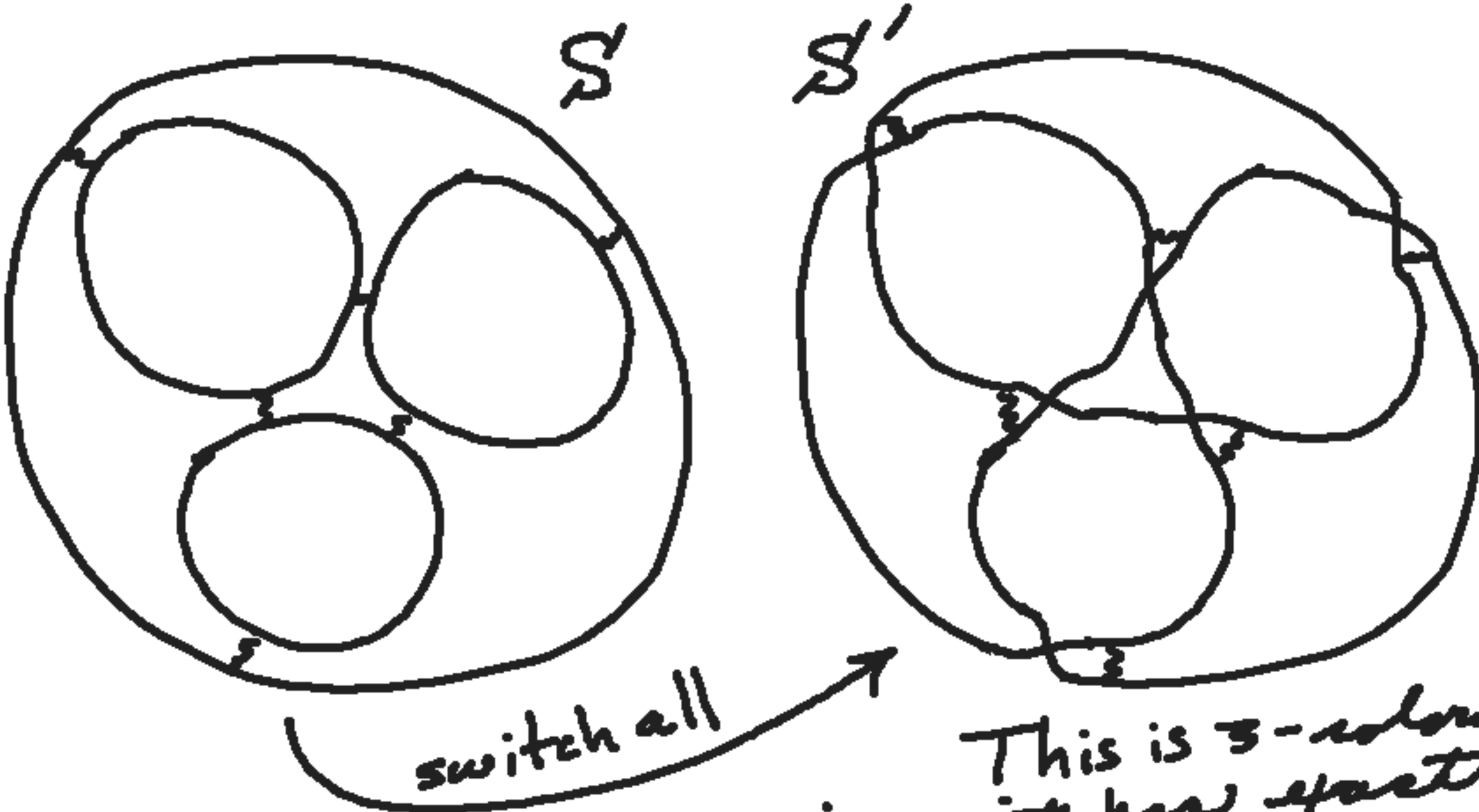
not sol

4CT $\Longrightarrow$

$\exists$ some switching

to get a 3 colorable state.

Exercise: Find the switch.
Find all such.



This is 3-colorable since it has exactly 3 loops.

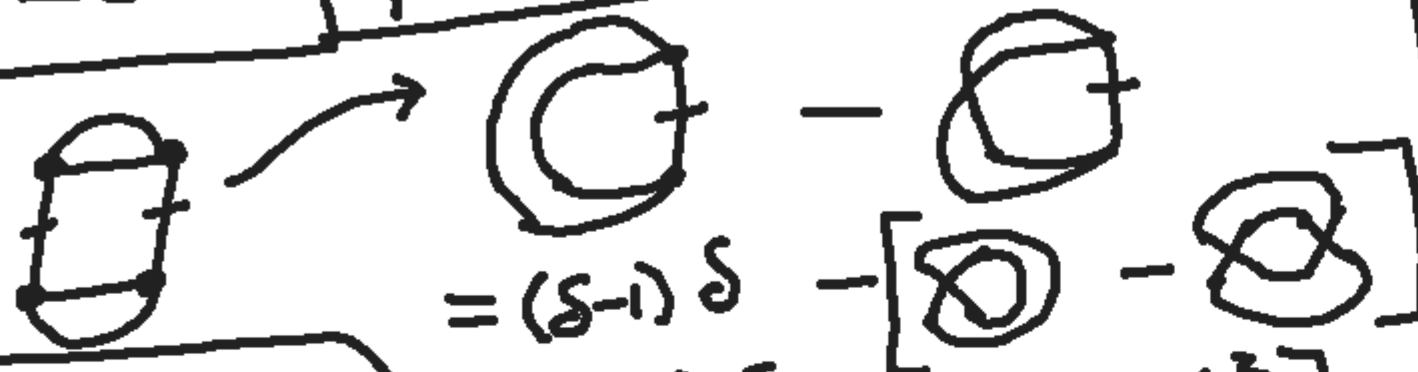
It is the only switching of S that is colorable.

(What is it about the plane that precludes uncolorables?)

Penrose Formula (defined ind. of PM choice)

What about a Penrose Polynomial?

This Penrose Poly depends on PM.

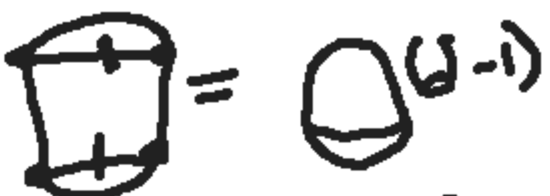


$= (\delta-1)\delta - [\delta - \delta^2]$

$= (\delta-1)\delta - \delta(1-\delta)$

$= \underline{2(\delta-1)\delta}$

$\delta = 3$: $\underline{2\cdot 2\cdot 3}$



$= \underline{\delta(\delta-1)^2}$

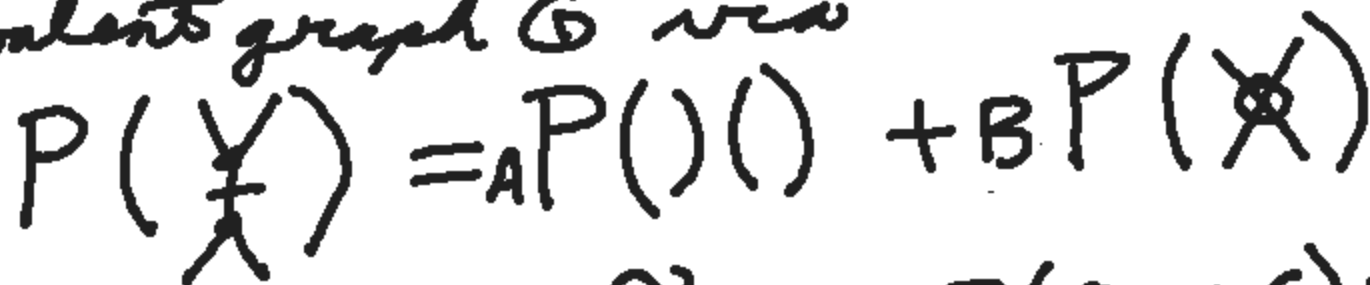
$\delta = 3$: $\underline{3\cdot 2^2}$

OK

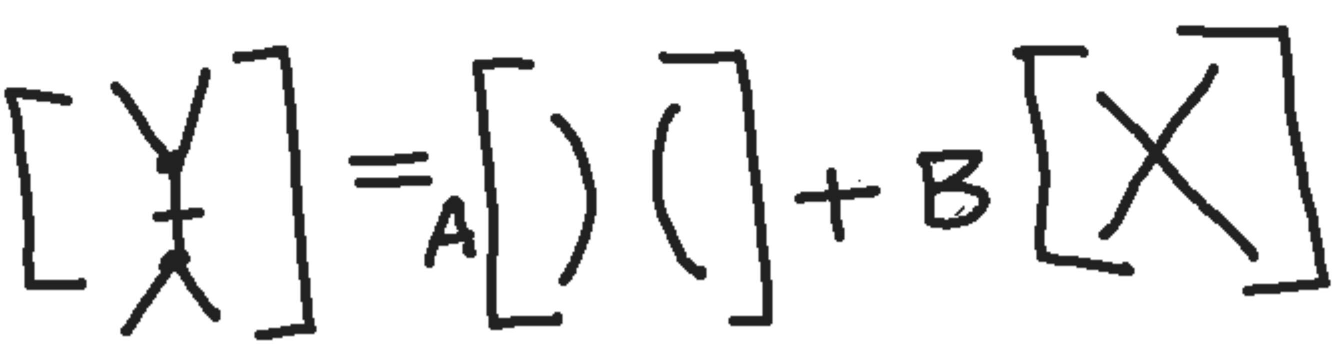
Scott Baldridge

We have a PM Poly!

We can define a **perfect matching polynomial** $P(G, M)$ for a virtual trivalent graph G via

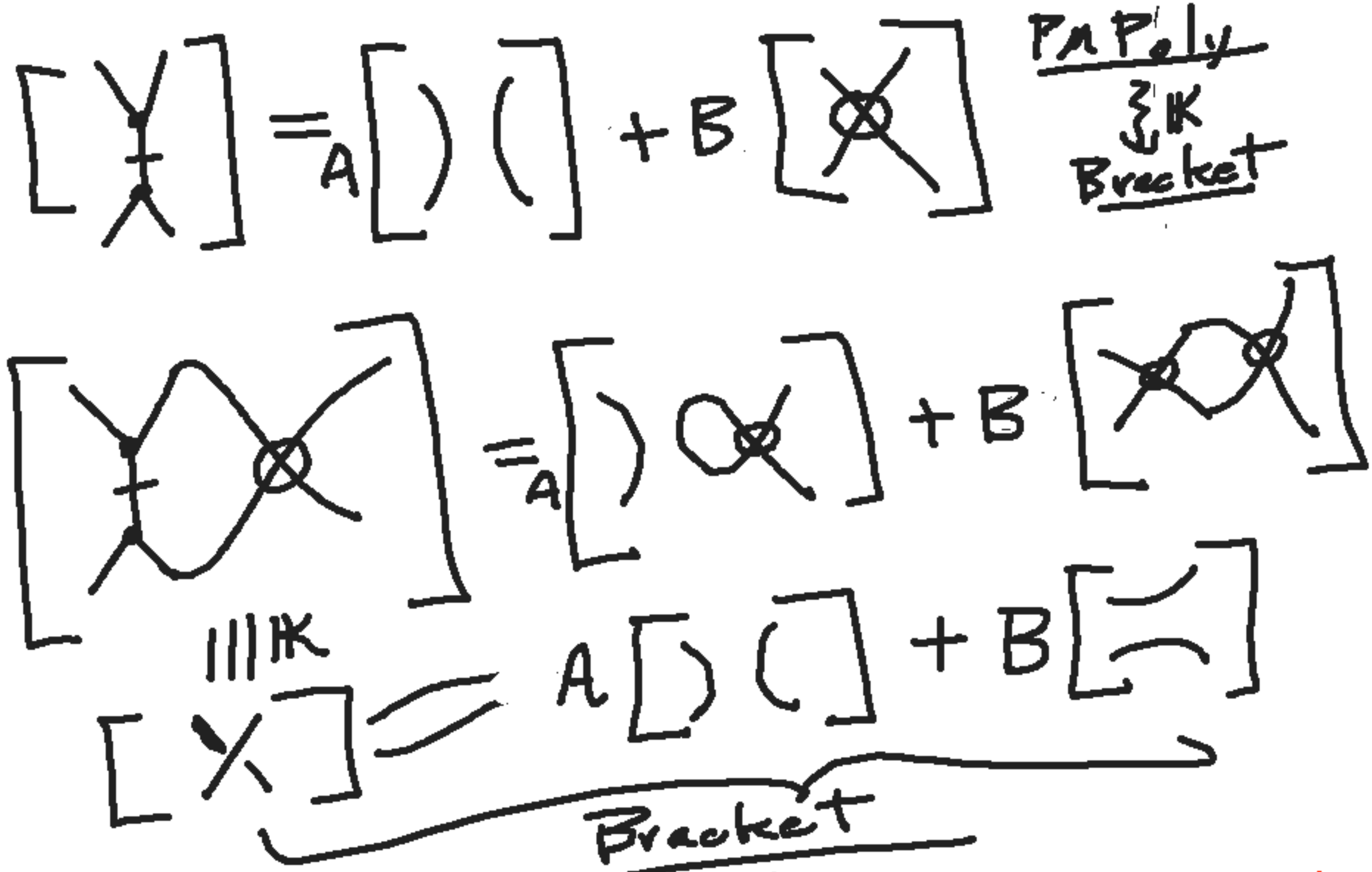


$$P(O) = \delta \qquad P(O \sqcup G) = \delta P(G)$$



$$[\bigcirc] = \delta$$

$\mathbb{K}$: Graphenes (trivalent with PM) $\longrightarrow$ Virtual Link Diagrams



$[\bigcirc] = \delta$

Thus K: Graphenes $\longrightarrow$ Virtual Links
translates bracket + PN Polynomial.