Penrose Formula, Perfect Matching Polynomials, Virtual Knot Theory and Khovanov Homology

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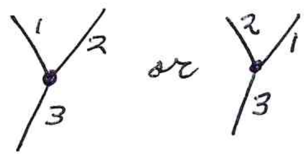
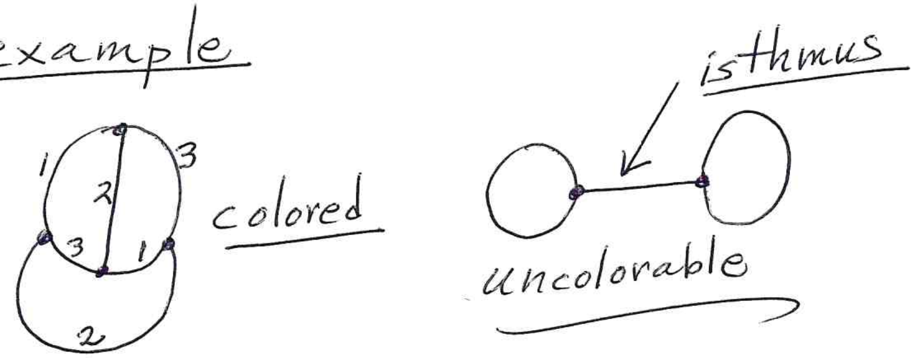
(joint work with Scott Baldridge, and William Rushworth)

1°. Coloring Trivalent Graphs



Rule: 3 distinct edge colors at each node.

example

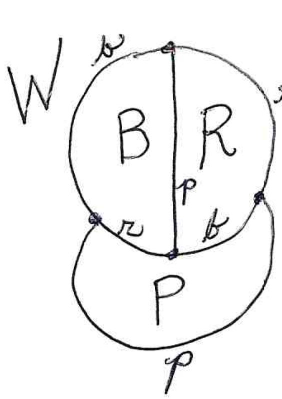


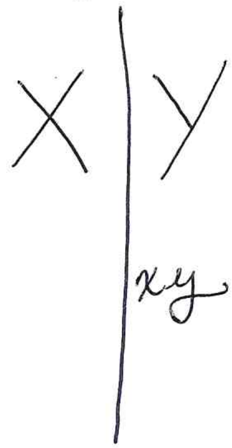
Theorem (Peter Guthrie Tait). 3-Colorability of isthmus-free plane trivalent graphs is equivalent to the Four Color Theorem (4CT).

example

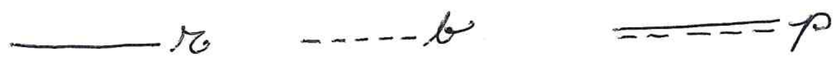
(a) Let colors be r (red), b (blue) and p (purple). Take four colors $\{W, R, B, P\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with

$$W = \text{Identity}, \quad R^2 = B^2 = P^2 = W$$
$$RB = P, \quad BP = R, \quad PR = B.$$



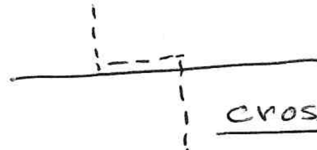


(b)

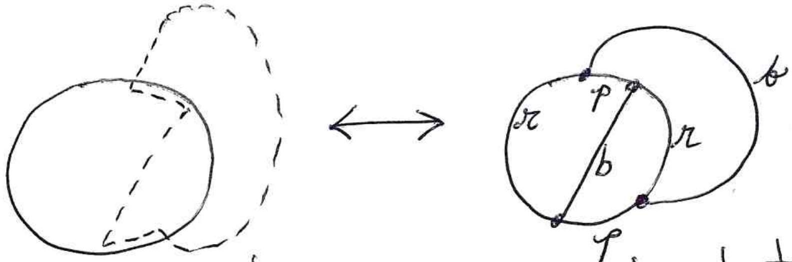


Let red and blue curves interact.

bounce

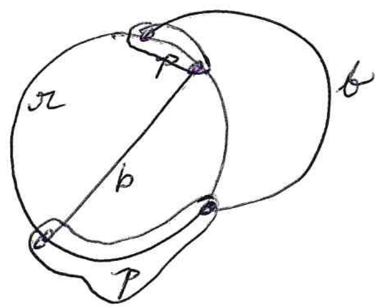


cross



a formation (G. Spencer-Brown) $\qquad$ colored trivalent graph

Note that every colored trivalent graph selects an <u>even perfect matching</u> (choose the purple edges).



(Γ, M)

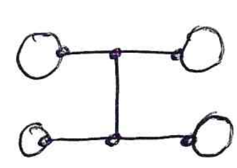
$(\Gamma - M)$

When $(\Gamma - M)$ is a collection of even cycles, we say M is an even perfect matching.

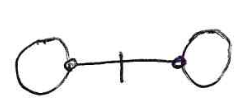
$\boxed{\Gamma \text{ is 3-colorable}} \Longleftrightarrow \boxed{\Gamma \text{ has an } \underline{\text{even}} \text{ PM.}}$

<u>Easier Than 4CT</u>: Every isthmus-free trivalent graph has a PM.

<u>example</u>

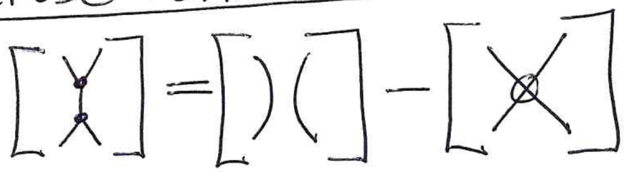


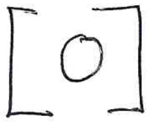
} no PM.



} only an odd PM

2° Penrose Formula

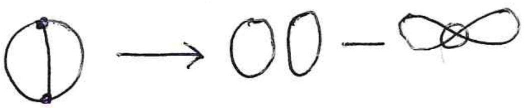


 $= 3$

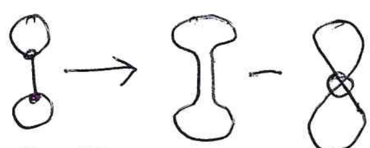
$\Rightarrow$ [G] = # of 3-colorings of G when $G \hookrightarrow \mathbb{R}^2$.

(From: R. Penrose, On Applications of Negative Dimensional Tensors, in "Combinatorial Mathematics and Its Applications", ed by D. J. Welsh (Academic Press 1971))

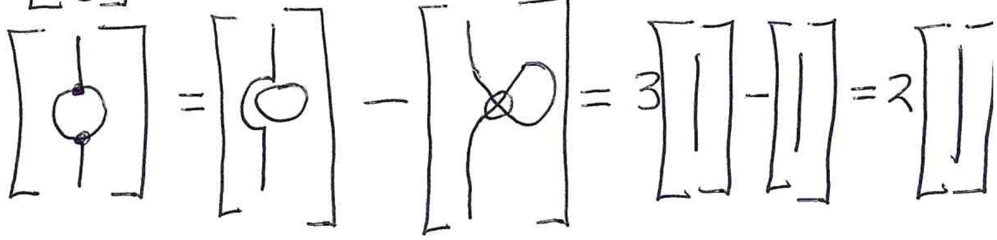
example

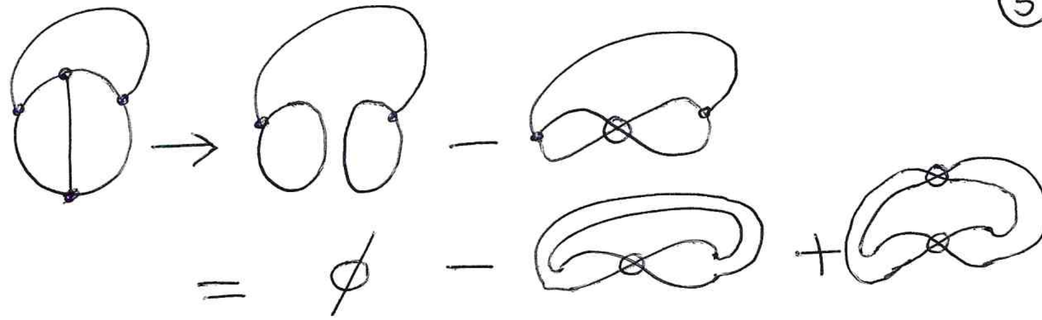
(a)  $= 3^2 - 3 = 6$

 $= 6$.

(b)  $= 3 - 3 = \emptyset$

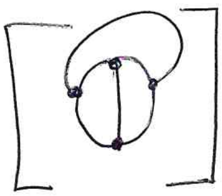
 $= \emptyset$.

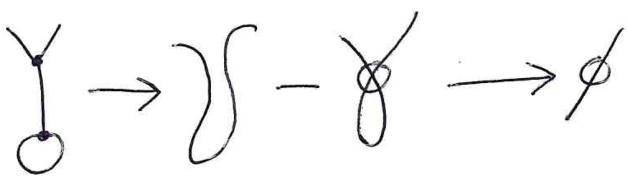
(c) 

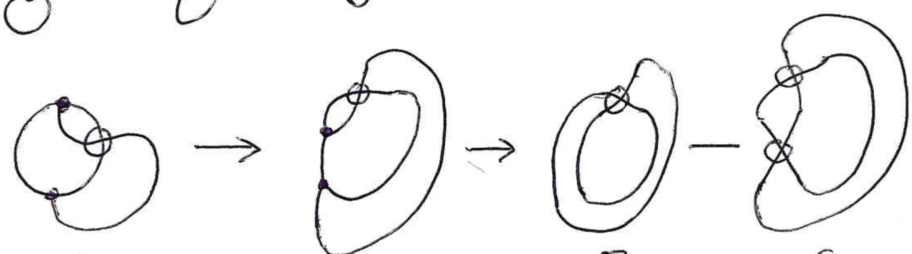
(d) 

$= -3 + OO$

$= -3 + 3^2$

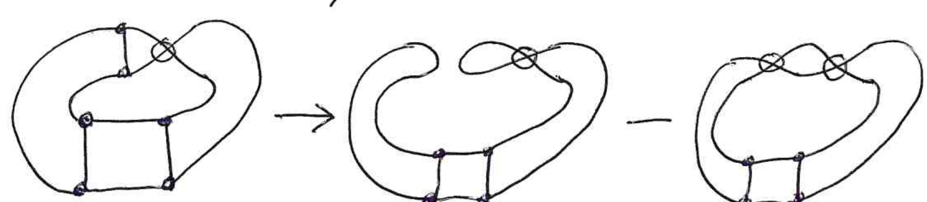
 $= 6.$

(e) 

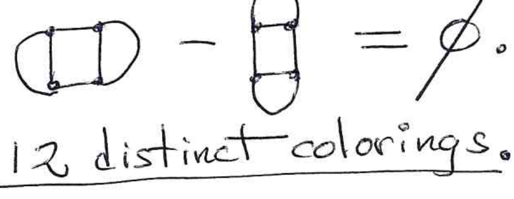
(f) 

$\longrightarrow 3 - 3^2 = -6$

not a plane diagram

(g) 

$K_{3,3}$
non-planar

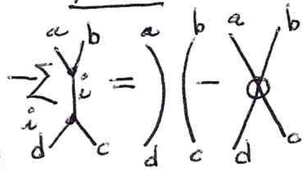
$\longrightarrow$ 

Note: $K_{3,3}$ has 12 distinct colorings.

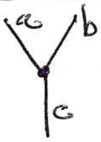
Proof of Penrose Formula

$$[\Gamma] \underset{def}{=\!=} \sum_{\sigma \in \mathrm{Colorings}(\Gamma)} \prod (\pm\sqrt{-1}) = \mathcal{C}(\sigma)$$

And



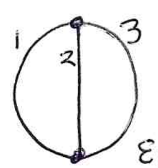
Epsilon Identity

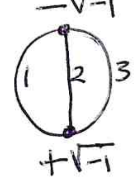
$\pm\sqrt{-1} = \sqrt{-1}$  $= \sqrt{-1}\,\varepsilon_{abc}$

ε_{abc}: $\varepsilon_{123} = +1$, $\varepsilon_{213} = -1$, $\varepsilon_{113} = 0$

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312}$

$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321}$

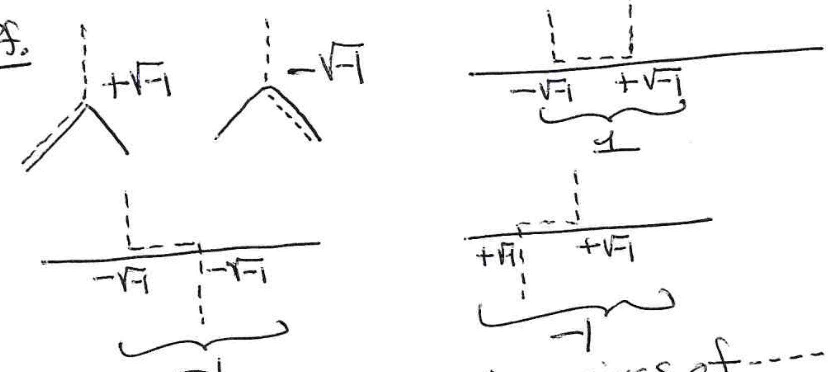
e.g. σ 

$\Rightarrow$ 

$\Rightarrow (-\sqrt{-1})(+\sqrt{-1}) = 1$ count for σ

Claim: $\mathcal{C}(\sigma) = 1$ for each σ.

Pf.



$\therefore\ \mathcal{C}(\sigma) = (-1)^{\#(\text{crossings of ---- and —— in } \sigma)}$ even by Jordan Curve Theorem.

$\mathcal{C}(\sigma) = 1.$ //

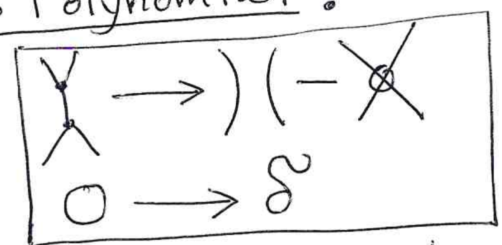
$$-\sum_i \varepsilon_{abi}\varepsilon_{icd} = \delta^a_d\delta^b_c - \delta^a_c\delta^b_d$$

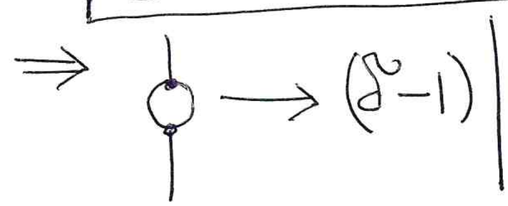
(The fact that $[K_{3,3}] = \emptyset$ proves that $K_{3,3}$ is not planar.)

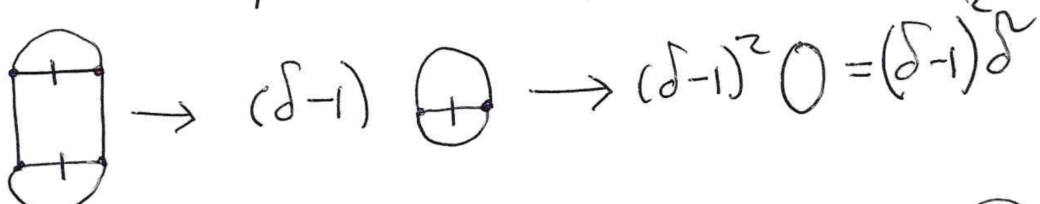
Penrose formula can be modified to count colorings for non-planar graphs. See (LHK, ILJ. Math. Vol. 60 No. 1, (2016) pp 251-271).

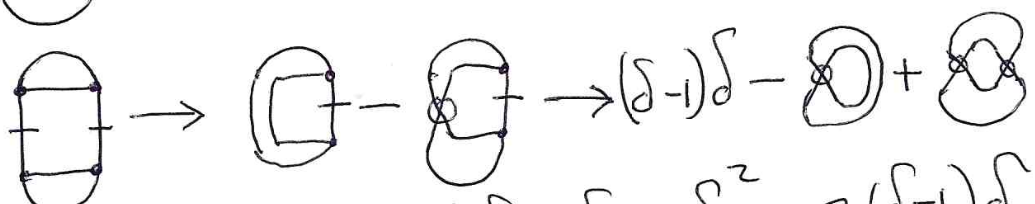
3° Penrose Polynomial?

Try









$$\longrightarrow (\delta-1)\delta - \delta + \delta^2 = 2(\delta-1)\delta$$

While $(3-1)^2 \cdot 3 = 2 \cdot (3-1) \cdot 3$,

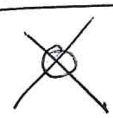
$$(\delta-1)^2\delta \neq 2(\delta-1)\delta$$

and $= \Leftrightarrow \delta - 1 = 2 \Leftrightarrow \delta = 3$.

Such an extension produces a <u>perfect matching polynomial</u>.

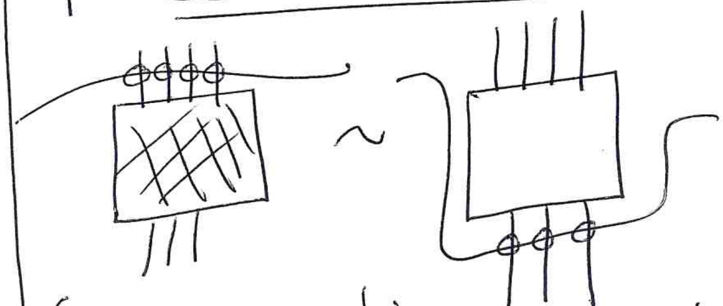
4. Virtuality of Graphs, Knots, Links

Remark. All our calculations involving



use virtual graph equivalence

planar isotopy (preserving cyclic orders at graphical vertices) + detour moves



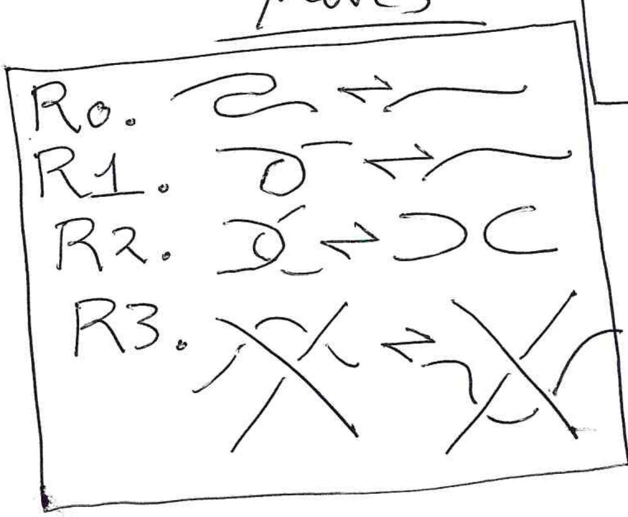
(a consecutive sequence of virtual crossings can be deleted and replaced elsewhere by another consecutive sequence of virtual crossings).

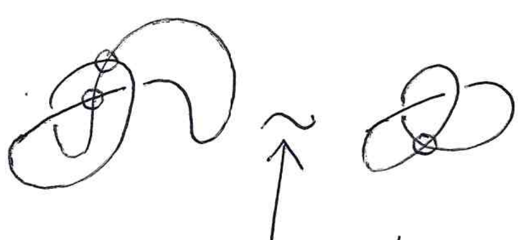
Virtual Knot Theory

Virtual Knot, Link Diagrams with

Reidemeister Moves + Detour Moves

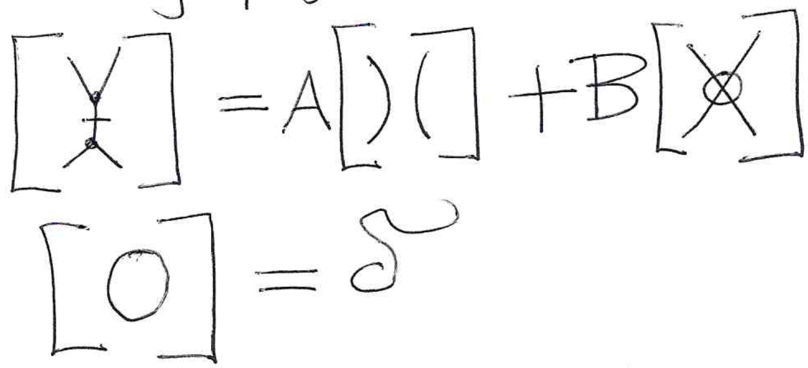
Reidemeister Moves





Detour Move

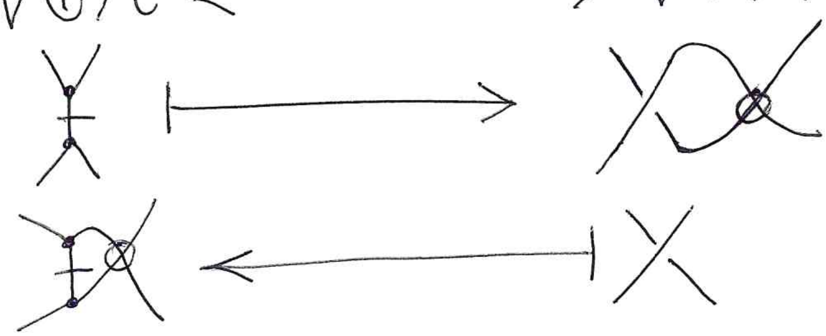
Let (G, M) be a (virtual) plane diagramatic trivalent graph with given perfect matching M.
Define a 3-variable perfect matching polynomial $[G, M]$ via:

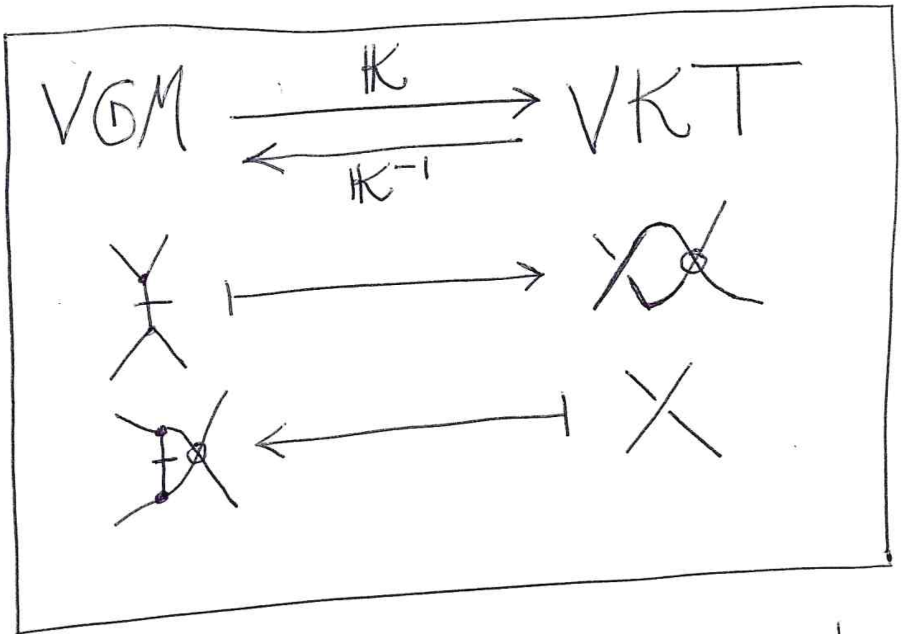


This can be used to discriminate many different perfect matchings on G.

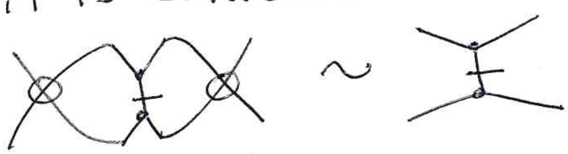
VGM = virtual graphs with perfect matching.
VKT = virtual knot and link diagrams.

$$VGM \xleftrightarrow{\mathbb{K}} VKT$$

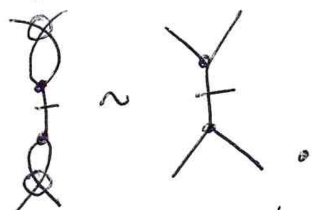


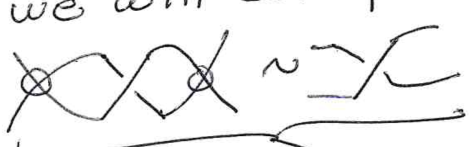
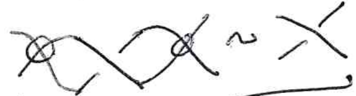


For some purposes, as we shall see, it is convenient to assume that



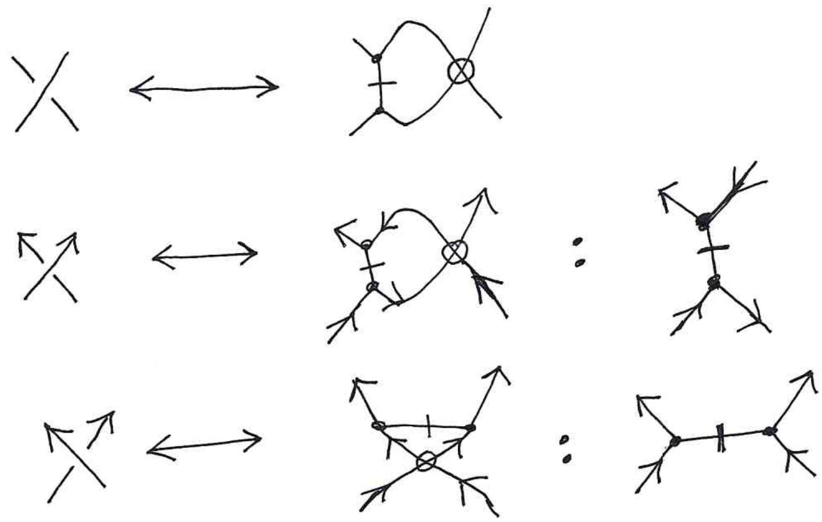
and that



I will do that here, but take note that we will correspondingly demand that  and 

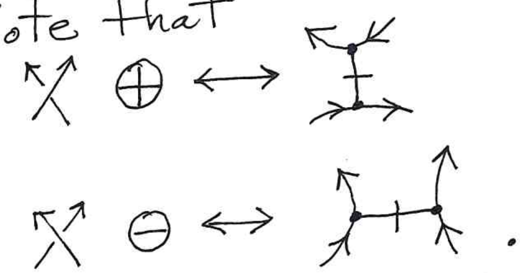
Z-equivalence

Orientation

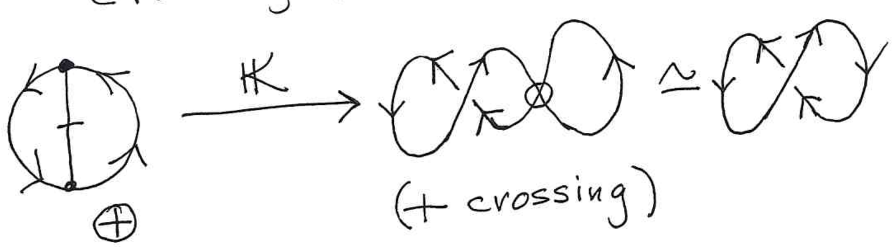


These two cases comprise the allowed orientations of the edges in $\mathbb{G}-M$ for $(\mathbb{G}, M)$ a perfect matching.

Note that



(ignoring the requisit virtual crossings)

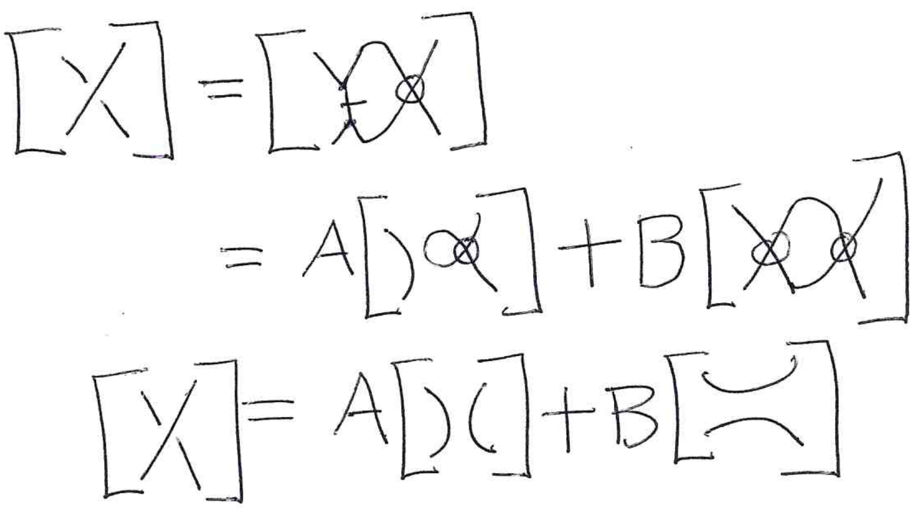


⑨

Define for $K \in VKT$,

$$[K] = [\mathbb{K}^{-1}(K)]$$

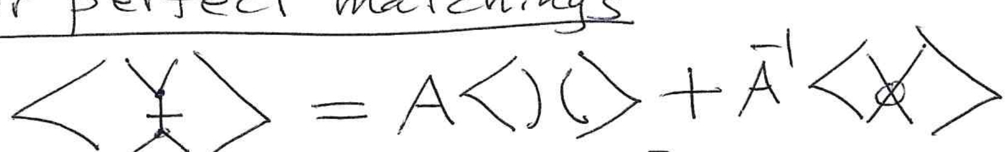
So:



and $[O] = d$.

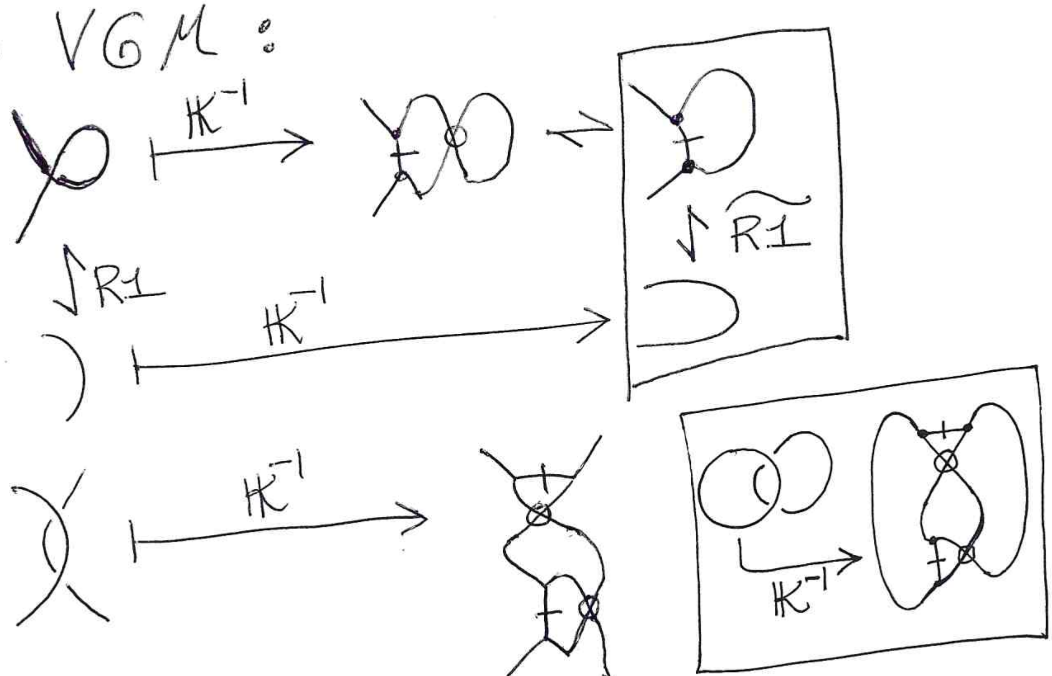
Thus the 3-variable PM poly on trivalent graphs transfers to the well-known 3-variable bracket on virtual link diagrams.

Taking $B = A^{-1}$ and $d = -A^2 - A^{-2}$, we then have a bracket polynomial for perfect matchings

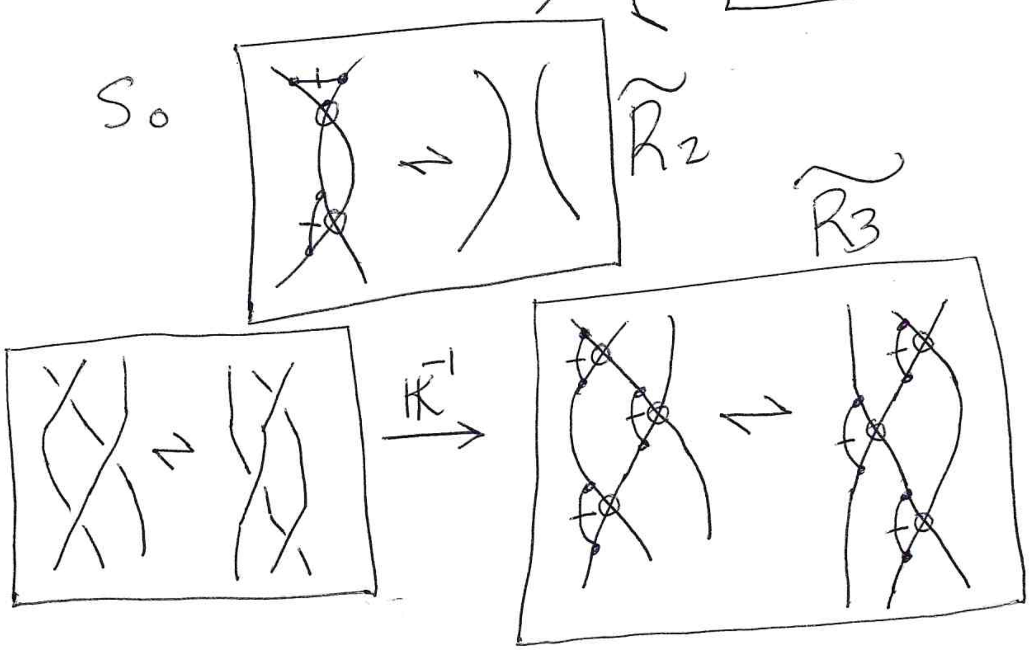


$$\langle O \rangle = -A^2 - A^{-2}.$$

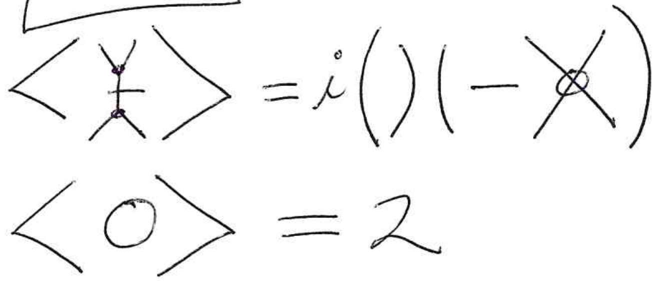
and correspondingly a Jones polynomial that is invariant under all 3 R moves. Perfect matching polynomials have topological properties + we can transfer the R-moves over to VGM:



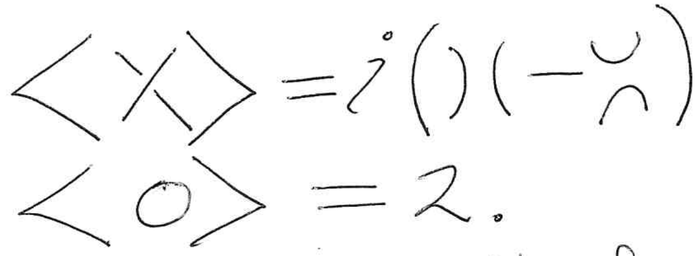
So



At $\boxed{A=i}$ we have



and



arXiv: 1812.10346 and 1810.07302

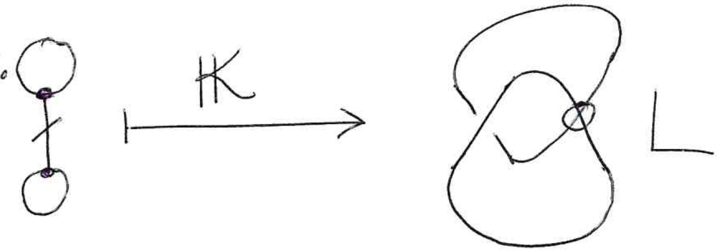
Transferring a result of Scott Baldridge about the perfect matching polynomial, we find

$\langle K \rangle \neq 0 \Leftrightarrow$ every link component of K meets K in an even number of virtual crossings

$\langle \Gamma \rangle_M \neq 0 \Leftrightarrow$ the perfect matching M on Γ is even.

When (Γ, M) is an even PM then $|\langle \Gamma, M \rangle| = 2^{\#\text{cycles}(\Gamma - M)}$.

example.

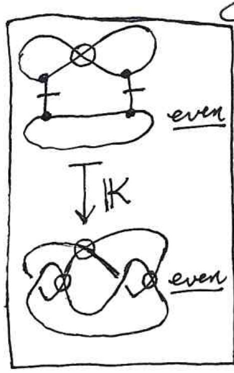


Compute $\langle L \rangle$ at $A = i$:

$\langle L \rangle = i \langle$  $\rangle - i \langle$  $\rangle$

$= \emptyset.$

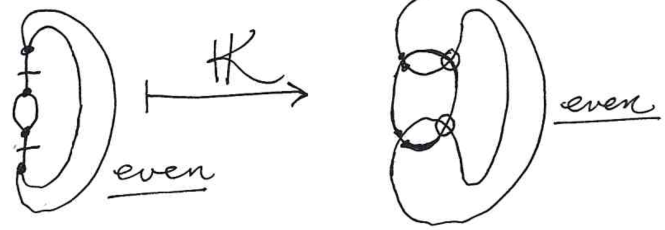
example. Call a virtual link even if any component shares (with the other components) an even number of virtual crossings. Then $\mathbb{K}^{-1}$(any virtual knot diagram) and $\mathbb{K}^{-1}$(any even virtual link diagram) are even perfect matchings. Thus we produce many already colorable graphs.



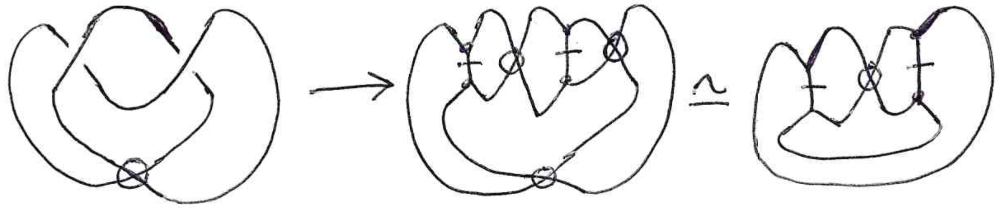
The 4CT says that, among them, are all isthmus-free plane graphs.

Will this topological framework yield deeper insight into the 4CT?

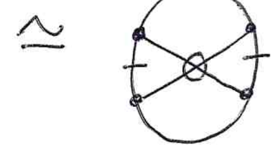
example:



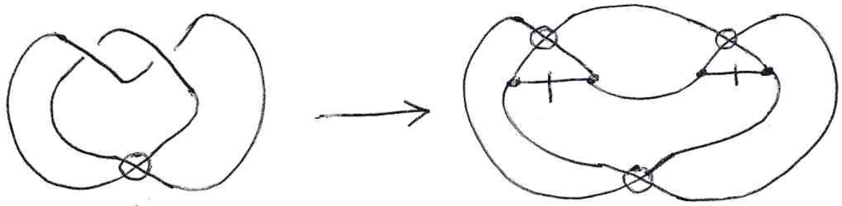
example



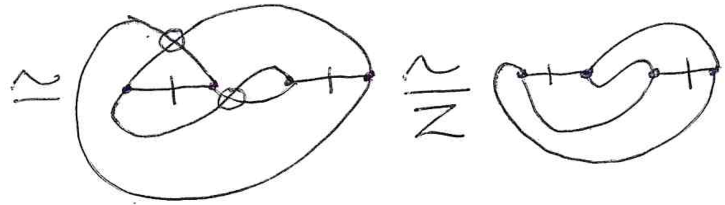
K^*

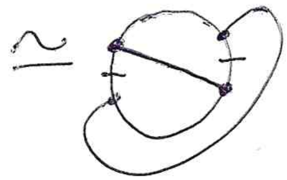


Γ



K





Γ'

Since $\langle K \rangle = A^2 + 1 - A^{-4}$

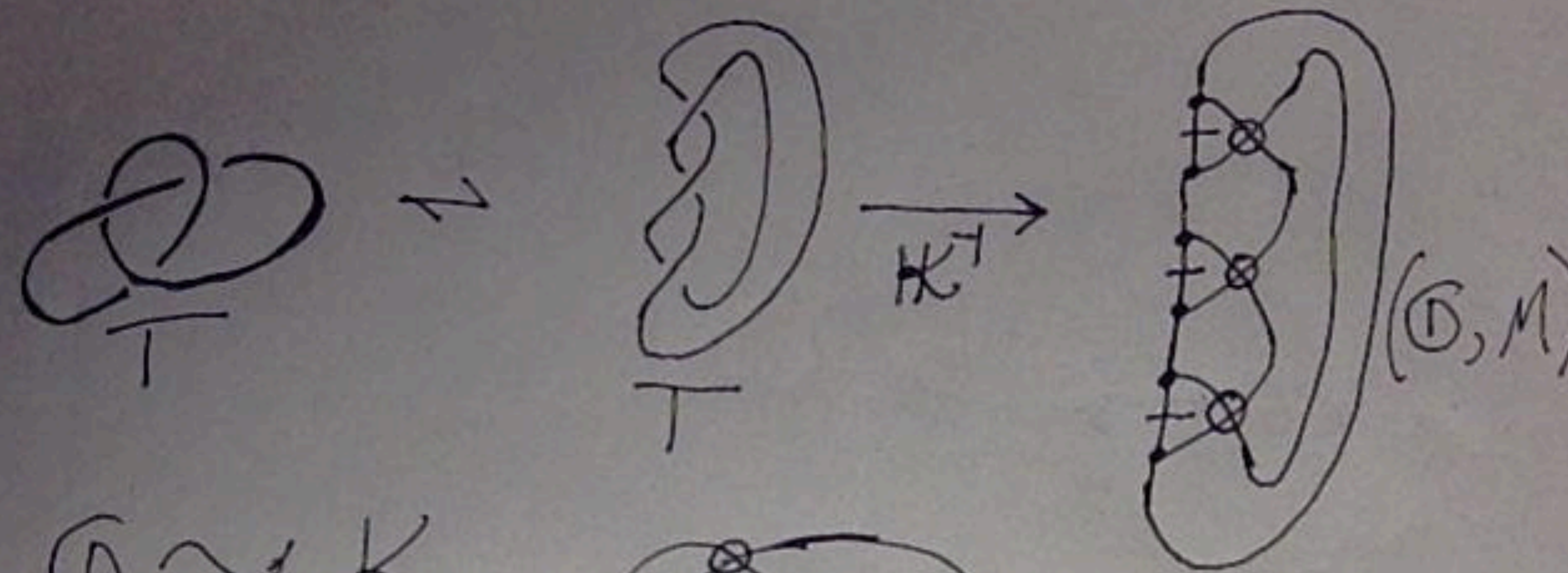
$f_K = A^{-4} + A^{-6} - A^{-10}$

$f_{K^*} \neq f_K$

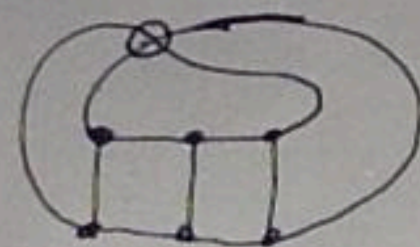
$\Rightarrow K^* \not\simeq K$

$\Rightarrow$ Γ and Γ' are distinct perfect matching graphs.

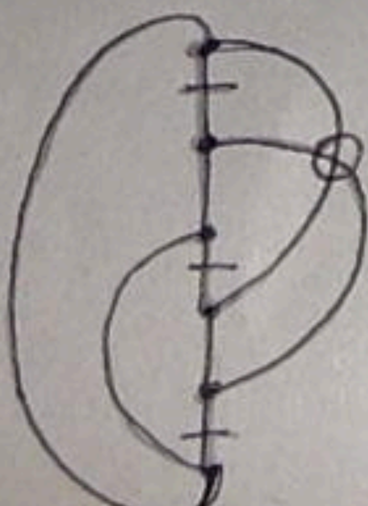
Note that both Γ and Γ' are even perfect matchings.



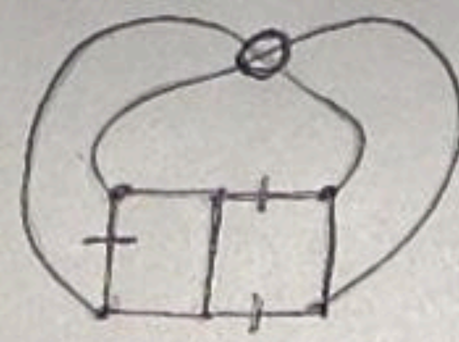
$G \underset{\text{graph}}{\sim} K_{33}$



$G \simeq$

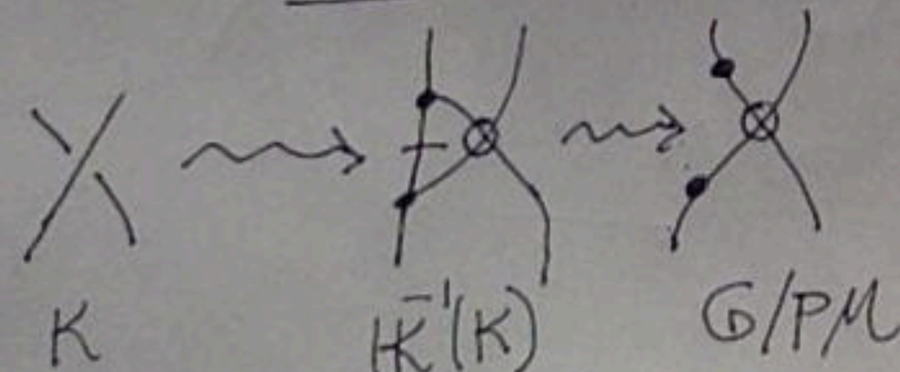


$\simeq$



Note that $\mathbb{K}^{-1}(K) - \{PM \text{ edges}\} = G/PM$

$\equiv \overline{K} = $ <u>K with all crossings made virtual.</u>



<u>Theorem</u>. K a classical knot or link diagram, then $\mathbb{K}^{-1}(K) = (G, M)$ is an even perfect matching.

Hence G is 3-colorable.

<u>Proof</u>. By the Jordan Curve theorem any component in G/PM will have an even number of virtual crossings.

$\therefore$ an even number of dots as above.

This proves that (G, M) is even. //

Proposition. Given a virtual trivalent graph G with a PM and a single circuit in $(G - M)$, then (G, M) is an even PM, and hence G is 3-colorable.

Proof. $e = e' + e_{PM}$ where $e' = \#\text{edges}(G - M)$. $e_{PM} = \#$ of PM edges.

$3v = 2e$, $2e_{PM} = v$.

$\therefore \frac{3v}{2} = e' + v/2 \Rightarrow v = e'$

and $3v = 2e \Rightarrow v$ even.

$\therefore e'$ is even. //

Corollary. Let K be any virtual knot diagram (one component), then $G(K)$ ($H^{-1}(K) = (G(K), M)$) is a 3-colorable trivalent graph.

Proving Scott Baldridge's result via the virtual link category.

Let $[\times] = [\asymp] - [\supset\subset]$

$[O] = 2.$

Use oriented diagram with

$wr = n_+ - n_-$, $n_+ = \#(\ )$, $n_- = \#(\ )$

Define $J_K = (-1)^{n_-}[K]$ $(\Rightarrow J = J)$

Then J_K is invariant under all RM's.

(e.g. $J = J - J = 2J - J = J.$

$J = -(J - J) = -(J - 2J) = J.$)

Theorem. $J_K = 2^{|K|}$ if K is <u>even</u> and $|K| = \#$ components of K. $J_K = \phi$ if K is <u>odd</u>.

(See previous defn of even and odd.)

Proof. The proof is by induction on the number of classical crossings in the diagram K. First suppose that K_i is a component of K and that $= c$ is a crossing between K_i and the rest of K. We can assume that all other crossings in K are <u>positive</u> (since J is unchanged by crossing switch). Then $J = J - J$ and each link on the right has oneless component, with K_i now welded to a component K_j of K. In the second case we must <u>re-orient</u> the diagram to $\supset\subset$.

If the K_i contribution $\lambda(K_i) = 1 + k$ is the count of classical intersections of other components with K_i, then $\lambda \longrightarrow k$ as contribution to $\rightrightarrows$ and $\lambda \longrightarrow -k$ as contribution to $\supset\subset$. Thus $\supset\subset$ acquires k negative crossings and we have

$J_K = J - (-1)^k J.$

Note that $(-1)^{k+1} = (-1)^{\pi(K_i)}$ where

$\pi(K_i) = \#$ of virtual crossings between K_i and rest of K.

Thus, by induction

$$J_{\rightrightarrows} = 2^{|\rightrightarrows|} \text{ or } \phi = 2^{|\rightrightarrows|}\cdot\varepsilon$$

$$J_{\supset\subset} = 2^{|\supset\subset|} \text{ or } \phi = 2^{|\supset\subset|}\cdot\varepsilon'$$

and $|\rightrightarrows| = |\supset\subset| = |\times| - 1$

and $\varepsilon' = \varepsilon = \phi$ or 1.

Thus $J_K = \left(1 + (-1)^{\pi(K_i)}\right) 2^{|\rightrightarrows|}\cdot\varepsilon$.

The same argument applies when both lines of $\times$ are on the same component (so that $\rightrightarrows$ and $\supset\subset$ have extra components). The theorem the follows by induction since

$$\left(1 + (-1)^{\pi(K_i)}\right) = \begin{cases} 2 & \text{when } \pi(K_i) \text{ is even,} \\ 0 & \text{when } \pi(K_i) \text{ is odd.} \end{cases}$$

<u>Remark</u>. Generalize to the following form of the Jones polynomial.

$$[\times] = [\asymp] - q[\supset\subset]$$

$$[O] = q + q^{-1}$$

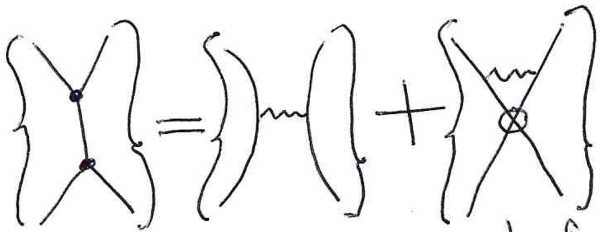
$$J_K = (-1)^{n_-} q^{n_+ - 2n_-}[K]$$

This version is useful for keeping track of grading in Khovanov Homology, and corresponds (by $\mathbb{K}^{-1}$) to the perfect matching polynomial

$$\boxed{\begin{aligned} P_{\succ\!\!-\!\!\prec} &= P_{)(} - q P_{\times} \\ P_O &= q + q^{-1} \end{aligned}}$$

.

5.° Return to Coloring



The Basic Tautology

This is a general formula for counting 3-colorings. Here

 means that the two arcs are colored differently.

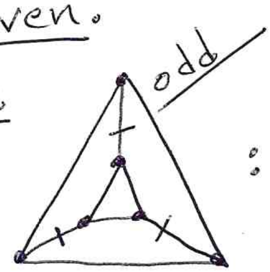
example. 

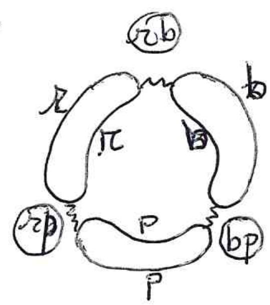
6 possibilities impossible

We can implement this method by choosing a perfect matching and expanding on its edges. The perfect matching need not be even.

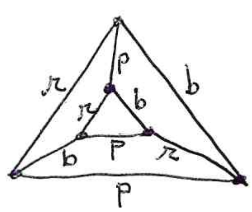
example

We start with any PM and locate all even PM's!

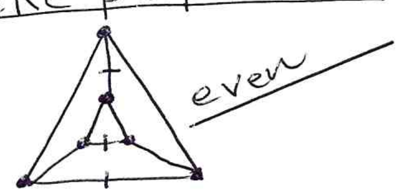
Γ 

: 

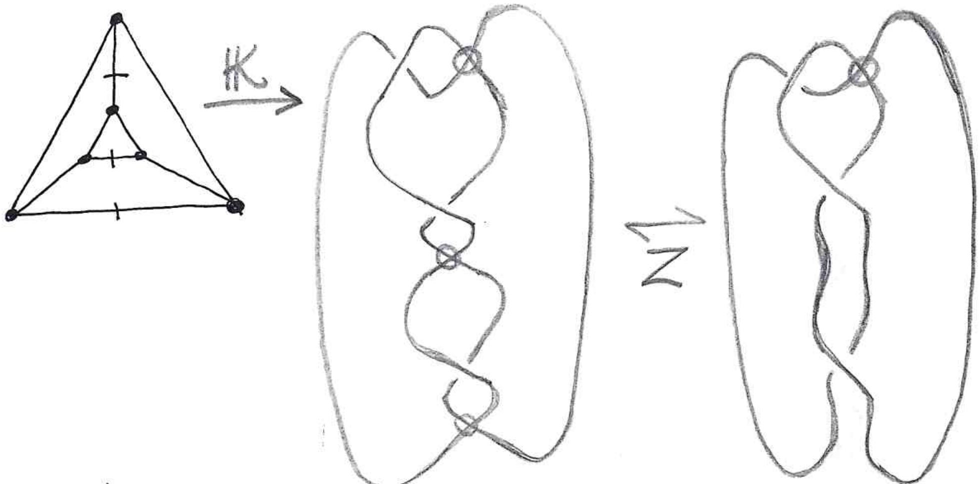
The "Astate" of Γ is colorable with 3 colors.



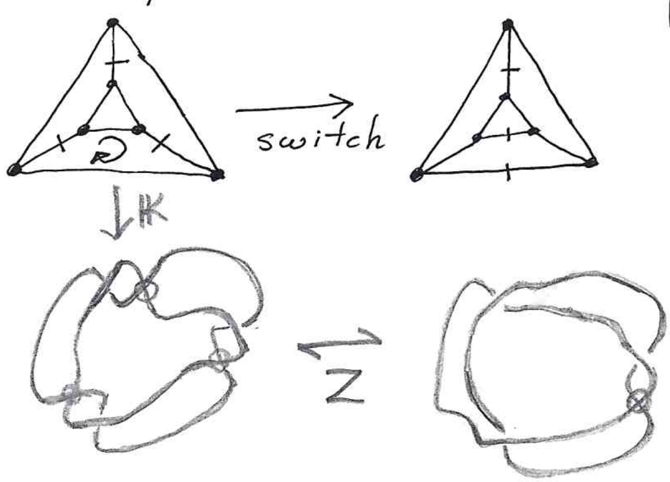
Take purple PM:



Whenever a graph is colorable, it has an even PM, and a corresponding even virtual link diagram.

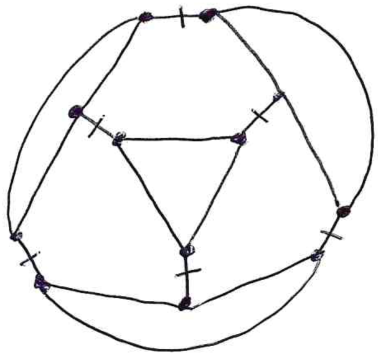


The four-color theorem asserts the colorability of planar isthmus-free trivalent graphs. From any given PM of a graph we can obtain other PM's by switching cycles consisting of alternating edges and PM edges. For example:



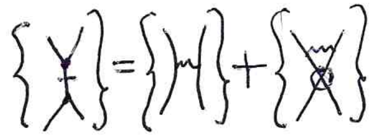
Problem

Understand relations among the links $\mathbb{K}(\Gamma, M)$ as M runs over PM's of a given graph Γ.

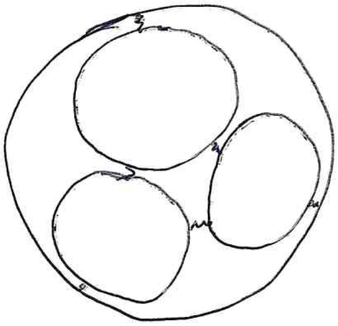


(G, M)

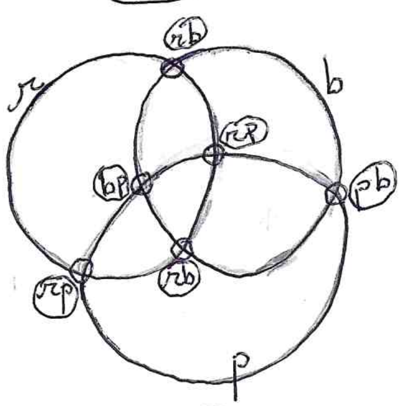
odd ~~perfect~~ matching



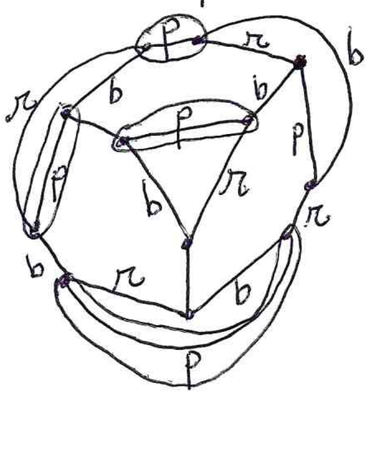
Basic Tautology applied with 3 colors to any perfect matching.



S state not colorable with 3 colors.

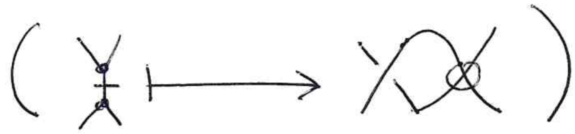


S' a 3-colored state.



an even perfect matching

Transfer <u>Basic Tautology</u> to Virtual Link Diagrams:



$$\{O\} = 3.$$

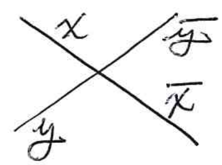
THIS WILL LOCATE ALL POSSIBLE 3-COLORINGS, BUT IT IS <u>NOT</u> INVARIANT UNDER REIDEMEISTER MOVES.

<u>BUT</u>: We can restrict to <u>2 colors</u>.

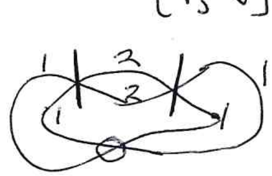
This corresponds to counting all possible colorings for a perfect matching where all matching edges have a constant color (e.g. 3).

$$[K] = \sum_{\substack{\text{all 2-colorings of } K \\ \sigma \\ \{1,2\}}} 1_\sigma$$

$2 = \overline{1}$
$1 = \overline{2}$



e.g.



And generalize to

$$[K] = \sum_{\sigma \in 2\,\text{colorings of } K} \langle K \mid \sigma \rangle$$

$$\langle \times \mid)\sim(\rangle = A$$

$$\langle \times \mid \asymp \rangle = A^{-1}$$

<u>Binary</u>
<u>Bracket</u>

$$\boxed{\begin{array}{l} [\times] = A[\,)\sim(\,] + A^{-1}[\asymp] \\ [\bigcirc] = 2 \end{array}}$$

$\Longrightarrow$ $[K]$ is <u>invariant under</u> R2 and R3.

$$[\,\circ\!\sim\,] = A[\,\frown\,]$$

$$[\,-\!\circ\!\sim\,] = A^{-1}[\,\frown\,].$$

<u>At $A=1$</u>: $[K] \neq \phi \iff K$ corresponds to an even PM.

When $[K] \neq 0$ ($A=1$)

$$\Longrightarrow |[K]| = 2^{\#\text{components of } K}$$

$$= 2^{\#\text{cycles in } (G-M)} \quad (G = \mathbb{K}^{-1}(K)).$$

Thus the binary bracket (A=1) and the standard bracket (A=i) both detect even PM's. The binary bracket and the standard bracket (aka Jones polynomial) both contain very definite topological information about PM's.

And the binary bracket is a specialization of the Basic Tautology that handles all possible 3 colorings of a trivalent graph.

Can we use this relationship of topology and combinatorics to unlock the coloring problem for trivalent graphs?

6.° And Khovanov Homology

Γ a trivalent graph with PM, M.

Define $Kh_o^*(\Gamma, M) = Kh_o^*(\mathbb{K}(\Gamma, M))$ where Kh_o^* is your favorite version of Khovanov Homology for virtuals. We are presently working with William Rushworth's Doubled Khovanov Homology and relating the Lee Homology part of that to the 2-coloring of virtual link diagrams.

arXiv: 1704.0734