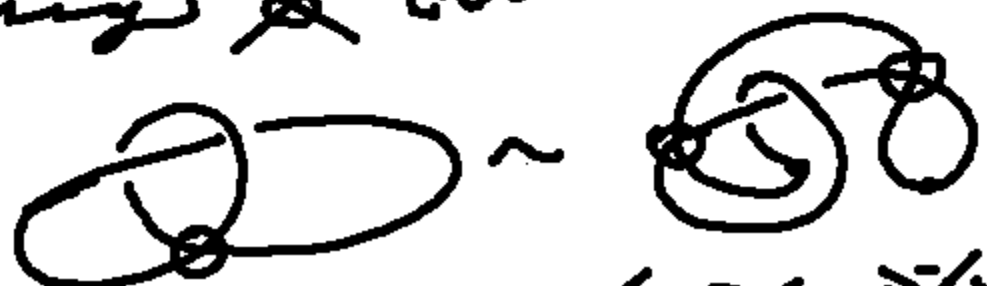
Virtual Knots + Khovanov Homology

LK

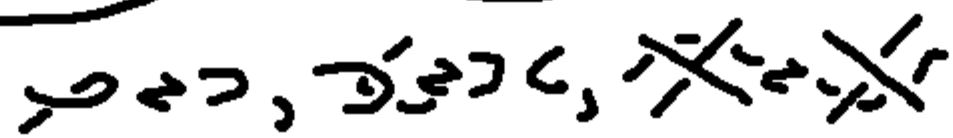
1. Virtual Knots

Classical diag knot theory & extend it with a new crossing (virtual crossing)

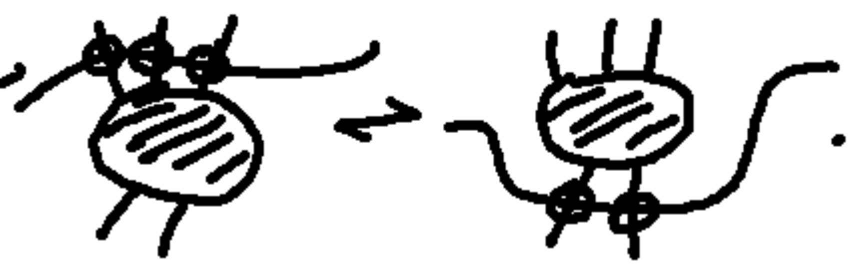
Diagrams



1. Reid Moves:



2. Detour Moves

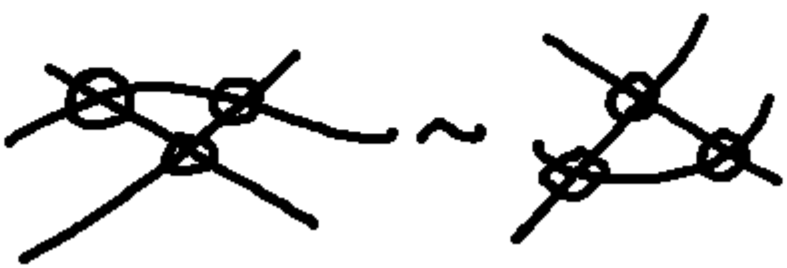


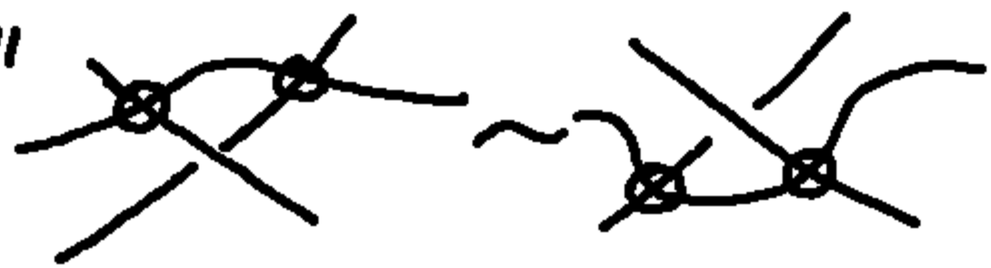
Small Moves that generate Detour Moves :

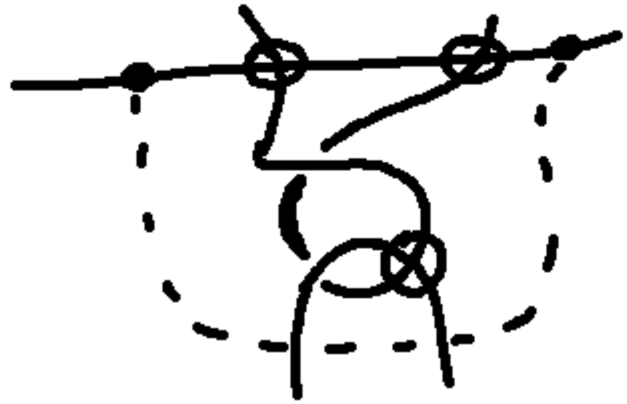
I'. 

(part of Reid Moves that we allow planar isotopy)

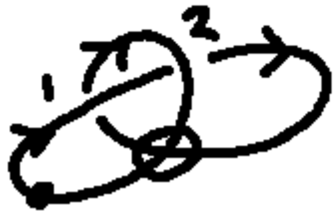
II'. 

III'. 

III''. 



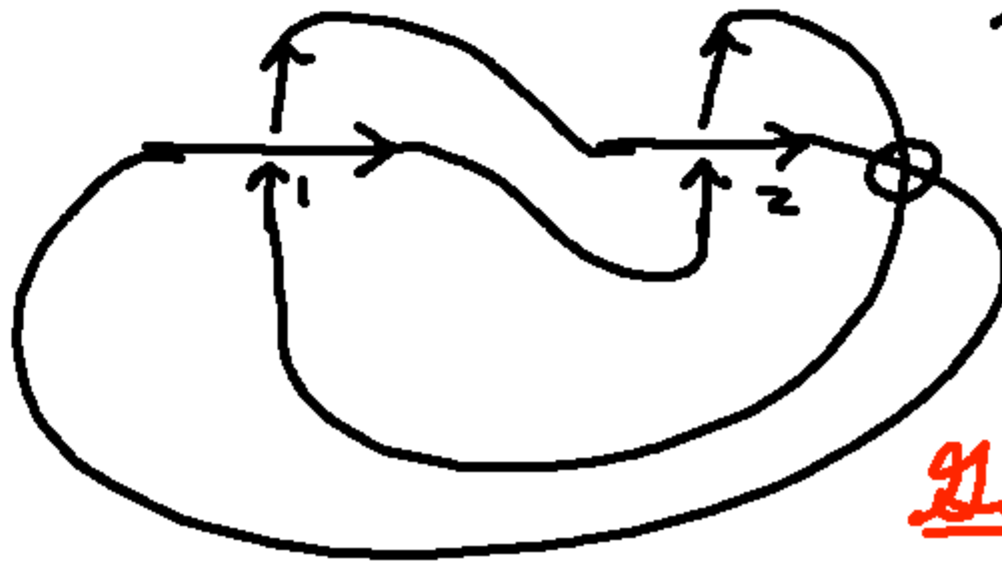
Can factorize any detour move into a composition of small moves.



Gauss Code

O1+ U2+ U1+ O2+

no virtual crossings

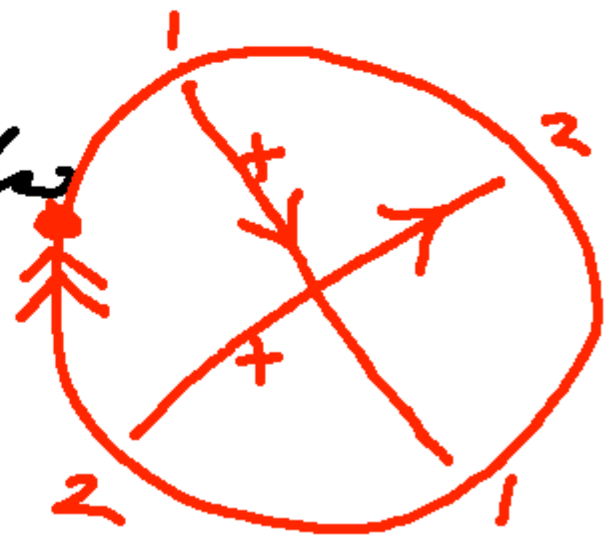


$$VKT = \left(\frac{\mathcal{GC}}{RM}\right)$$

analogy

$$VKT/CKT \equiv \frac{\text{All Graphs}}{\text{Planar Graphs}}$$

Gauss Diagram



Bare Gauss Code

1 2 1 2

1 and 2 are odd crossings.

K

BGC: 1212

Both crossings are odd.

Odd Writhe of K = J(K)

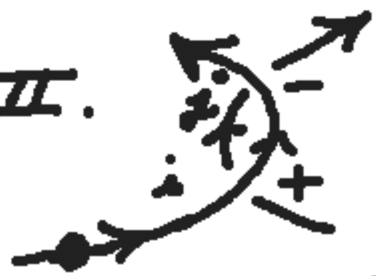
$$J(K) = \sum_{c \in \text{Odd crossings}(K)} \mathrm{sgn}(c)$$

$$\boxed{\begin{array}{l} \mathrm{sgn}(\;) = +1 \\ \mathrm{sgn}(\;) = -1 \end{array}}$$

Claim: J(K) is a VKT invariant.

I.  $\cdots i\, i \cdots$ even ✓

III. Exercise!

II.  $\cdots i\, j\, \overbrace{\alpha}^{1+|\alpha|}\, i\, j \cdots$ 

Claim: i & j both odd or both even.

If both counted, then $\boxed{+1 + (-1) = 0}$

$K^* =$ reverse all crossings in K

K

K^*

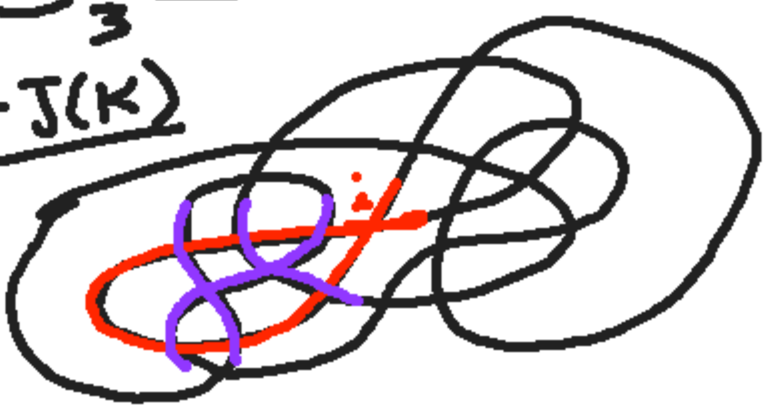
$J(K) = 2$, $J(K^*) = -2$

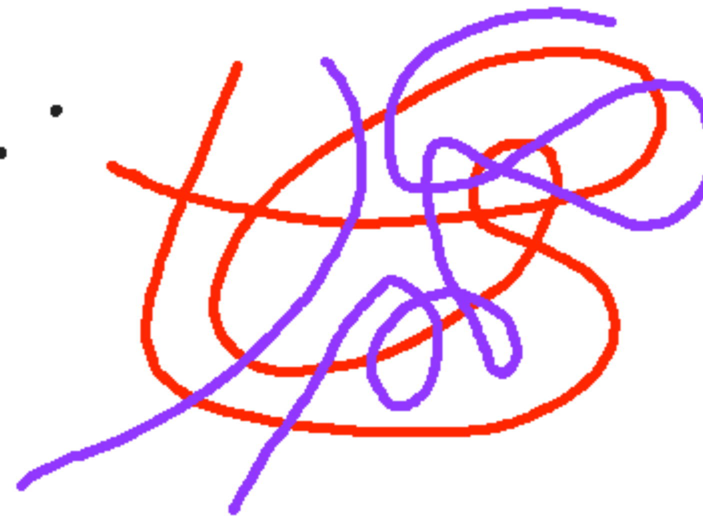
$\Rightarrow K$ non-trivial & non-classical

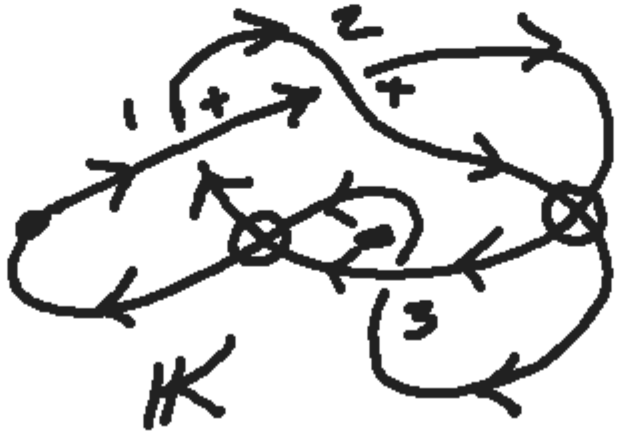
1. $J(K)$ invar of VK.
2. $J(K \text{ classical}) = 0$ because classical knot diagrams do not have any odd crossings

123123.

3. $J(K^*) = -J(K)$





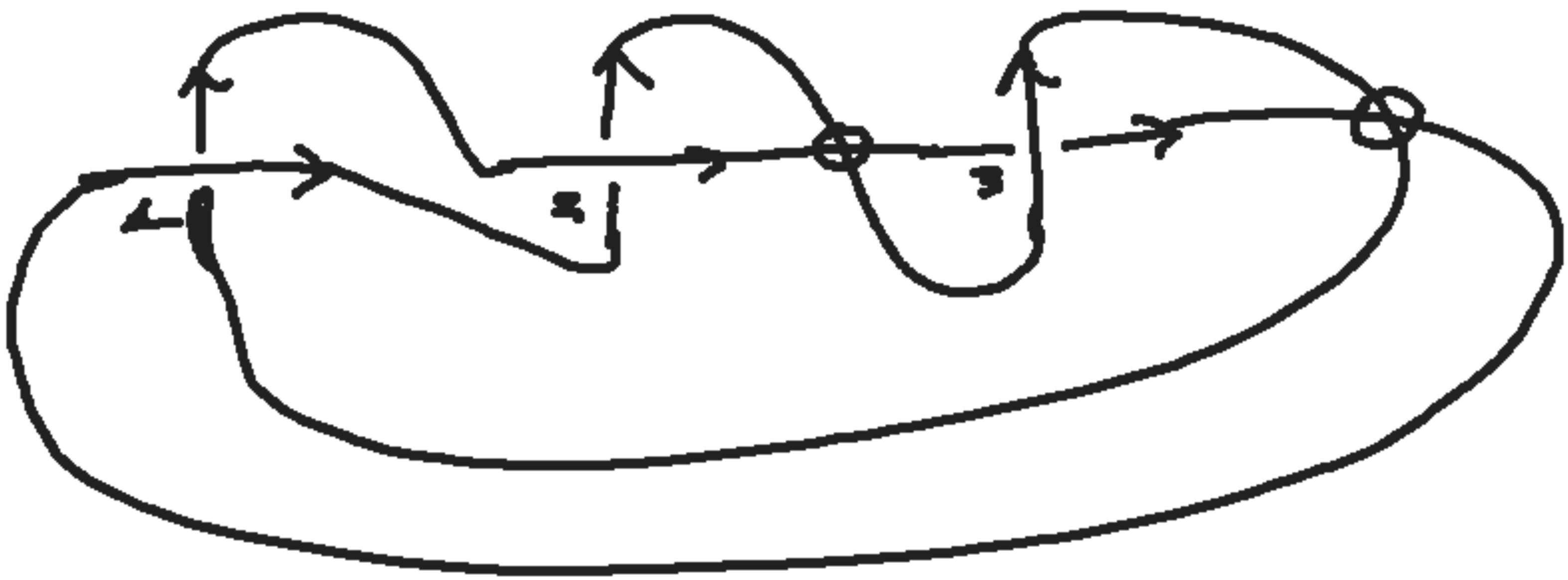


BG: 123123

all crossings even.

$J(K) = 0$

O1+ U2+ O3- U1+ O2+ U3-



Knotoids (V. Turaev)

$\boxed{1212}$

odd crossing

Knot diag with endpoints.

endpoint can be in diff regions.

Use RM's + do not allow

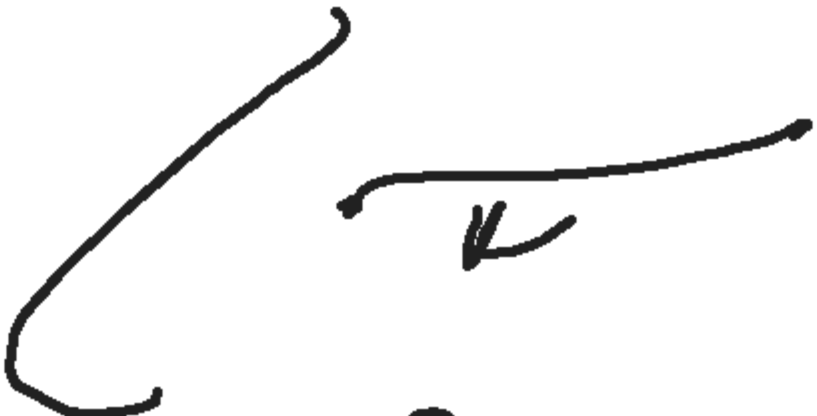
or

1-1 tangle

Knot Type Knotoid

Can define $J(K)$ same way.

$J(k) = 2$

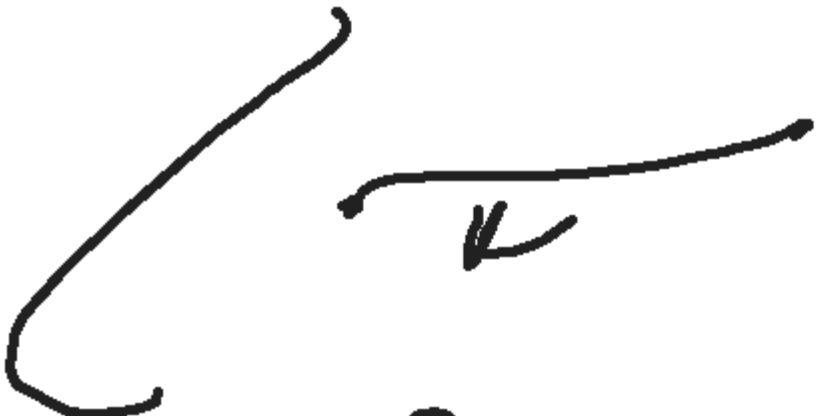


$J(k) = 0$

So k is nontrivial & $k \neq k^*$

12 • 12

$J(k) = 2$



$J(k) = 0$

So k is nontrivial

& $k \neq k^*$

12●12

$\Theta(k) \longleftarrow$  k

$\cap$

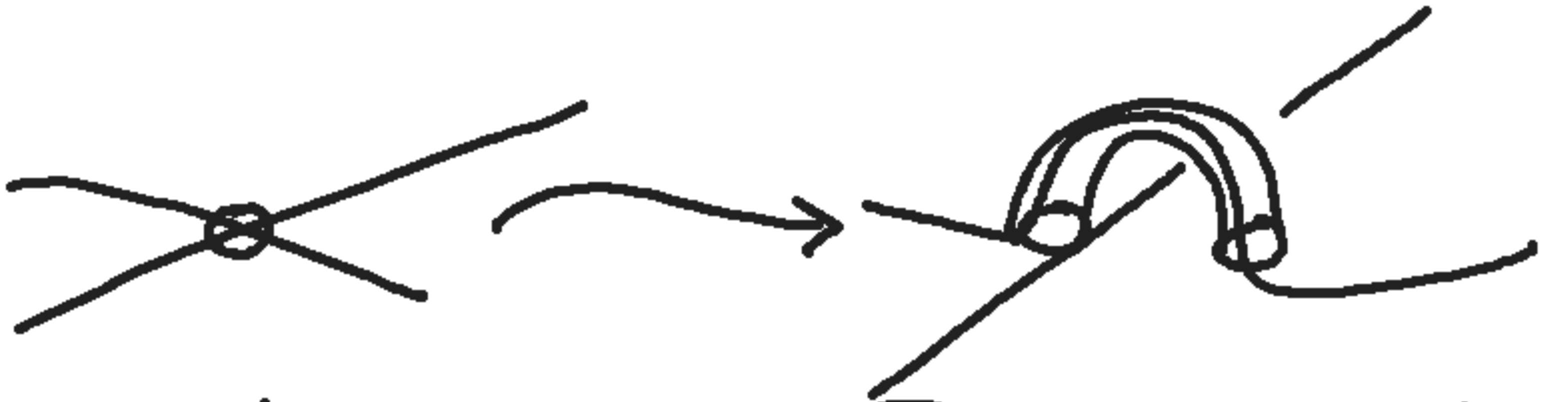
$\mathbb{R}^3$

Turaev observed that $\Theta(k)$ classifies k

We do have virtual knotoids.

e.g. 

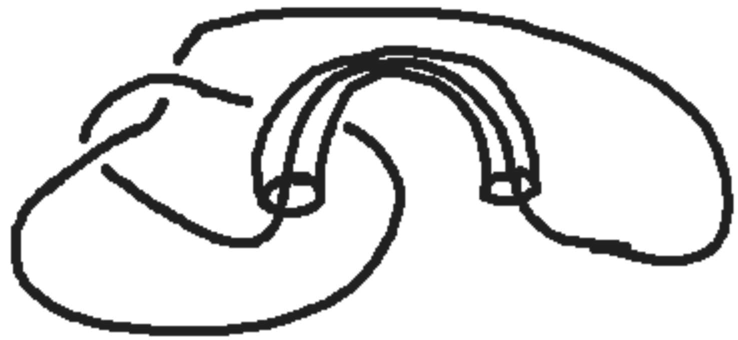
I do not know at this time a generalization of the $\Theta(k)$!



$K \longrightarrow D_K \subset \text{Surface} \times I$

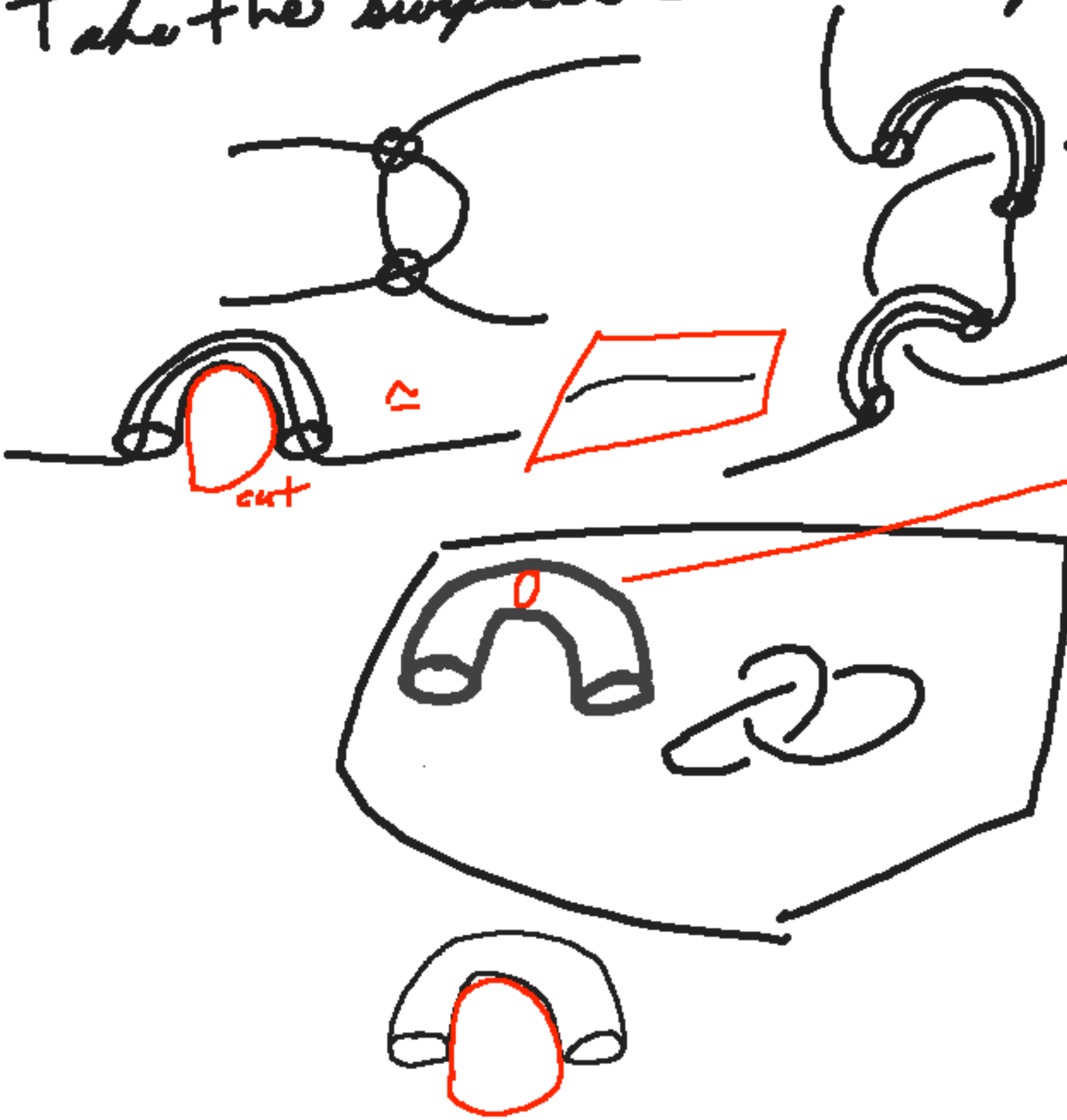
$D : VK \longrightarrow$ Diag on Surface or K diagram in a surface.

K diag on S^2



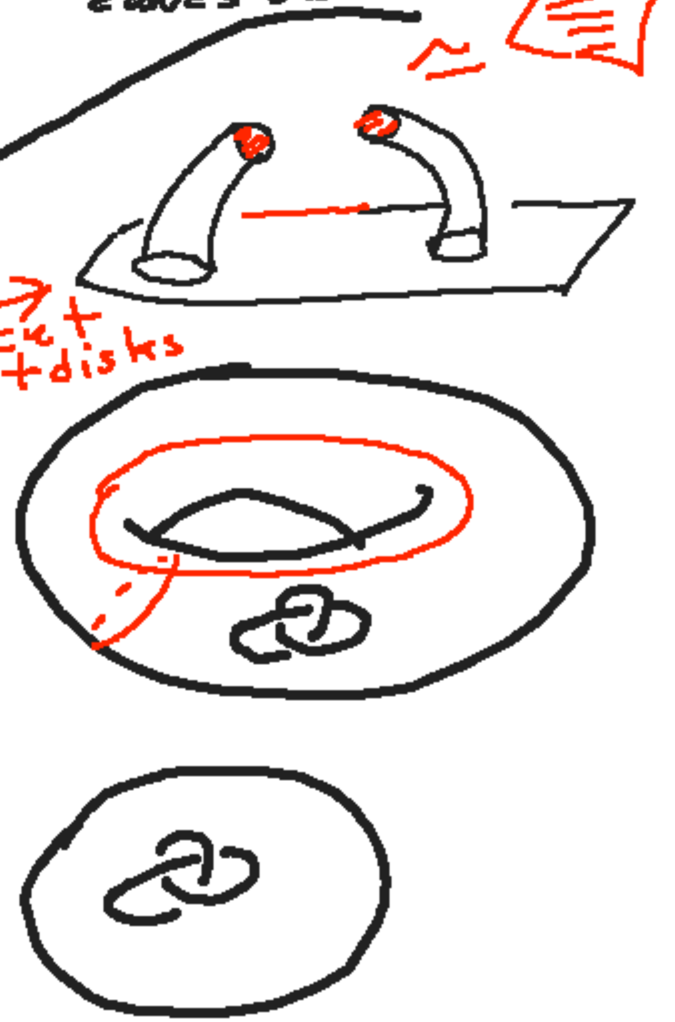
D_K diagram on a torus.

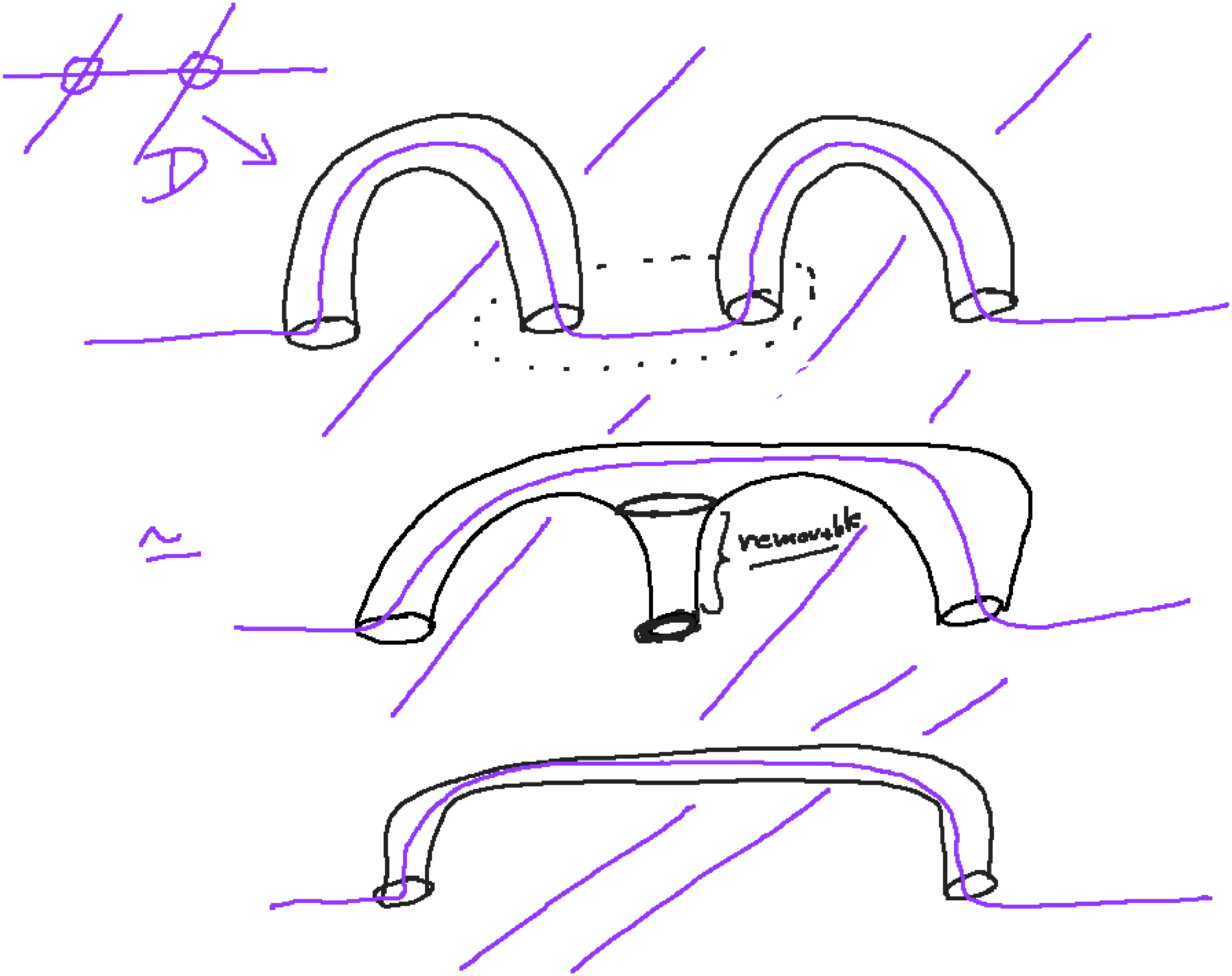
Take the surface ⊃ diagram up to



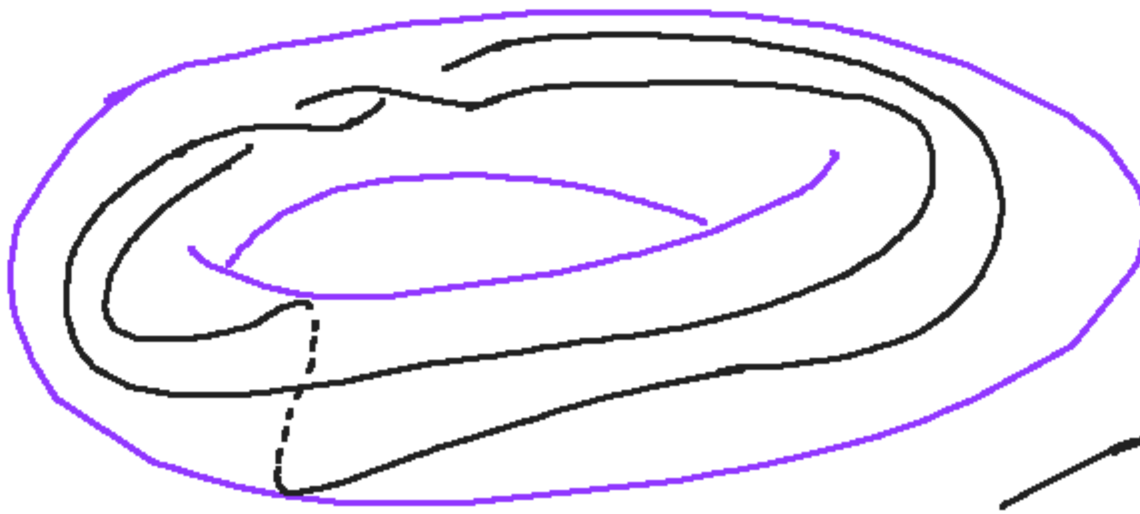
Handle Stabilization

± handles by using surgery curves outside the knot!




D
removable

$D : VK \longrightarrow SK$ (knot diagrams in surfaces)
modulo stabil.

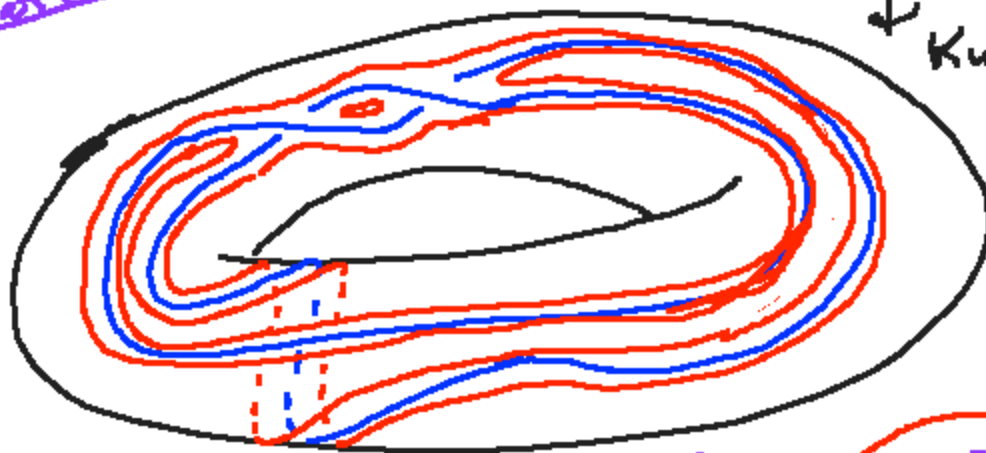


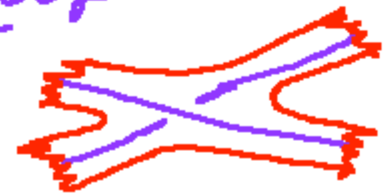
Claim

VKT $\leftrightarrow$ (Knots in $S_g \times I$)/handle stabilization

good diagrammatic formulation

What about other 3-mflds + Knots in them?





add disks to bdry circles