

What is a Virtual Crossing?

XX

1. Recall that a graph is specified by an edge set E of a node set V . Each edge has two (sometimes equal) associated endpoints that belong to V . e.g.



$$E = \{e, f, g\}$$

$$V = \{1, 2, 3\}$$


$$\partial e = \{1, 2\}$$

$$\partial f = \{1, 3\}$$

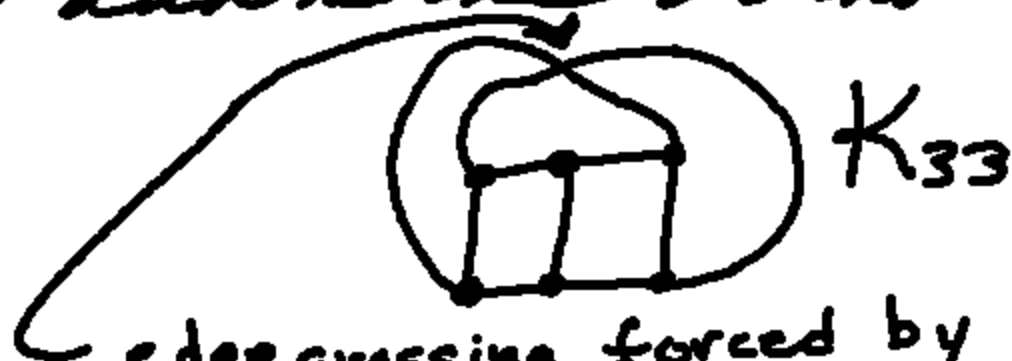
$$\partial g = \{2, 3\}$$

Denote endpoints of edge e by ∂e . Then above

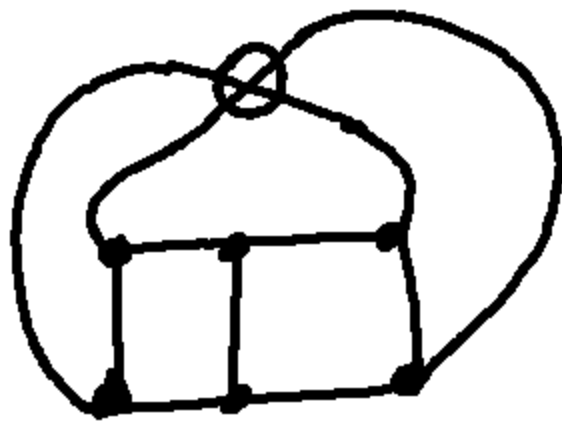
Graphs are often represented so that their edges are intervals homeom $[0, 1] = \{x \mid 0 \leq x \leq 1\} \subset \mathbb{R}$ real line. In that case the edge intervals are all disjoint in an abstract graph G .

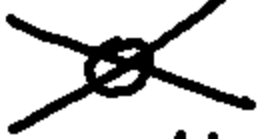
So we will assume that graphs have edges that are disjoint intervals (possibly circles if the edge is a loop as in )

Given topological graphs in this sense, we can ask if a graph G embeds in \mathbb{R}^2 (the plane). Some do and some do not.

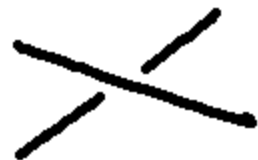


edge crossing forced by the fact that $K_{3,3}$ cannot be embedded in the plane.



For a given immersion of a graph G in the plane, there may be some crossings of edges . We will call these artifacts of the embedding of G in the plane virtual crossings. They are not nodes of G . The least number of virtual crossings in an immersion of G reflects its non-planarity.

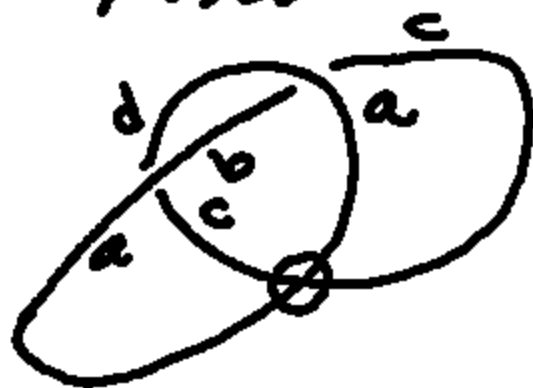
In the diagrammatic theory
of virtual knots we have
standard crossings



and virtual crossings



The abstract virtual knot (or link)
diagram has only crossings
& no virtual crossings.



$$\left\{ \begin{array}{c|c} & d \\ \hline a & b \\ \hline & c \end{array} , \begin{array}{c|c} & d \\ \hline b & c \\ \hline & a \end{array} \right\}$$

The abstract
virtual knot
diagram is
an analogue
of an abstract
graph.

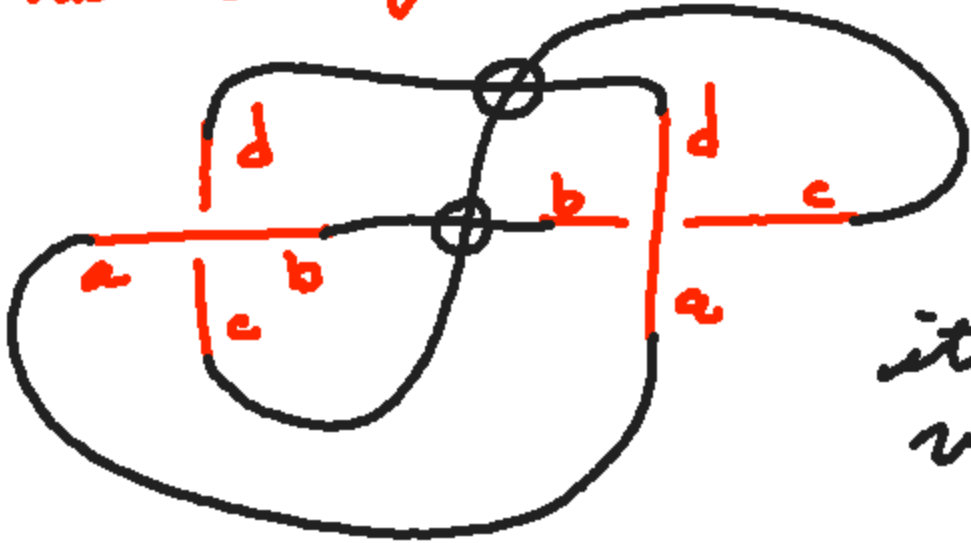
This data (labelled crossings
with their given cyclic orders)
specifies the abstract virtual
knot diagram.

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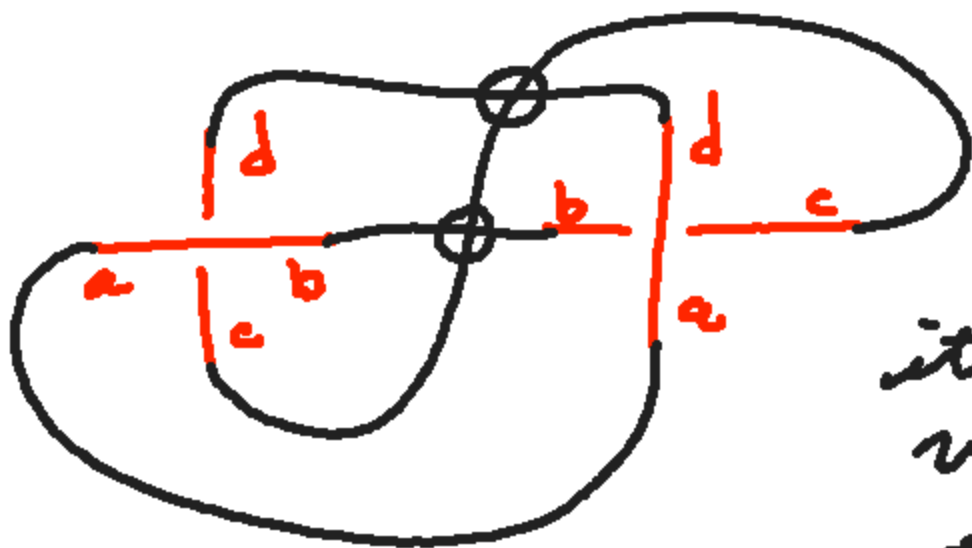
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Try to reconstruct a planar
knot diagram from the abstract diagram.

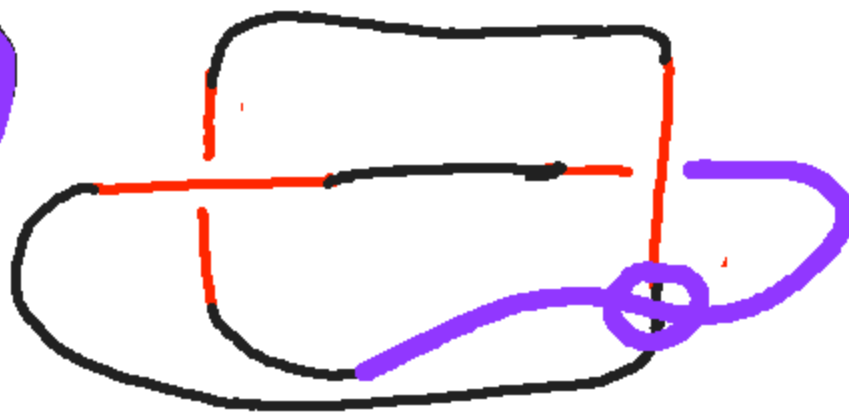
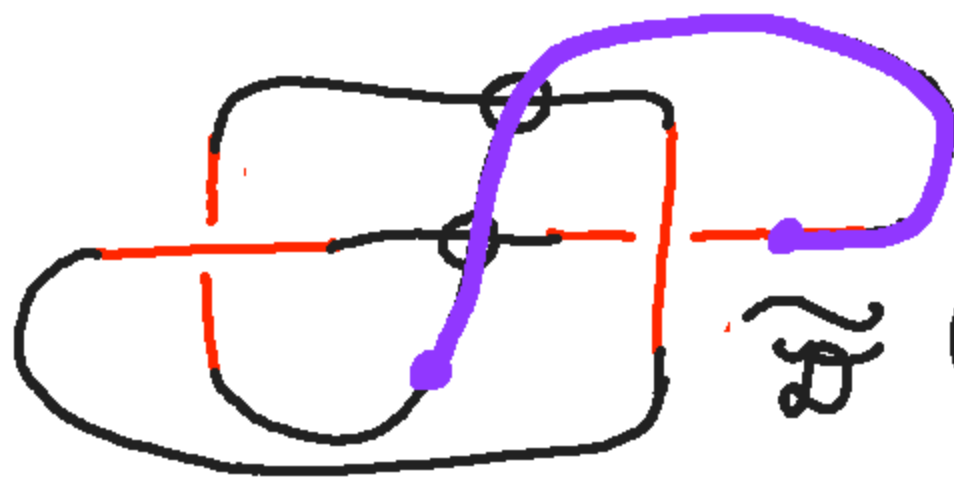
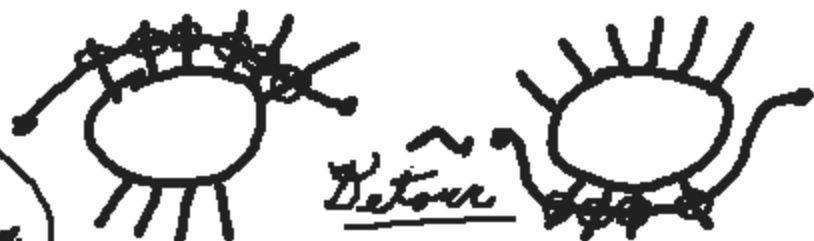


This is
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from the
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But we allow detector moves on virtual crossings.
 (since we only care about the abstract knot diagram)



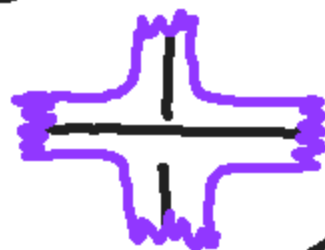
So the virtual crossings
are antithesis of planar representation.
Diagrammatic virtual knot
theory uses diagrams with
crossings and virtual
crossings.

Moves = Reidemeister Moves


+
Below Moves for
Virtual Crossings.



Constructing a surface with boundary on which the abstract diagram is embedded.

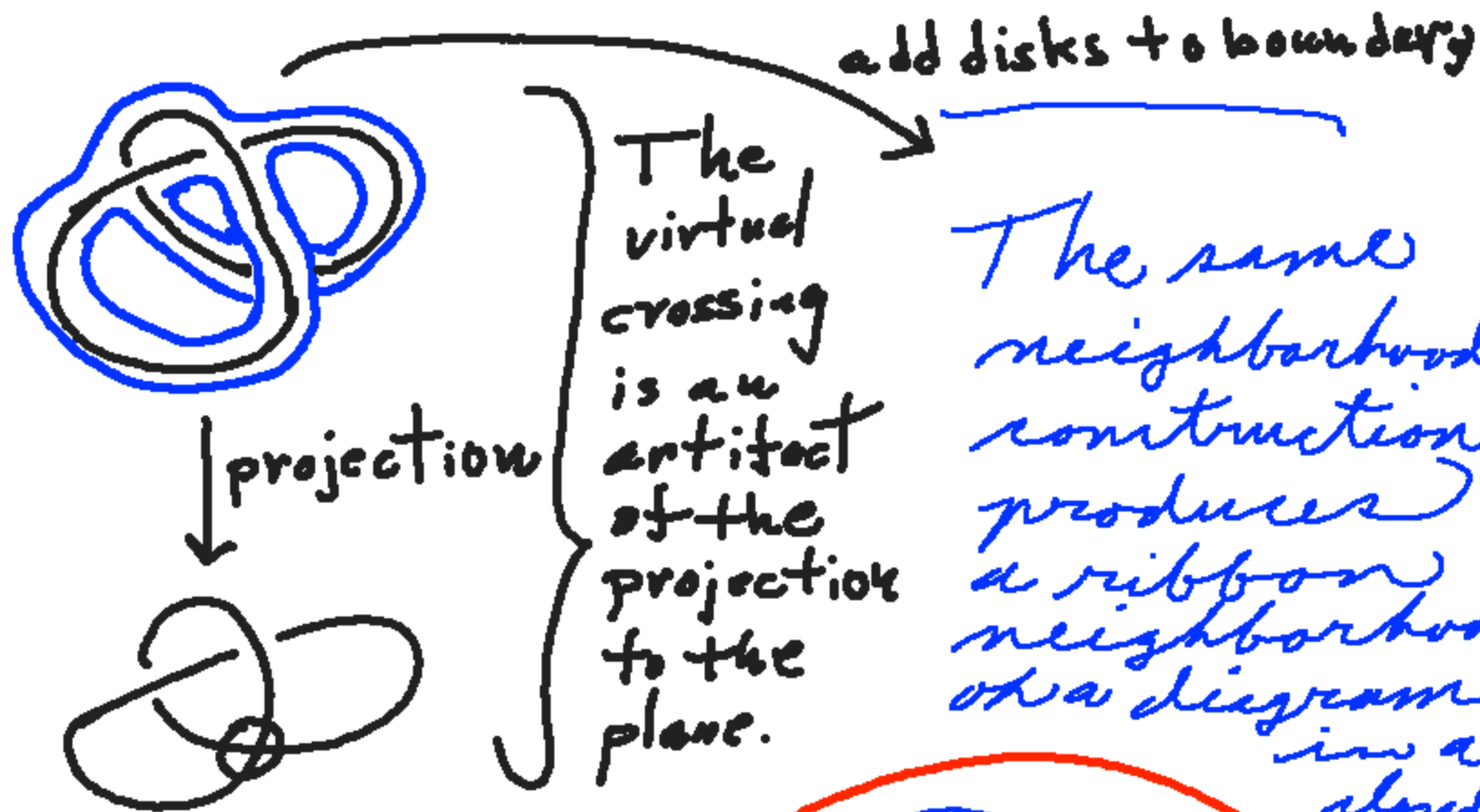


ramps can go over or under because we make an abstract surface.



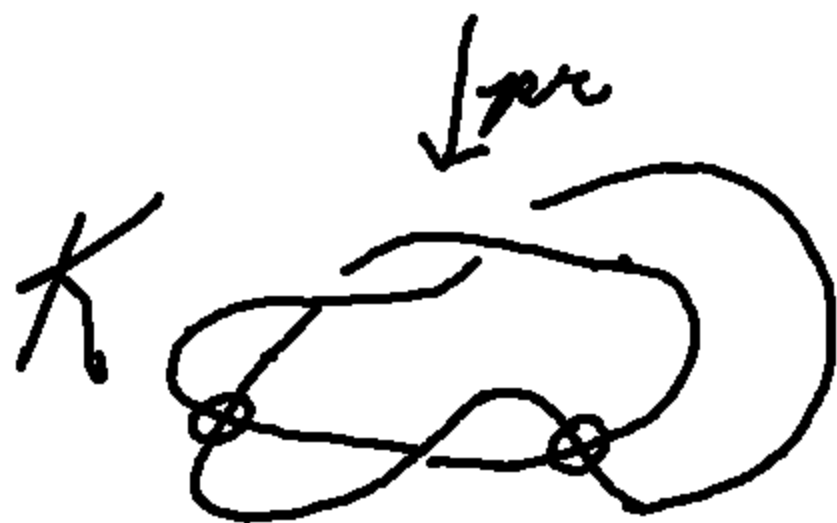
or



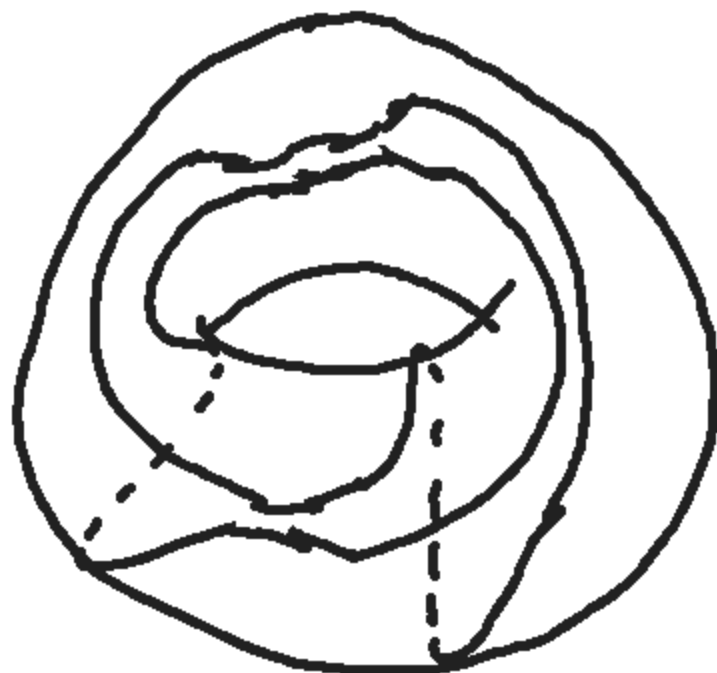
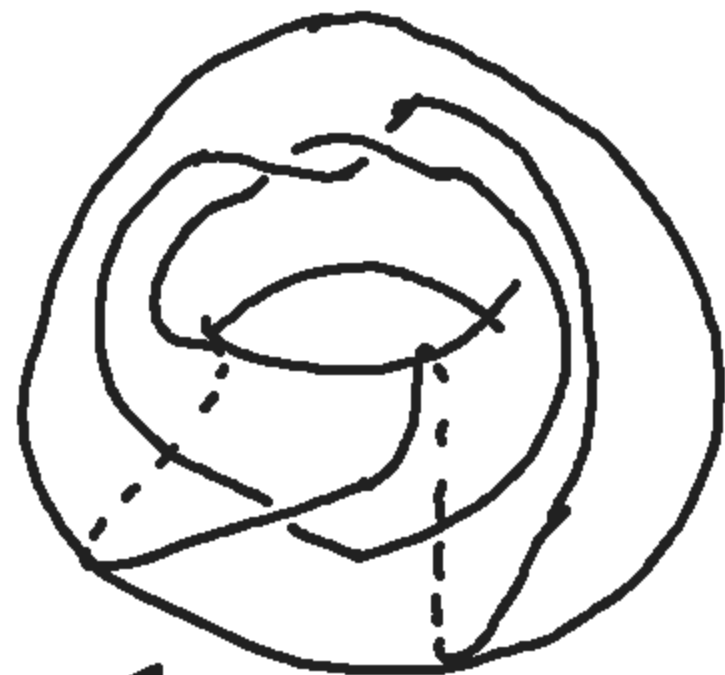


The upshot of this correspondence is that virtual knots up to diagram \sim are same as knots in thickened surfaces up to handle stabilization (next notes)





We can study
the surface basket
of this knot.
States are
collections of
loops $\subset T_{\text{area}}$.



There
are ways
to study
the knots
on a
given
surface.

Then study

- { • homology classes of
state curves.
- { • isotopy classes of state
curves.

Ask: Do the systems of
curves force the knot to
have a given underlying genus?


(we will put his
paper in Dropbox)

Kuperberg's Theorem . $1/2$

by surgery) or embedding a
virtual knot or link in place
in a surface S_g + g is minimal
then the topological type of $K \subset S_g$
is unique.

\Rightarrow If $K \subset S_g$ minimal
then its type in S_g
determines the virtual
knot type.

What is a virtual crossing?
(again)

you can regard  as
shortcuts for



the plane with a handle attached
so that one line goes thru
the handle and the other
goes underneath the handle.



(so you can use either line for handle)



These knots on surfaces are taken up to handle stabilization. an empty handle can be added or removed,

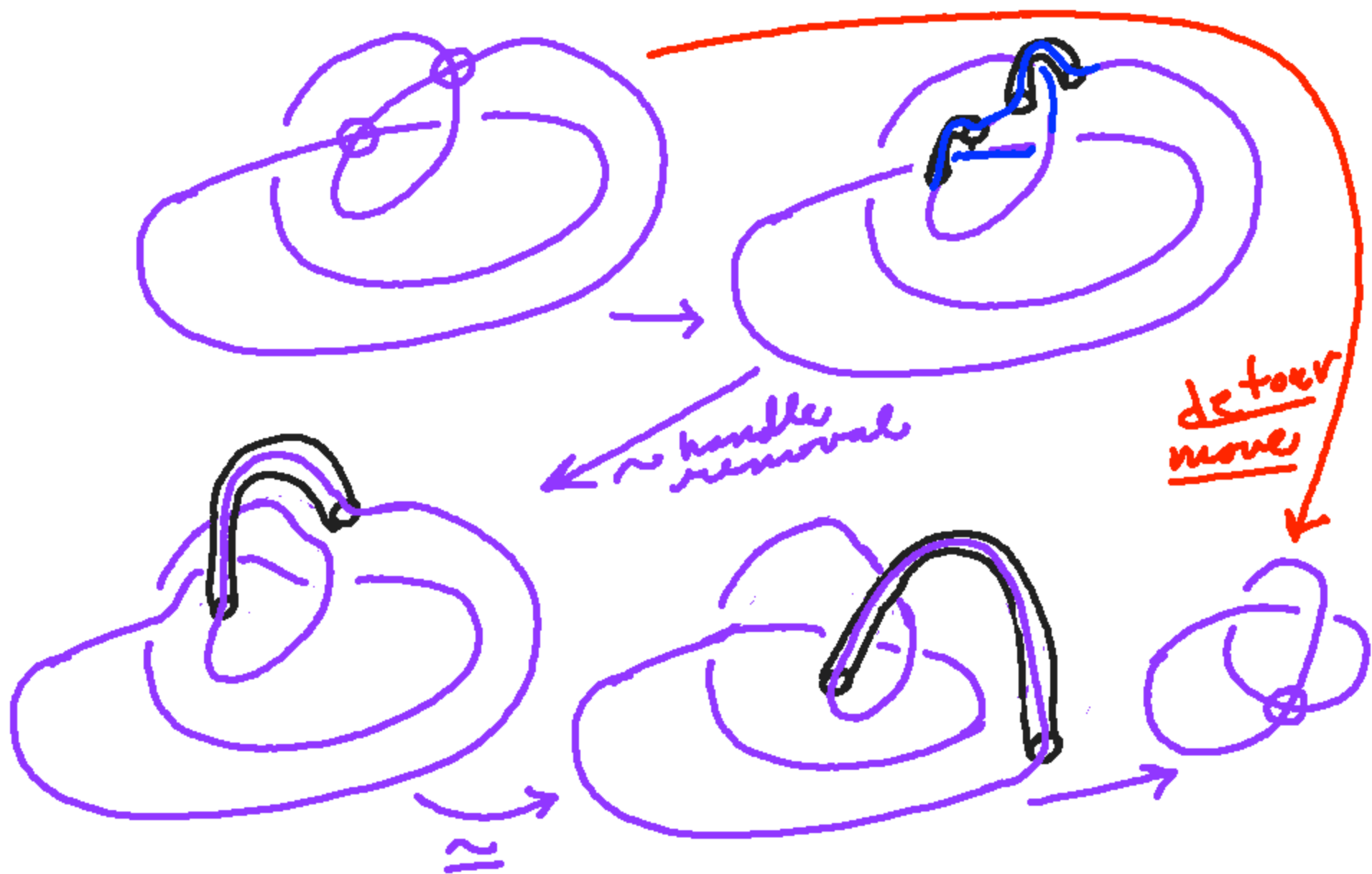




handle removal.

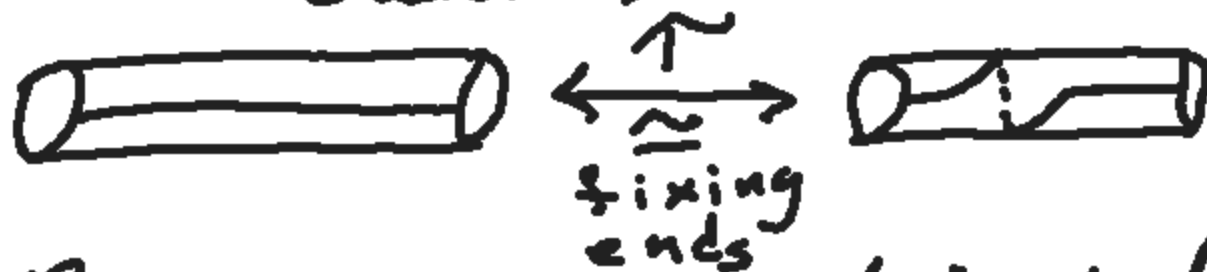


This corresponds to the detour move, since the big handle can be moved around.





Since we are looking at knots on the surface, we allow Dehn twists.






\therefore Can remove twist in red line and get (next page)



now we can eliminate
the handle by doing
surgery along the
blue curve. (cut
& paste in two disks)

We get

so we
have shown how in the handle
picture

As in this model for
 virtual knot theory, each
 diagram with virtual
 crossings is shorthand
 for a diagram on a surface
 of genus $g = \# \text{ of } \bigcirc \bigotimes$. The
 moves, , ,
 moves to handle slides plus
 surgeries on handles. The
 move  requires
 a Dehn twist.

There is a special version
of the theory where we do
not allow $\varphi \sim \supset$
(so no Dehn twists allowed on
the surfaces).

For that theory this one
handle per φ is a good
interpretation. I call
the theory where $\varphi \sim \supset$ is
not allowed, rotational
virtual knot theory.

Note that if \otimes were an ambiguous crossing (either over or under, but we do not know which) then we would have



since this would be true if $\otimes = \nearrow$ or $\otimes = \searrow$.

But this move is not allowed in VKT.

Remark. Can also regard
~~X~~ as a permutation



More Invariants

Parity



minimum
 Parity?
 Are Hamiltonian
 $[212]$ 142 are odd

odd and even crossings.



123123
 all even!

Odd Writhe $\xrightarrow{H} \xrightarrow{H} : \text{sign}$

K oriented virtual knots.

$H = \text{braid Homotopy}$

$$J_K = \sum_{C \in \text{Or}(K)} \text{sign}(C)$$

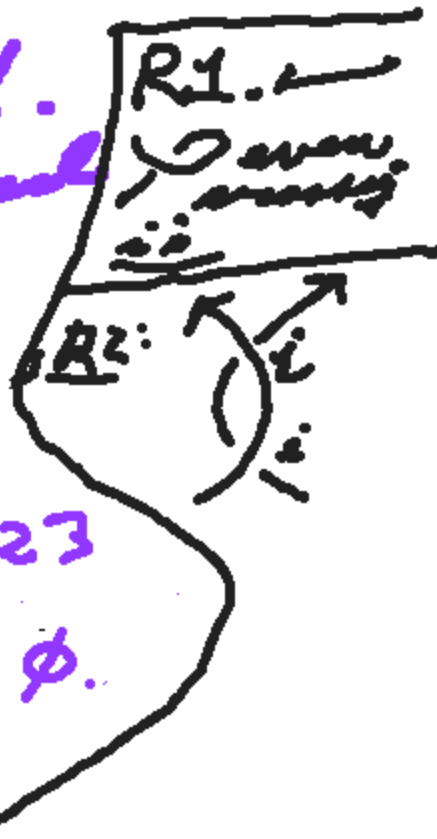
1. $K \sim K'$ as virtual $\Rightarrow J_K = J_{K'}$.

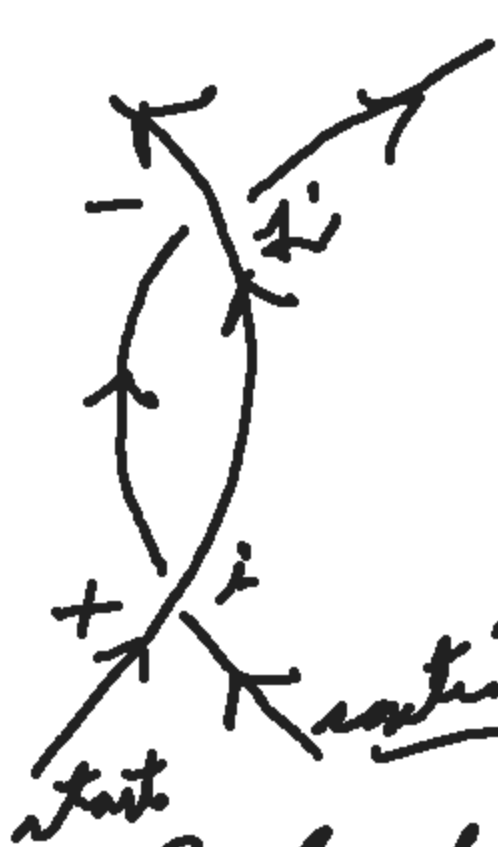
2. $K \sim \text{classical} \Rightarrow J_K = 0$.

3. $J_{K^*} = -J_K$

 $J_K = 2$.

 $J_K = 0$.





$$\begin{array}{c} |1+1\alpha| \\ \delta_{ij} \alpha_{ij} \beta \\ |1+1\alpha| \end{array}$$

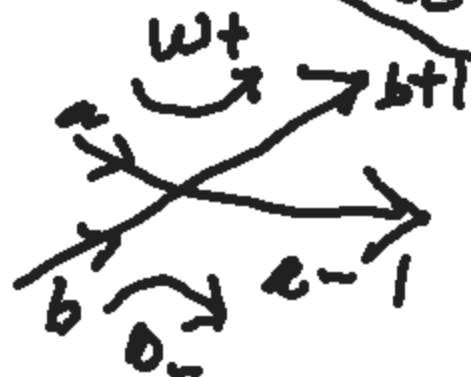
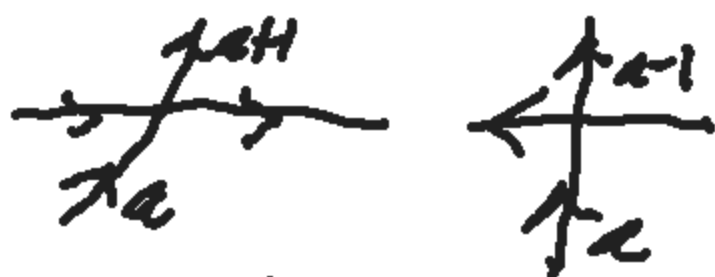
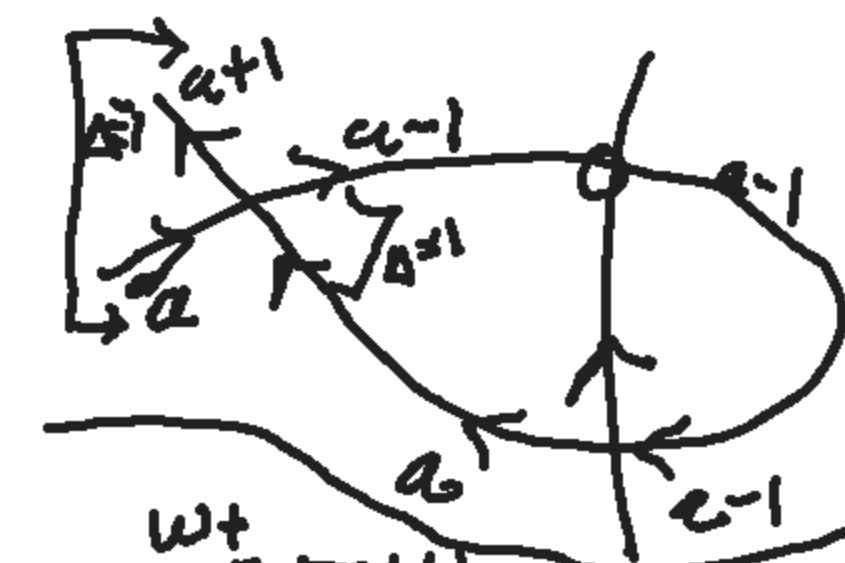
$\Rightarrow i, j$ are both even
or both odd.

Check R3 also.

Parity Bracket - next time

Affine Index Polynomial

Labd Solana ($a, b, m \in \mathbb{Z}$)



$$w_+ = a - b - 1$$

$$w_- = b - a + 1 = -w_+$$

$$w^t \left(\begin{array}{c} c \rightarrow b \\ a \rightarrow d \end{array} \right) = w_+ = \underline{c-b}, \quad w^t \left(\begin{array}{c} c \rightarrow b \\ a \rightarrow d \end{array} \right) = \underline{e-d}$$

$$P_K = \sum_{c \in \mathcal{C}_2(K)} \text{sgn}(c) t^{\text{wt}(c)} - \text{wt}(K)$$

$$\sum_{c \in \mathcal{C}_2(K)} \text{sgn}(c) \begin{bmatrix} t^{\text{wt}(c)} \\ -1 \end{bmatrix}$$

P_K is
an invariant of K
under RNS + Dehn twists.



Lemma.
you can
always
label any
diagram.

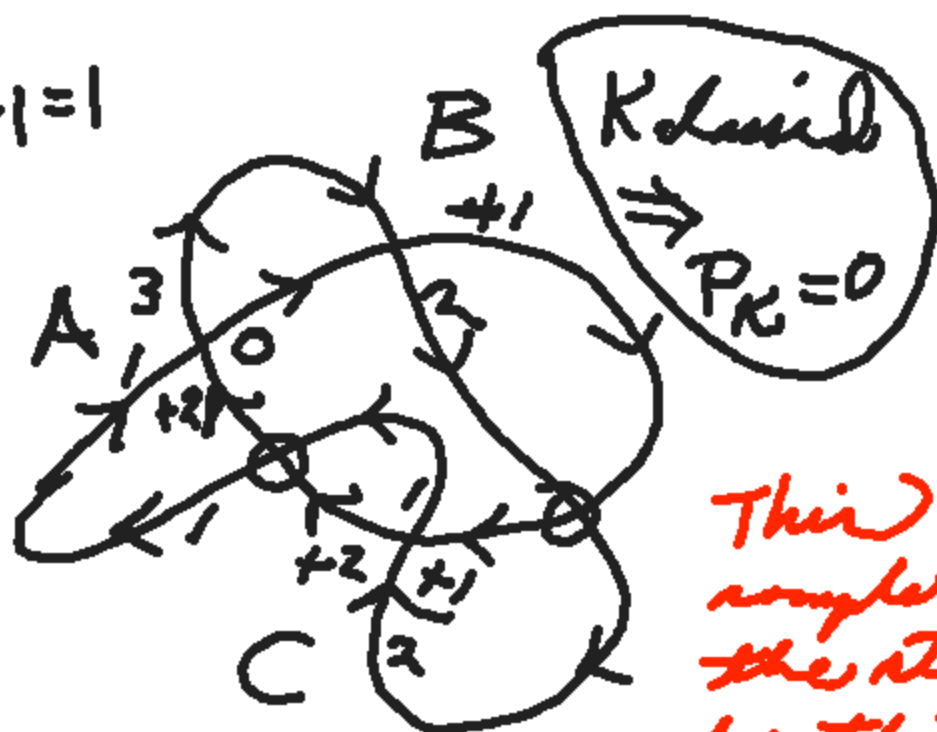
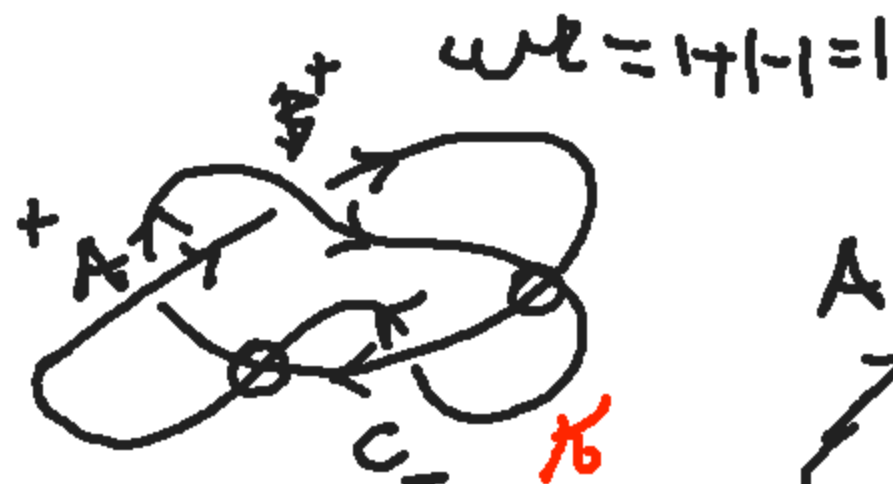
	w_+	w_-
A	-1	1
B	1	-1

both +
so $P_K = \bar{x}^1 + x^1 - 2$



$$P_{K^*} = -x^1 - \bar{x}^1 + 2$$

$\Rightarrow K \not\sim K^*$



K is trivial
 $\Rightarrow P_K = 0$


	w_+	w_-
A	-2	2
B	2	-2
C	0	0

$$P_K = \tau^{-2} + \tau^2 - 1 - 1$$

$$P_K = \tau^{-2} + \tau^2 - 2$$

$\therefore K$ is not trivial
 $S_K = 1$ K is not trivial

This completes the story for this example.
 we have
 1) $S_K = 1$
 2) K non-trivial & non-trivial channel

Problem. Try another example
of this kind. e.g.  K

becomes unknotted if you switch c_1 and c_4 .
Let $\tilde{K} = \text{Virt}(K; c_1, c_4)$, so that $\int \tilde{K} = 1$.
Compute $P_{\tilde{K}}$ and see if it is $\neq 0$.

Problem. Same idea but for

