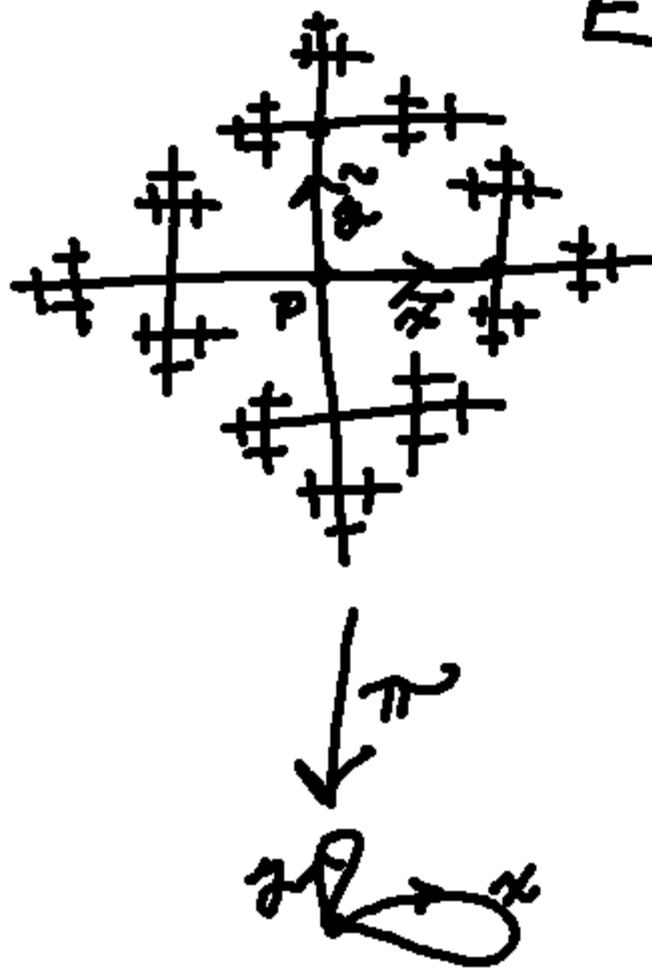
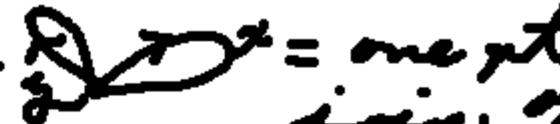
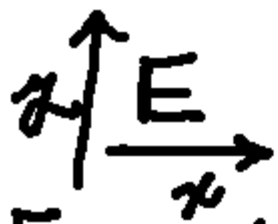
$$\begin{array}{c} E \\ \big\downarrow \pi \\ B \end{array}$$



E = Universal covering space of B

B = 

= one pt join of two circles.



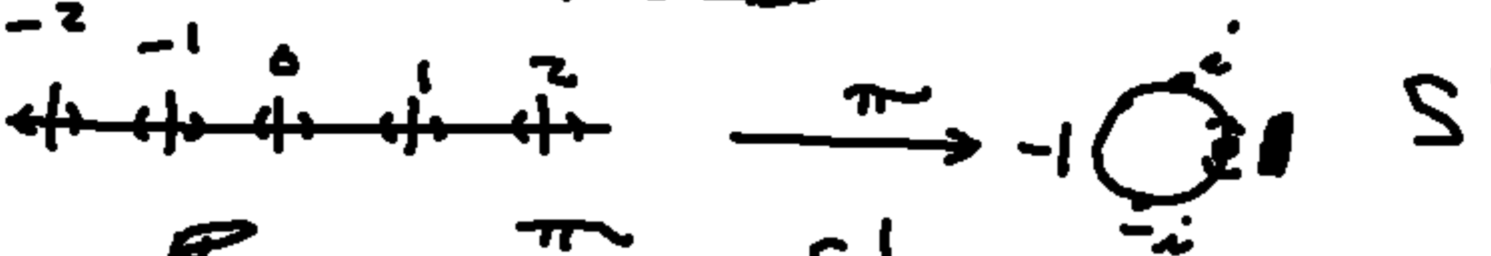
- E is acted on by $G = (x, y)$
- Every 1-cell in E is of the form $w\tilde{x}$ or $w\tilde{y}$, $w \in G$.
- If $w \in G$ + $\tilde{w}$ denotes lift of w to E starting at basept p, then

$$\boxed{\widetilde{w\tau} = \tilde{w} + w\tilde{\tau}}$$ Basic path lift formula.

Example of Covering Space

$E \xrightarrow{\pi} B$ $\boxed{\text{Covering space}}$

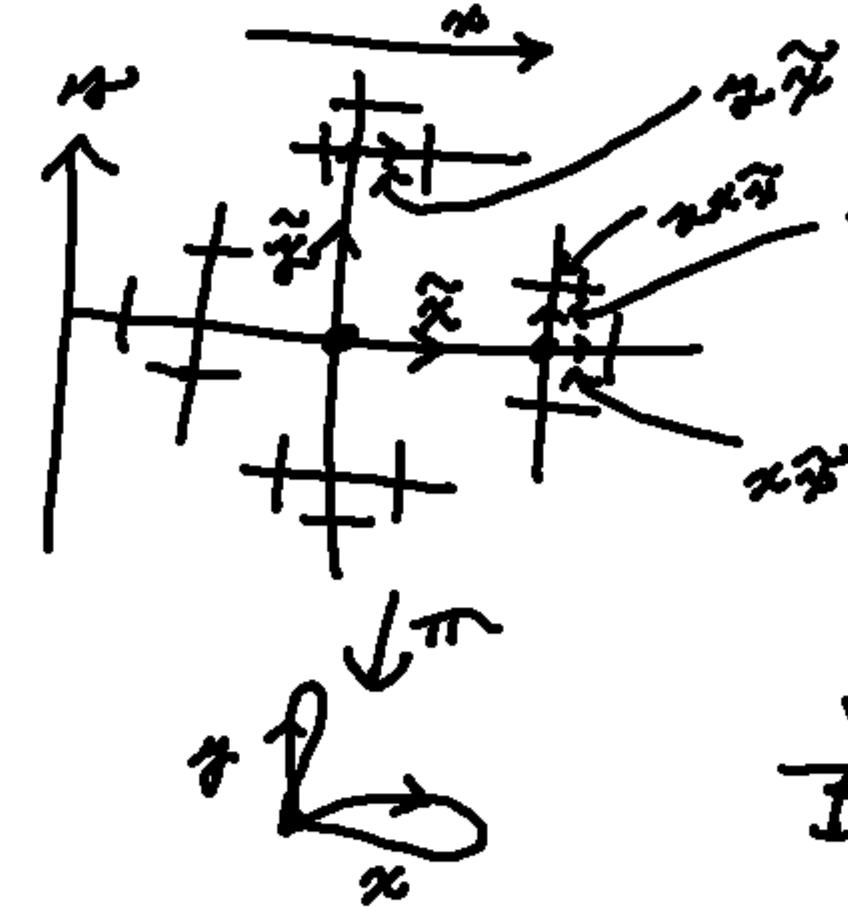
local homeom



$\mathbb{R} \xrightarrow{\pi} S^1$

$$\pi(r) = e^{2\pi i r}$$

$$\pi^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$



$$E \xrightarrow{\pi} B$$

Edges in E all of the form $w\tilde{x}$ or $w\tilde{y}$ where w is a word in $G = F(x,y)$.

$xyx\tilde{x}$

Lifting a path in B = lifting a word in G.

$$\left.\begin{array}{l} w,\ \tau \\ \widetilde{w\tau} = \tilde{w} + w\tilde{\tau} \end{array}\right.$$

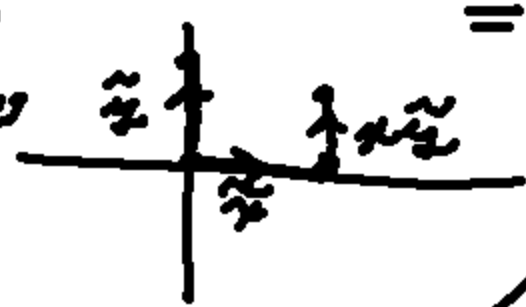
$$\boxed{\widetilde{w\tau} = \tilde{w} + w\tilde{\tau}}$$

$G = F(x, y)$

$\tilde{w} = \star\tilde{x} + \#\tilde{y}$

$\star, \# \in$ Group Ring of G

$\widetilde{xy} = \tilde{x} + x\tilde{y}$



$\widetilde{xyx}$

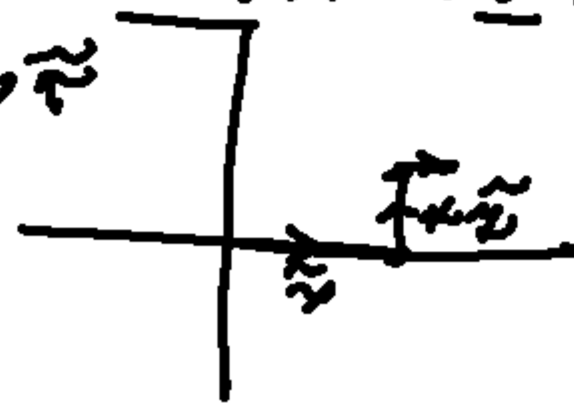
$= \tilde{x} + x(\widetilde{yx})$

$= \tilde{x} + x(\tilde{y} + y\tilde{x})$

$\widetilde{w\tau} = \tilde{w} + w\tilde{\tau}$

$= \tilde{x} + x\tilde{y} + xy\tilde{x}$

$= (1+xy)\tilde{x} + \underline{x}\tilde{y}.$



$$\boxed{\widetilde{w\tau} = \widetilde{w} + w\widetilde{\tau}}$$

<u>Fox's Defn.</u> $w \in F(x,y)$

$$\widetilde{w} = \left(\frac{\partial w}{\partial x}\right)\widetilde{x} + \left(\frac{\partial w}{\partial y}\right)\widetilde{y}$$

"<u>Free Derivative</u>"

$$\widetilde{x^2y} = \widetilde{xxy} = \widetilde{x} + x(\widetilde{xy})$$

$$= \widetilde{x} + x(\widetilde{x} + x\widetilde{y})$$

$$= (1+x)\widetilde{x} + x^2\widetilde{y}$$

$$\therefore \frac{\partial(x^2y)}{\partial x} = 1+x, \quad \frac{\partial(x^2y)}{\partial y} = x^2$$

$\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ defined by $\rightarrow$ $\boxed{\widetilde{w\tau} = \tilde{w} + w\tilde{\tau}}$

$$\tilde{w} = \frac{\partial w}{\partial x}\tilde{x} + \frac{\partial w}{\partial y}\tilde{y}$$

$$\frac{\partial (w\tau)}{\partial x} \qquad \widetilde{w\tau} = \tilde{w} + w\tilde{\tau}$$

$$= \frac{\partial w}{\partial x}\tilde{x} + \frac{\partial w}{\partial y}\tilde{y}$$

$$+ w\left(\frac{\partial \tau}{\partial x}\tilde{x} + \frac{\partial \tau}{\partial y}\tilde{y}\right)$$

$$= \left(\frac{\partial w}{\partial x} + w\frac{\partial \tau}{\partial x}\right)\tilde{x} + \underbrace{\left(\frac{\partial w}{\partial y} + w\frac{\partial \tau}{\partial y}\right)}_{= \frac{\partial (w\tau)}{\partial y}}\tilde{y}$$

$$\boxed{D(w\tau) = Dw + wD\tau}$$

$$\widetilde{x^3} = \tilde{x} + x\widetilde{x^2}$$
$$= \tilde{x} + x(\tilde{x} + x\tilde{x})$$
$$= (1 + x + x^2)\tilde{x}$$

$$\frac{\partial x^n}{\partial x} = 1 + x + x^2 + \cdots + x^{n-1}$$

Given $G = (g_1, \dots, g_n \mid r_1, \dots, r_m)$

+ surj $G \xrightarrow{\ \phi\ } \mathbb{Z} = (t \mid \ )$

knot groups always have this.

K

$G = \pi_1(S^3 - K) \longrightarrow \mathrm{Abel}(G) \simeq \mathbb{Z}$

$\Big(c = b^{-1}ab \longrightarrow \underset{\text{all to same } \mathbb{Z}}{c = a} \Big)$

Given $G = (g_1, \dots, g_n \mid r_1, \dots, r_m)$

+ surj $G \xrightarrow{\Phi} \mathbb{Z} = (t \mid \ )$

knot groups always have this.

K

$G = \pi_1(S^3 - K) \longrightarrow \mathrm{Abel}(G) \simeq \mathbb{Z}$

$\Big(c = b^{-1}ab \rightsquigarrow c = a$ all to one $\mathbb{Z}$ ✓ $\Big)$

$$J = \left(\frac{\partial r_i}{\partial g_j} \right)^{\Phi}$$

← apply Φ to all elements

Fox: $\Delta_K(t)$ = gen of the principal ideal in $\mathbb{Z}[t, t^{-1}]$ gen by all $(n-1) \times (n-1)$ minors of J.

$G = (a, b \mid aba = bab)$ *trefoil*

$$\begin{array}{ccc} G & & a \quad b \\ \downarrow \Phi & & \searrow \ \swarrow \\ (t) & & t \end{array}$$

$$\widehat{aba}^{\phi} = (1+t^2)\tilde{a} + t\tilde{b}$$
$$\widehat{bab}^{\phi} = t\tilde{a} + (1+t^2)\tilde{b}$$

$G = (g_1, \dots, g_n \mid r_1, \dots, r_m)$

examine $\tilde{r}_1, \dots, \tilde{r}_m$ *via* $\underline{\tilde{g}_1, \dots, \tilde{g}_n}$

$$\widehat{aba} = \tilde{a} + a(\widehat{ba}) = \tilde{a} + a(\tilde{b} + b\tilde{a})$$
$$= \tilde{a} + a\tilde{b} + ab\tilde{a}$$

$$\boxed{\widehat{aba} = (1+ab)\tilde{a} + a\tilde{b}}$$

$$\widehat{bab} = \tilde{b} + b\,(\widehat{ab}) = \tilde{b} + b(\tilde{a} + a\tilde{b})$$

$$\boxed{\widehat{bab} = b\tilde{a} + (1+ba)\tilde{b}}$$

$G = (a, b \mid aba = bab)$ *trefoil*

$$\begin{tikzcd} G \arrow[d, "\Phi"] \\ (t) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[dr] & & b \arrow[dl] \\ & t & \end{tikzcd}$$

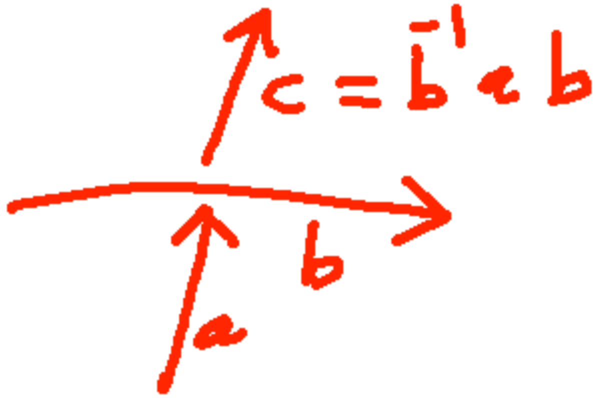
$$\widetilde{aba}^{\phi} = (1+t^2)\tilde{a} + t\tilde{b}$$
$$\widetilde{bab}^{\phi} = t\tilde{a} + (1+t^2)\tilde{b}$$

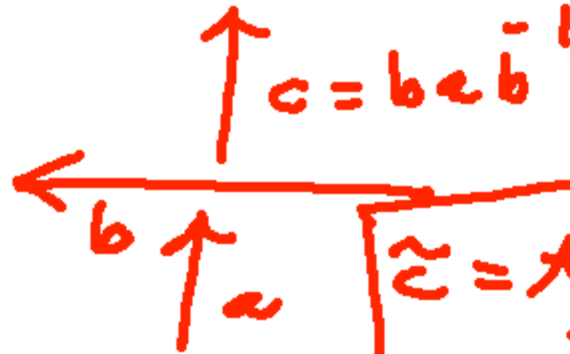
$$(1+t^2)\tilde{a} + t\tilde{b} = t\tilde{a} + (1+t^2)\tilde{b}$$

$$(t^2-t+1)\tilde{a} = (t^2-t+1)\tilde{b}$$

$$\underline{(t^2-t+1)}(\tilde{a}-\tilde{b}) = 0$$

$\Delta_K(t)$ <u>Alex Polyn</u>





$$\boxed{\tilde c = x \tilde a + (1-x)\tilde b}$$

$$\circlearrowleft \longrightarrow (x \| \ )$$

$$a, b, c, \ldots \rightsquigarrow x$$

$$\tilde c = \widetilde{b^{-1}} + b^{-1}\widetilde{(ab)}$$

$$= \widetilde{b^{-1}} + b^{-1}(\tilde a + a \tilde b)$$

$$= -b^{-1}\tilde b + b^{-1}\tilde a + b^{-1} a \tilde b$$

$$= (-b^{-1} + b^{-1} a)\tilde b + b^{-1}\tilde a$$

$$\tilde c = (\cancel{-x^{-1}} + 1)\tilde b + x^{-1}\tilde a$$

$$\boxed{\tilde c = x^{-1}\tilde a + (1 - x^{-1})\tilde b}$$

$$\boxed{\widetilde{b^{-1}} = -b^{-1}\tilde b}$$

$$b b^{-1} = 1$$

$$\widetilde{b b^{-1}} = \tilde 1$$

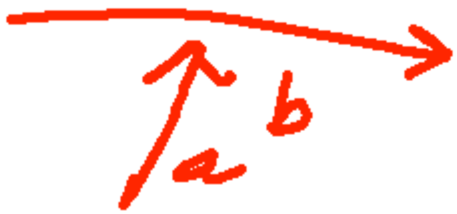
$$\tilde b + b \widetilde{b^{-1}} = 0$$

$$b \widetilde{b^{-1}} = -\tilde b$$

$$\widetilde{b^{-1}} = -b^{-1}\tilde b$$

Change of Notation

$$c = a * b = t^{-1}a + (1-t^{-1})b$$



$$a \bar{*} b = ta + (1-t^{-1})b$$

This is (Alex) quandle.

One more change. Flip $t \to t^{-1}$.

$$a * b = ta + (1-t)b$$
$$a \bar{*} b = t^{-1}a + (1-t^{-1})b$$

Calc Fox

Quandle

$t = -1$

$\Rightarrow$ Fox Coloring Quandle

1. $c = tb + (1-t)a$
2. $b = ta + (1-t)c$
3. $a = tc + (1-t)b$

$$\begin{array}{ccc} a & b & c \\ \hline 1-t & t & -1 \\ t & -1 & 1-t \\ -1 & 1-t & t \end{array}$$

$$\begin{pmatrix} -1 & 1-t \\ 1-t & t \end{pmatrix} \xrightarrow{\det} -t-(1-t)^2$$

$$\begin{aligned} &\parallel \\ &-t-(1-2t+t^2) \\ &\parallel \\ &-t-1+2t-t^2 \\ &\parallel \\ &-1+t-t^2 \\ &= -(t^2-t+1) \end{aligned}$$

poly
upto
$(\pm t^N)$

Torus Knot of type m, n

$$G = (a, b \mid a^m = b^n)$$

Fox Calculus $\rightsquigarrow$ alex polyn (fast!)

$$\widetilde{a^m} = \widetilde{b^n}$$

$$(1 + a + \dots + a^{m-1})\tilde{a}$$

$$\parallel$$

$$(1 + b + \dots + b^{n-1})\tilde{b}$$

$$\downarrow \Phi$$

$$?$$

$K_{4,3}$

$$\boxed{\widetilde{w\vec{r}} = \tilde{w} + w\vec{r}}$$

(do it or look in box)