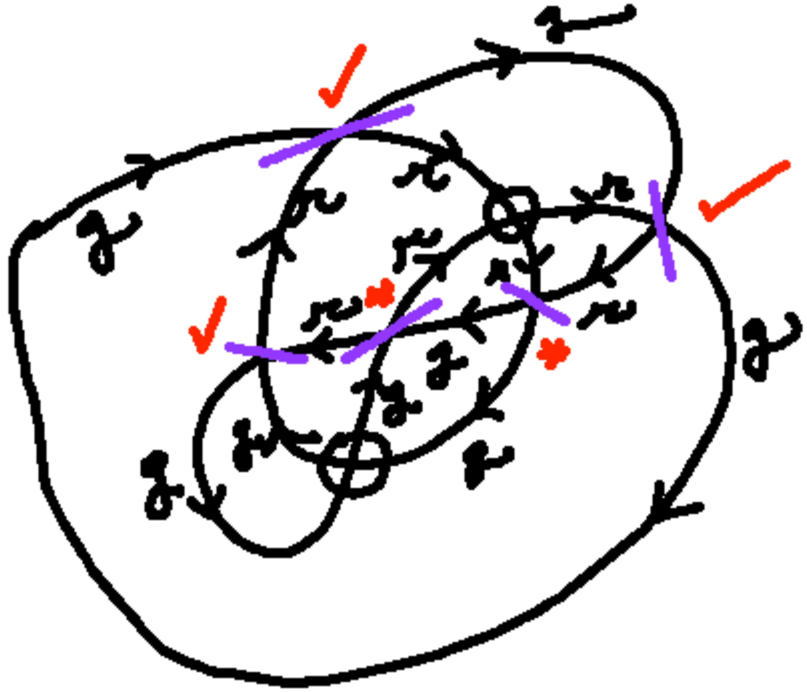
Lecture 10



Binary Brackets

$$[\,\times\,] = A[\,\asymp\,] + A^{-1}[\,)(\,]$$

$$[\,O\,] = 2$$

colored diff

there are 2 colors
$\underbrace{\text{red}}_{r}$, $\underbrace{\text{green}}_{g}$

$$[K] = \sum_{\text{colorings}} A^{\text{power}(col)} \cdot \underline{\text{loop eval}}$$

$$[\times] = A[\asymp] + A^{-1}[)\,(]$$

use colors r, g.

$$[O] = [O^r] + [O^g] = 2.$$

$$[K] = \sum_{\sigma \in \text{Colorings}(K)} [K^\sigma] \quad \text{where } [K^\sigma] = \text{product of } A + A^{-1}.$$

If K is classical, 1-component, then there are 2 colorings + $[K^\sigma] = A^{wr(K)}$:

(Pf. Each coloring $\leftrightarrow$ Seifert smoothing).

e.g.   $\longleftrightarrow A^3$

Thus K a classical knot, then

$$[K] = 2A^{wr(K)}.$$

If K is a 2 components classical link, then there are two orientations.

$$K^{\sigma}, \; K^{\sigma'} \qquad wr\,K^{\sigma'} = -wr\,K^{\sigma}$$

$$K^{\sigma} = K_1, \quad K^{\sigma'} = K_2$$

$$wr\,K^{\sigma'} = 2\,lk(K_2), \qquad wr\,K^{\sigma} = -2\,lk(K_2)$$

$$2\,lk(K_1) \qquad -2\,lk(K_1)$$

$$\Rightarrow [K] = A^{2\,lk(K_1)} + A^{-2\,lk(K_1)}.$$

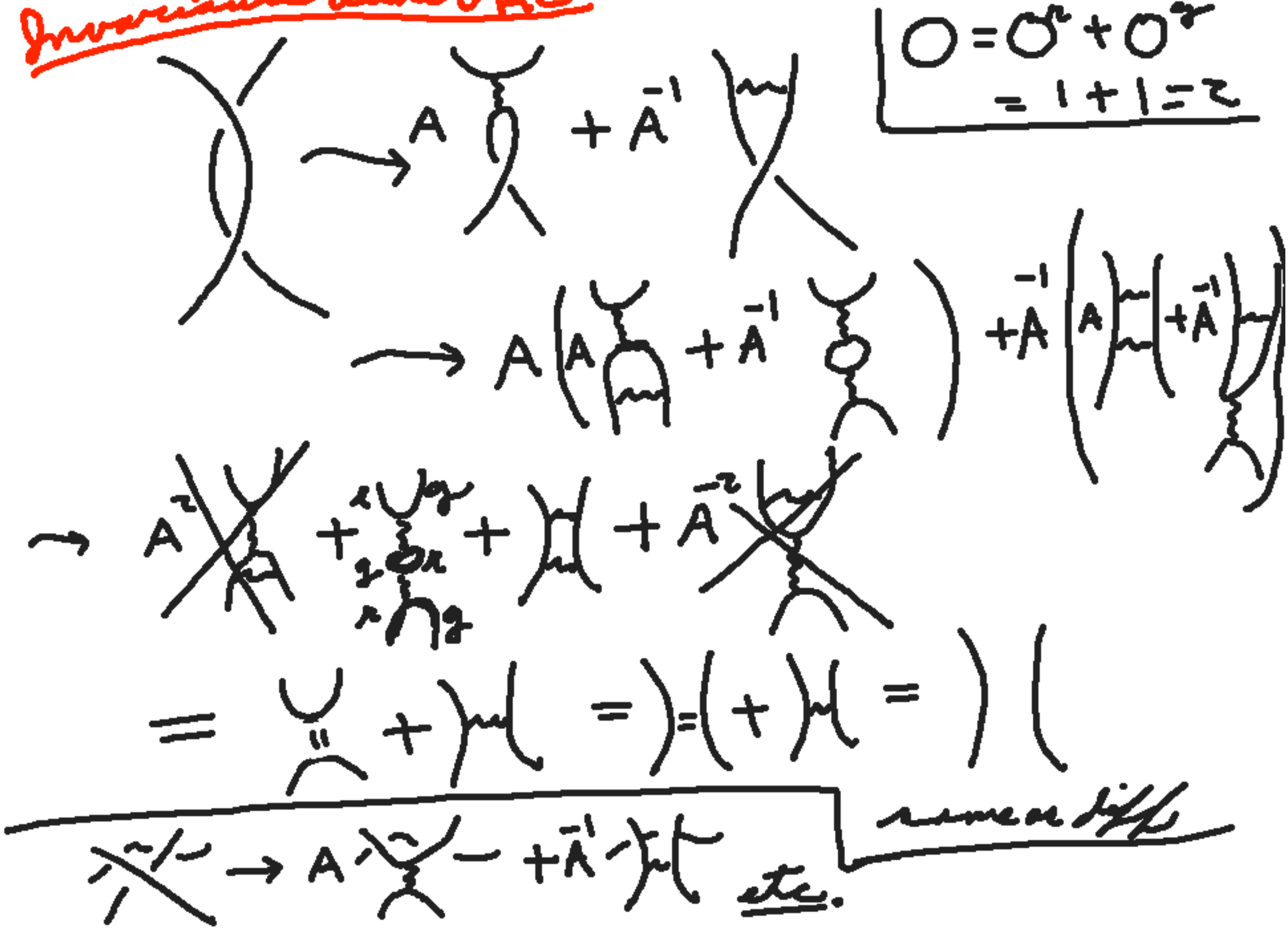
Virtual Knots + Links are more interesting.

K

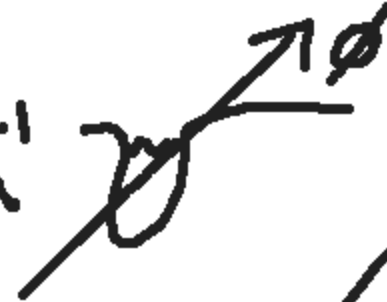
$$[K] = 2A^{-2}$$

N.B. $\underline{wr(K) = 2}$

Invariance under R2



Behaviour under R1

 $\longrightarrow A$  $+ A^{-1}$  $\emptyset$

 $\longrightarrow A$ 

 $\longrightarrow A^{-1}$ 

Define

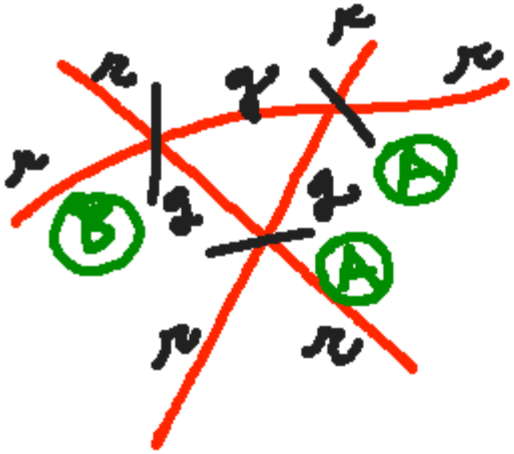
$B_K = A^{-wr(K)} [K]$

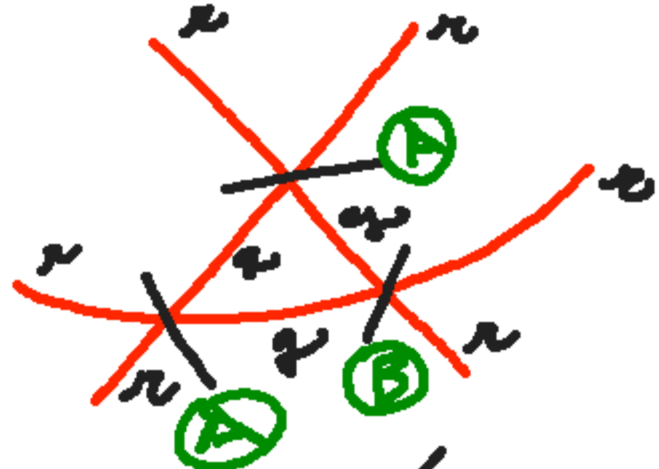
B_K invar under R1, R2, R3

Check $[K] \equiv A^{wr(K)}$ up to 2 — K classical (even writhe(K) / odd writhe(K))

one comp $[K_{virt}] \equiv A$

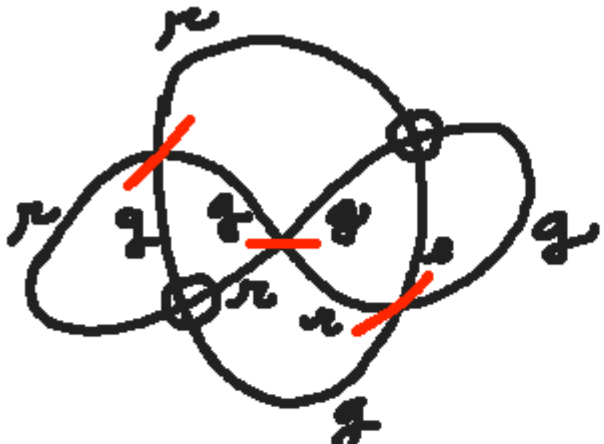
When K has many components, then $[K]$ = Sum over contributions from the orientations of K.





This verifies one of the cases of invariance under R3. We leave the other cases to you as an exercise.

L



AAA

$A^{-1}AA^{-1}$

$$\Rightarrow [L] = 2(A^3 + A^{-1})$$

Taking the above orientation on L

$wr(L) = -1$ So if we let

$$\underbrace{B_L = A^{-wr(L)}[K]}$$

Normalized Binary Bracket

Invariant under all 3 RM's

Here $B_L = A(2(A^3+A^{-1})) = 2(A^4+1)$.

$\Rightarrow L \neq L^*$ and L non-trivial virtual link.

Thm. K virt knot $\Rightarrow B_K = 2A^{-2J(K)}$, $J(K) =$ odd writhe of K

If K is a virtual knot with one component then the coloring result at a crossing depends upon whether it is even or odd.

odd crossing

We have

$\varepsilon(c) = +1$
$\varepsilon(c') = -1$
standard crossing signs

$$\boxed{A^{-wr(K)} = \prod_{\text{all } c} A^{-\varepsilon(c)}}$$

$$[K] = 2\prod_{\text{even } c} A^{\varepsilon(c)} \prod_{\text{odd } c} A^{-\varepsilon(c)} \Rightarrow B_K = A^{-wr(K)}[K] = 2\prod_{\text{odd } c} A^{-2\varepsilon(c)}$$

$\therefore B_K = 2A^{-2J(K)}$ where $J(K) =$ odd writhe of K.

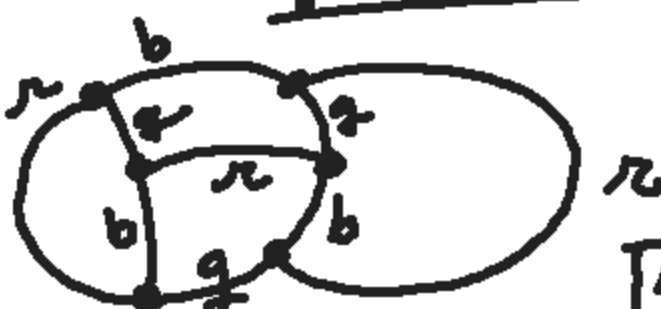
$$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$$

$$\langle O \rangle = 2 \quad , \text{ 2 colors}$$

What about more colors?

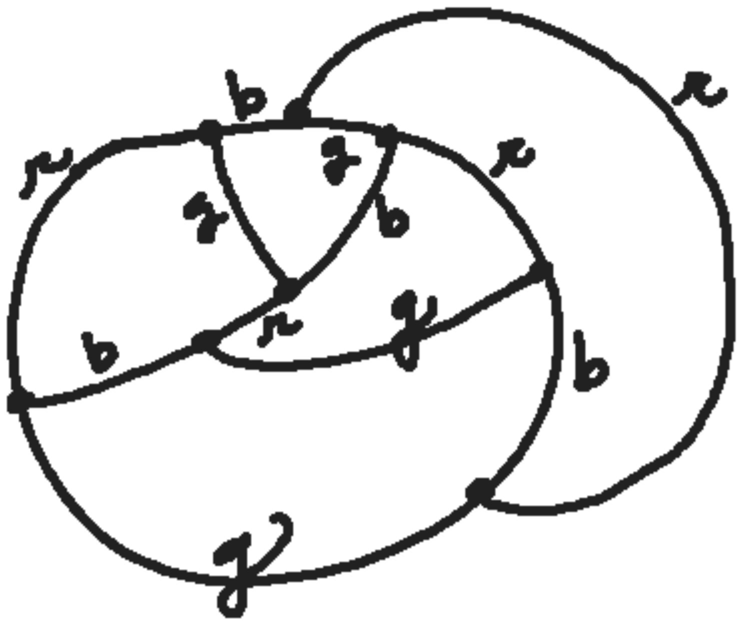
<u>Ans.</u> Harder to make invar under RM. But something to think about for graphs.

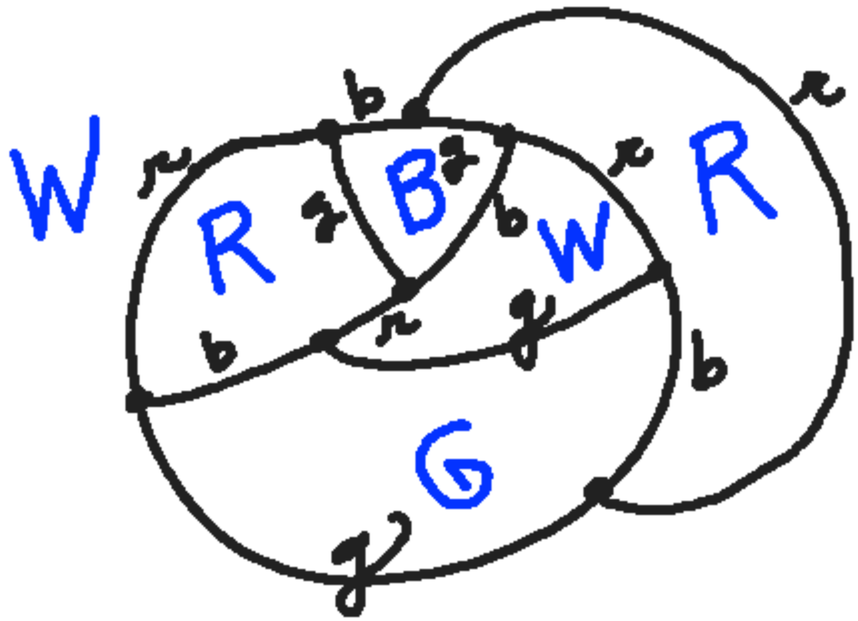
<u>Digression</u> 3 coloring trivalent graphs. $\boxed{r, g, b}$



3 dist cols at each node

$\boxed{4CT} \Leftrightarrow$ Planar no isthmus is colorable.





To color the map use

$\{W, R, G, B\}$

$R^2 = W$ W = identity

$G^2 = W$ Commutative

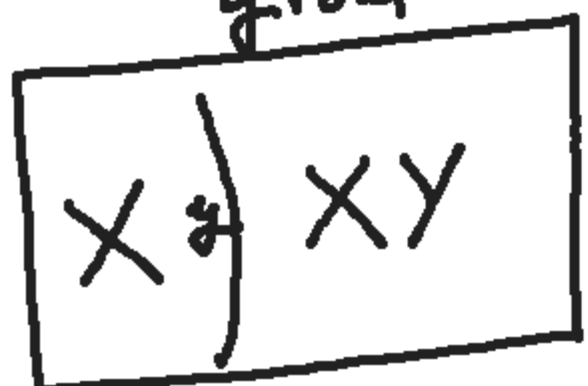
$B^2 = W$

$RG = B$

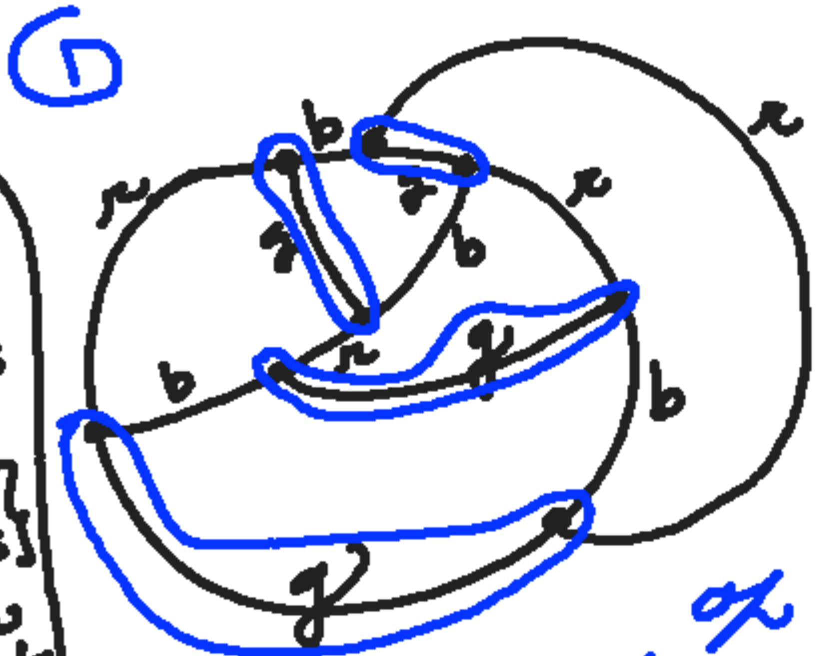
$GB = R$

$BR = G$

group of 4 colors



**
Even PM means all cycles in $G - \{\text{PM edges}\}$ are even (even # of edges).

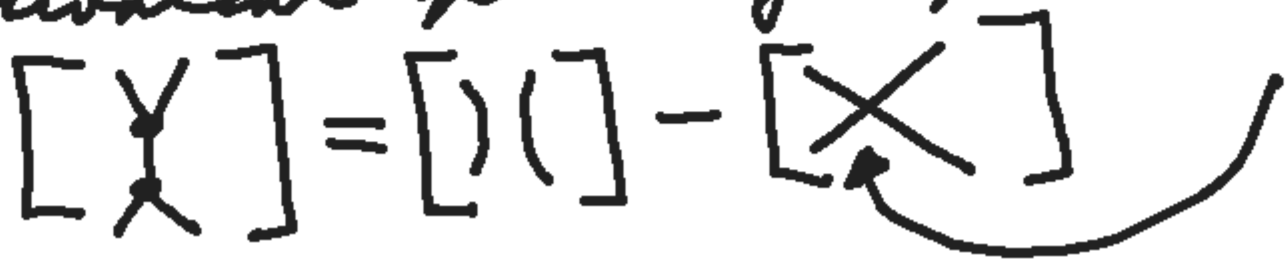


Coloring in 3 colors, the third color selects an even number of disjoint edges that exhaust all the nodes in the graph.

This is an even perfect matching for G. It is not hard (look it up! on Wiki) to prove that every trivalent isthmus free graph has a perfect matching, but hard to show (= 4CT) that planar graphs have even PM's.

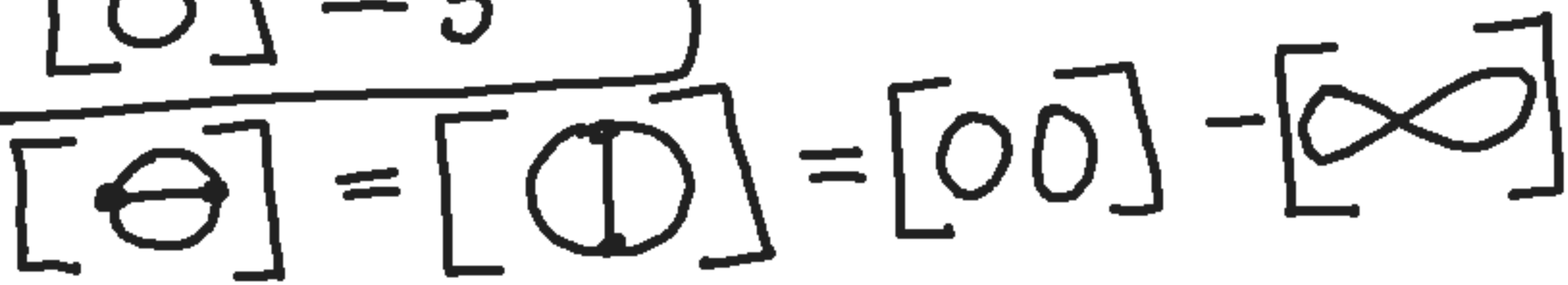
Penrose Formula

Counts 3-colorings of trivalent plane graphs.

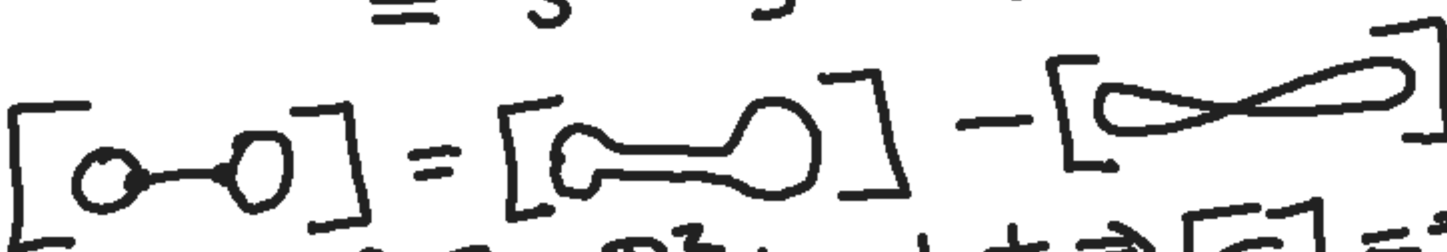


virtual crossing

$[\bigcirc] = 3$



$= 3^2 - 3 = 6$ ✓

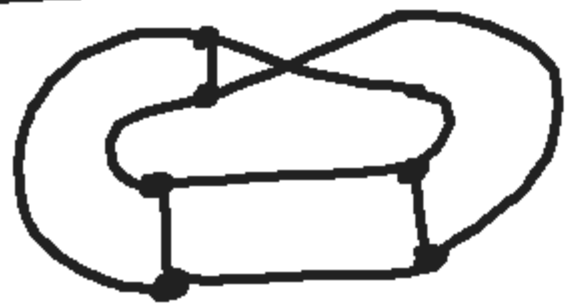


$= 3 - 3 = 0$

If $G \subset \mathbb{R}^2$ trivalent $\Rightarrow [G] = \#$ 3 cols of G.

[G] does not work (always) when G is not planar.

<u>ex</u>.



$G = K_{33}$

Gas Elec Water Problem Graph.

<u>12 colorings</u>

But $[G] = \phi$.

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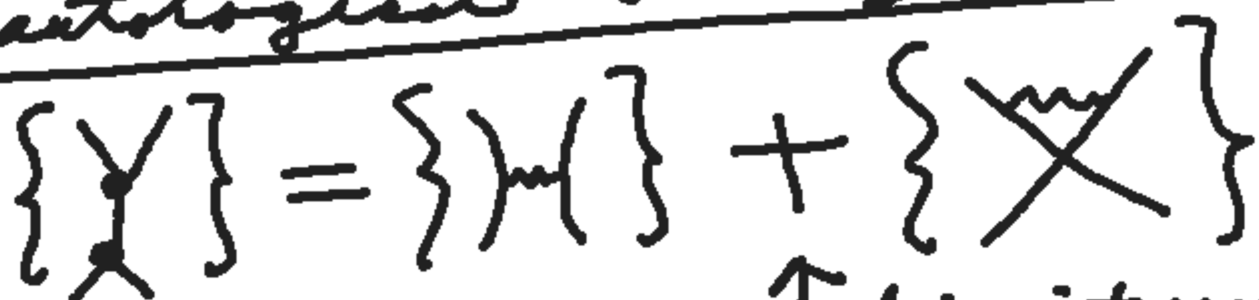
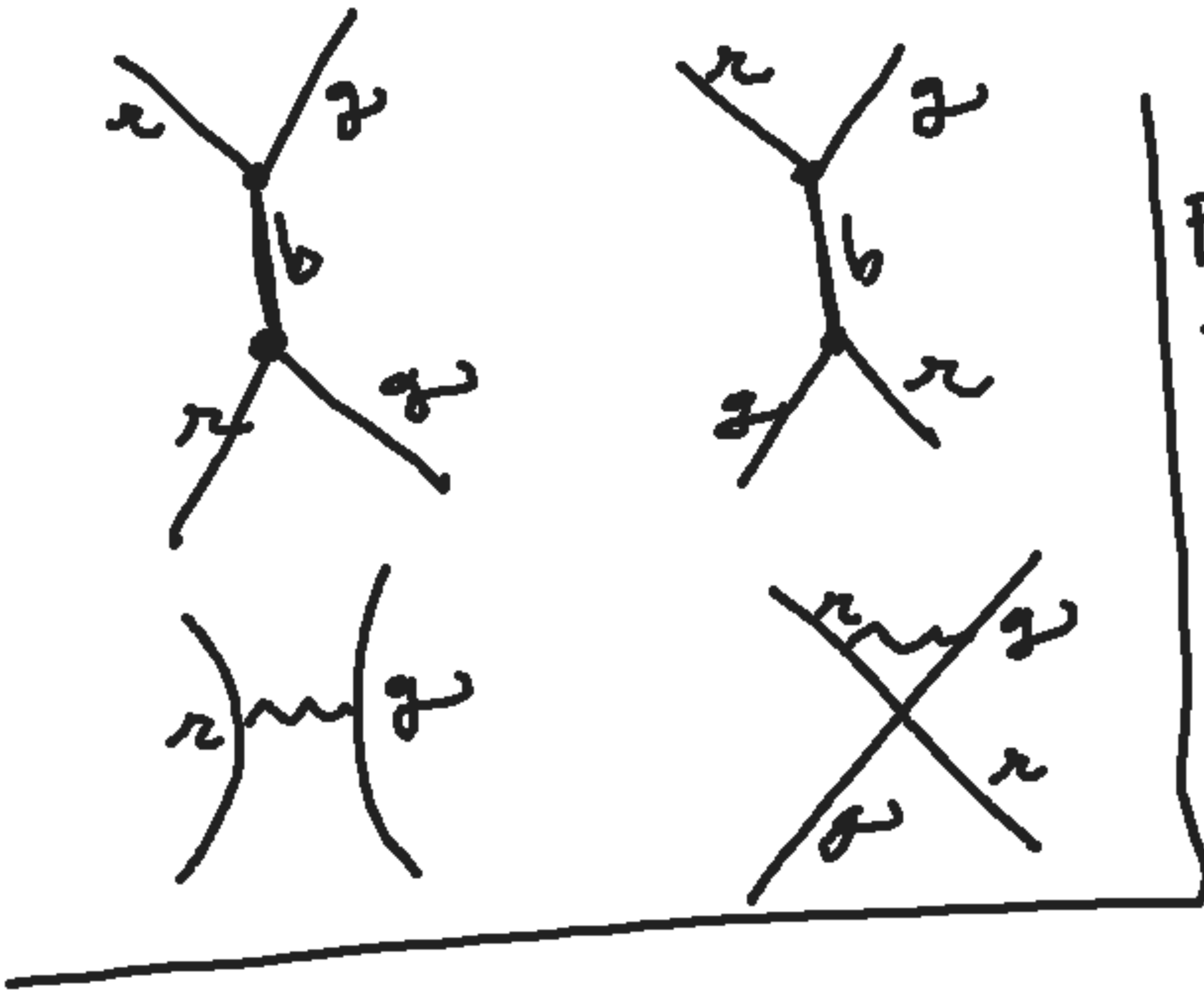
* <u>Tautological Coloring Formula</u>

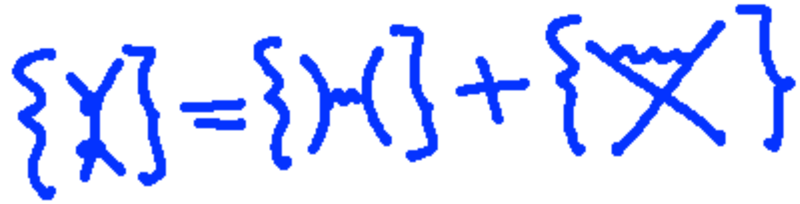


↑ disjoint union.

T colorings <u>set</u>



Proof of the Tautological Identity.

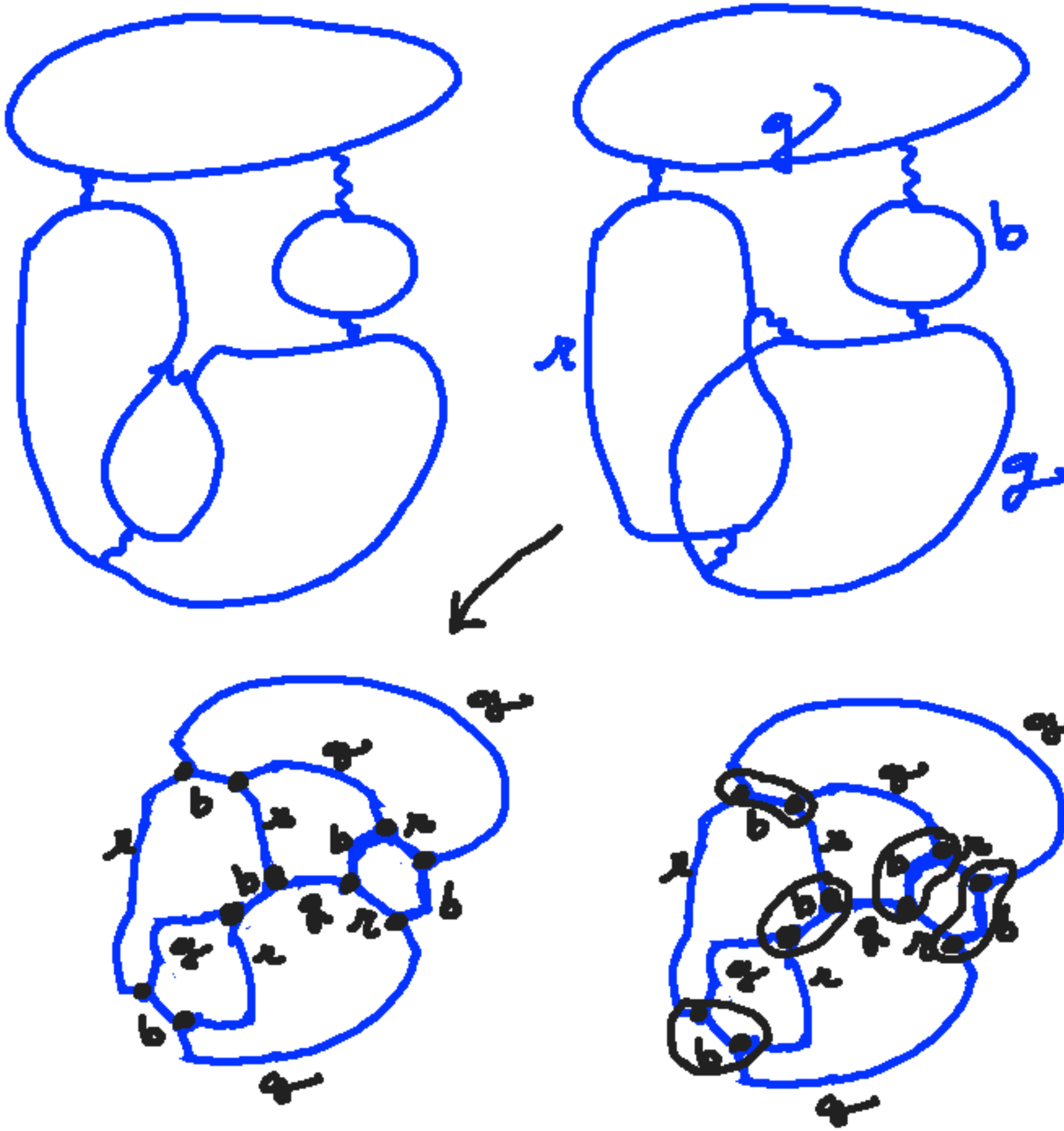


Choose a PM and expand on at each  .

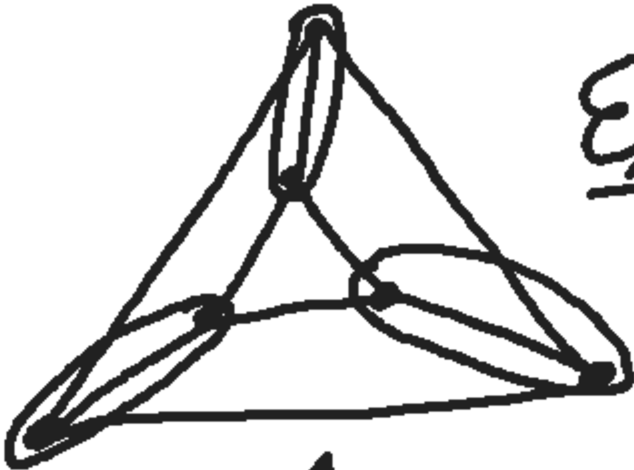
These loops can't be colored with 3 colors so that $x \ne y$.

4CT says some loop state obtained via $\rightarrow$  is colorable.

Exercise: Find the colorable loop state.



Note that this is an even P.M.

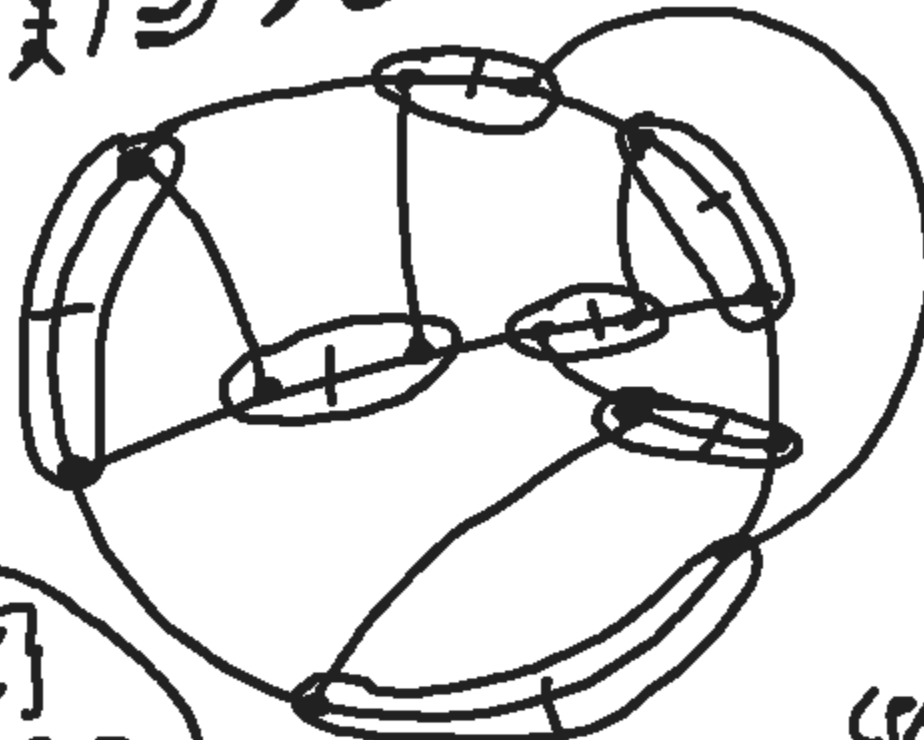


Exercise. Work out the colorings of

G

starting with the PM

Map Graphs with PM $\xrightarrow{\mathbb{K}}$ Virtual Links

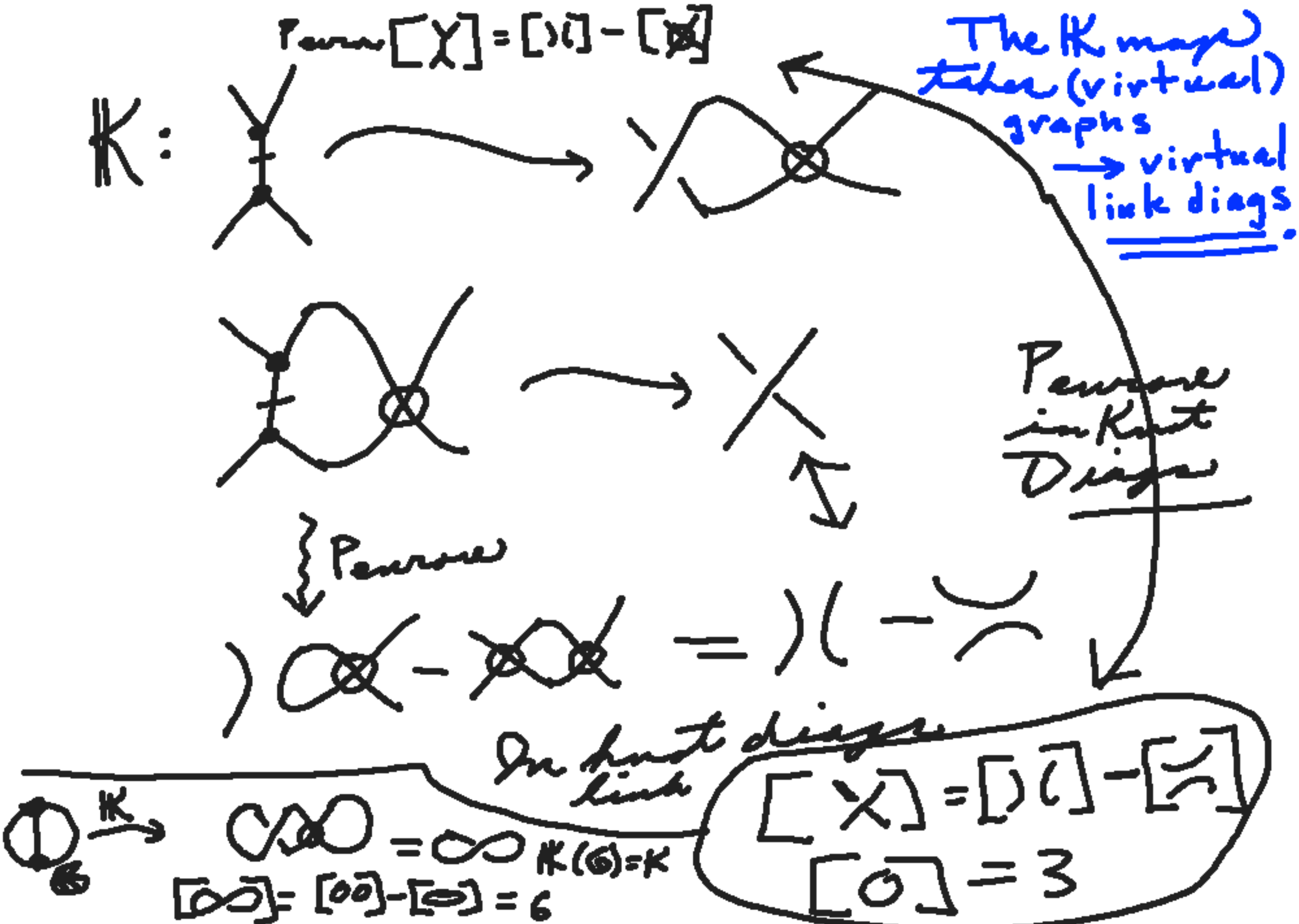


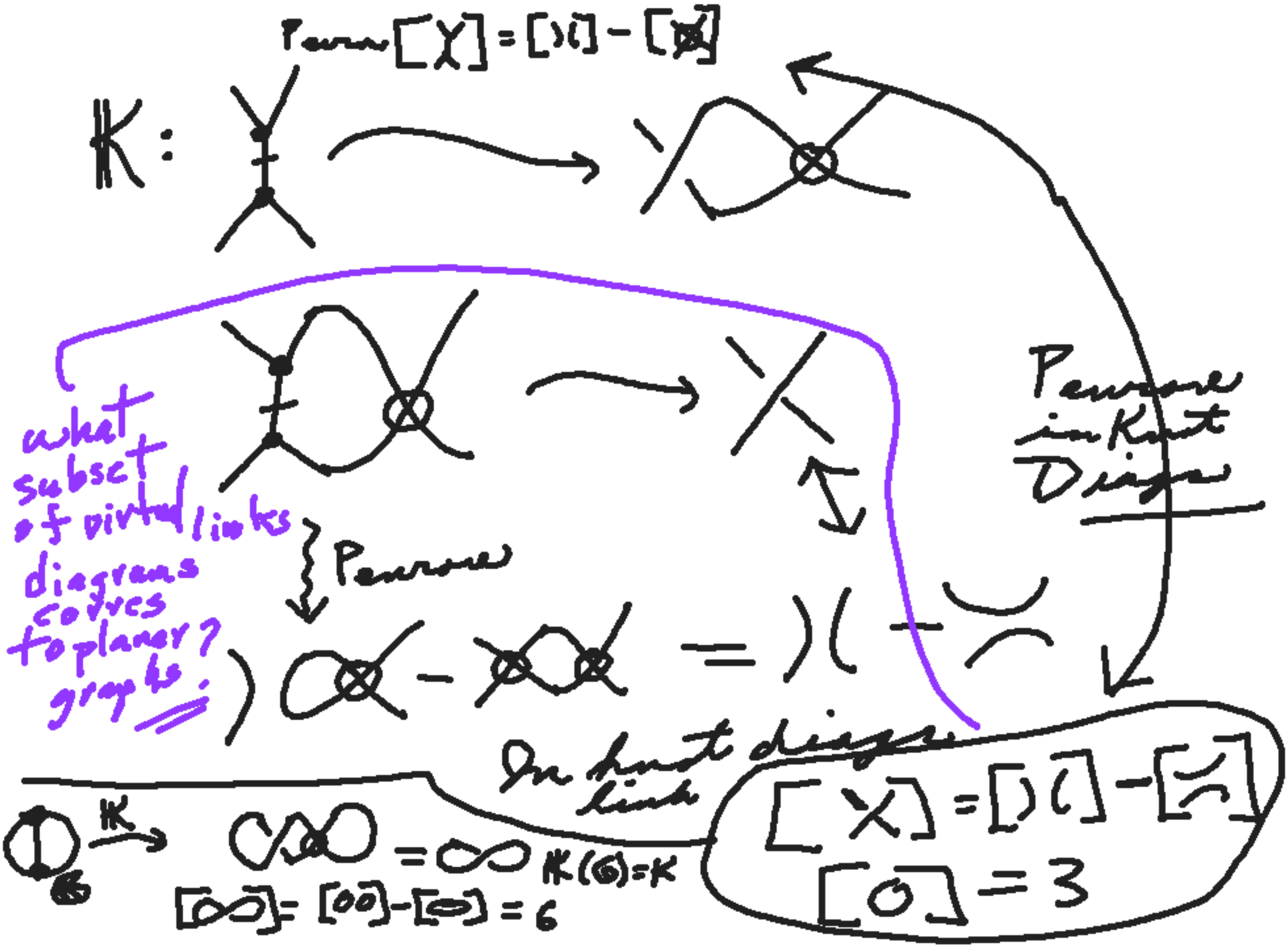
$\{\times\} = \{)(\} + \{\times\}$

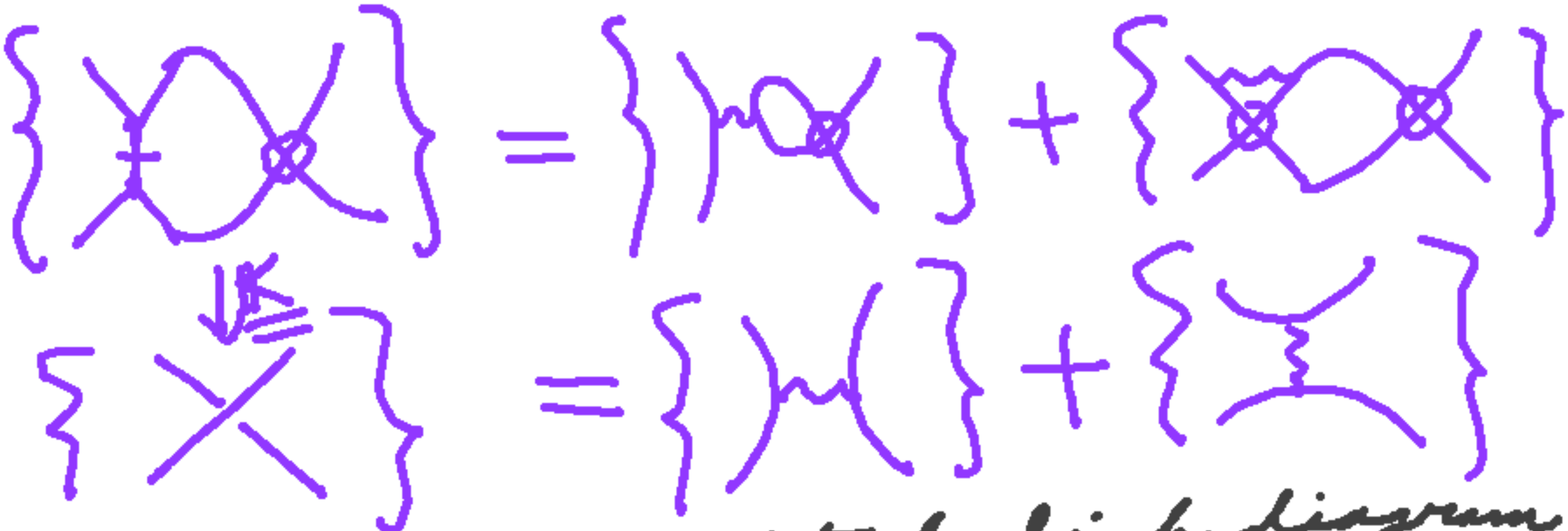
Perfect Matching (PM)

good subset of edges to smooth

Thm (Tutte/Petersen) Every trivalent (no isthmus) graph has a PM.

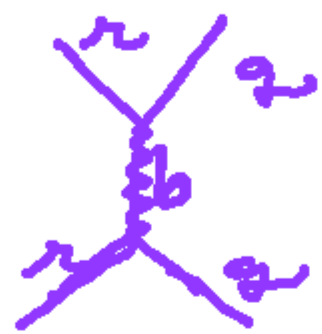

Penrose
K:
The K map takes (virtual) graphs → virtual link diags.
Penrose in Knot Diags
Penrose
In knot diags. link
= 3
= 6


Penrose[Y] = [)(] − [⊗]
K :
Penrose in Knot Diagrams
what subset of virtual links diagrams covers to planar graphs?
Penrose
=
In knot diags link
[X] = [)(] − [~]
[O] = 3
= ∞
K(θ)=K
[∞] = [OO] − [⊖] = 6



This is the virtual link diagram version of the tautological coloring expansion.

We use 3 colors



Full Coloring of a trivalent
Plane Graph $\longleftrightarrow$ many choices
of PM + binary bracket poly.

top invar

BBP

categorified

Lee Homology
via Rushworth.

Next:

1) More about
$\mathbb{K}$ map.

2) Rushworth
Doubled
Khovanov
Homology

$\neq 0$ by changing
module etc.