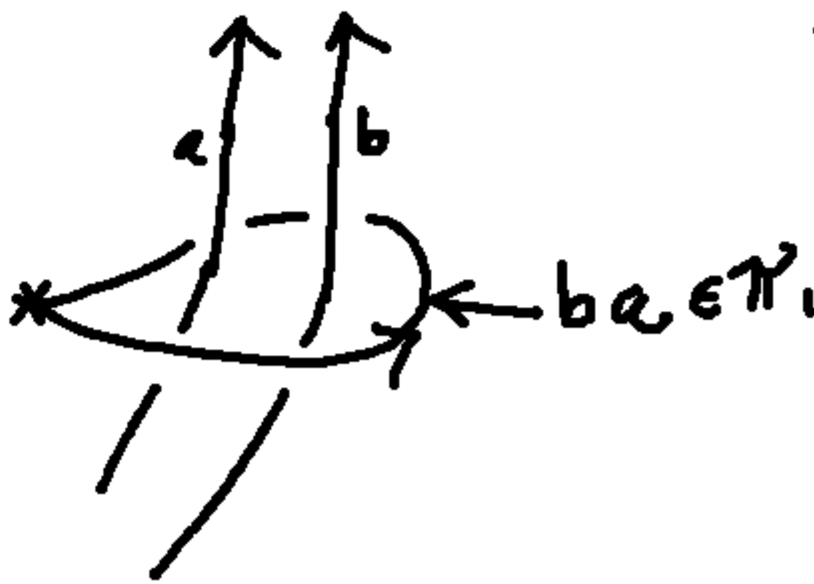
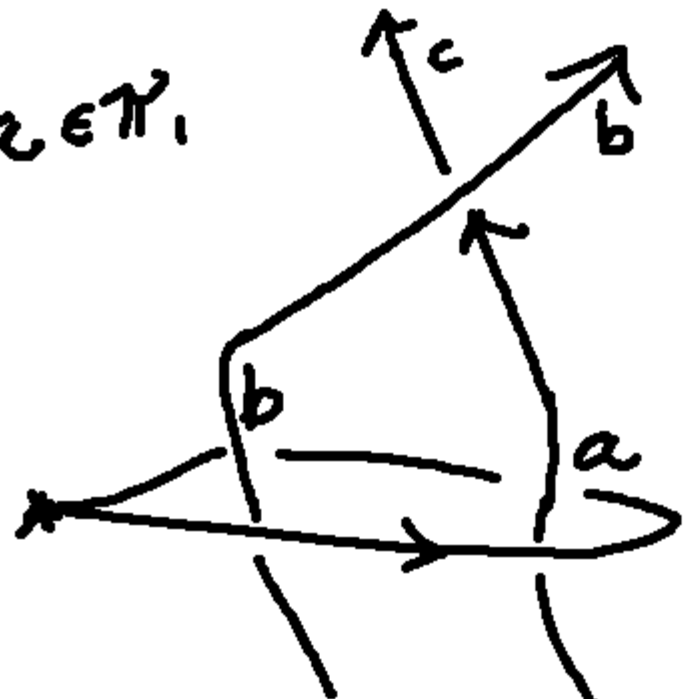
$\pi_1(S^3 - K, *)$

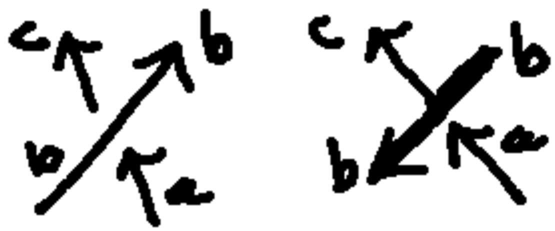


See book Intro to Alg. Topology by W. Massey.



$ab = bc$ in π_1

Van Kampen Theorem



$ab = bc$

$\boxed{c = b^{-1}ab}$

$ab^{-1} = b^{-1}c$

$\boxed{c = bab^{-1}}$

Fact: $\pi_1(S^3 - K)$:
- gens arcs
- 1 reln each crossing

K

1. $c = a^{-1}ba$
2. $b = c^{-1}ac$
3. $a = b^{-1}cb$

$G = \pi_1(S^3 - T) = (a, b, c \mid \quad)$.

$\pi_1(S^3 - O) \cong (a \mid \ ) \cong \mathbb{Z}$

Simplify: $b = (a^{-1}ba)^{-1} a (a^{-1}ba)$

$b = a^{-1}b^{-1}a\, \not{a}\not{a}^{-1} ba$

$a = b^{-1}a^{-1}bab$

$aba = bab$

$bab = aba$

$$G = (a, b \mid bab = aba)$$

repr perm by diag

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

This group is non abelian ∴ K is knotted.

$f : G \longrightarrow S_3$ ← permut of 3 objects

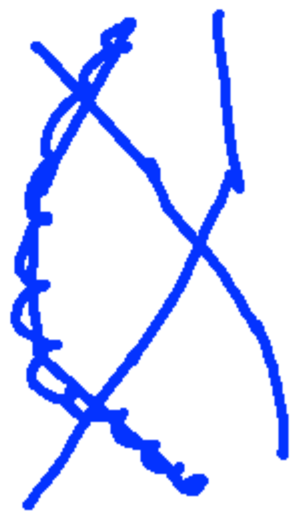
$a \longrightarrow X| = A$

$b \longrightarrow |X = B$

$ABA = BAB$

Claim

A =  , B = 

ABA =  $= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

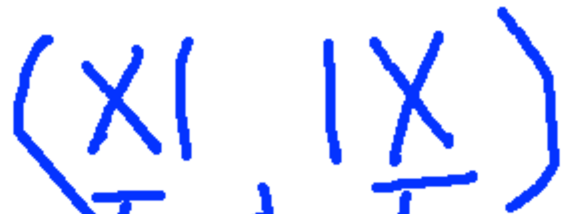
BAB =  $= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\{$  $\} = S_3$

$T_i =$ 

n strands

S_n



$i = 1, \ldots, n-1$

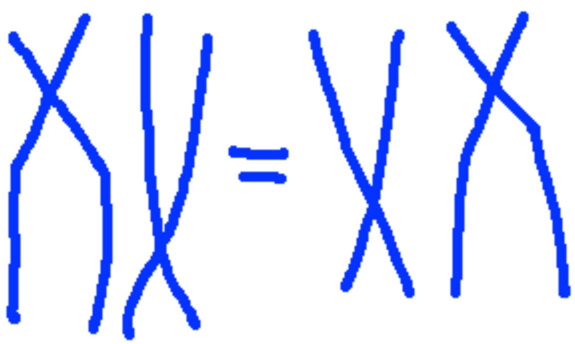
gens of S_n.

$$S_n = \left(T_1, \ldots, T_{n-1} \,\middle|\, T_i^2 = 1,\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ T_i T_j = T_j T_i \ (|i-j|>1) \right)$$

$i = 1, \ldots, n-1$

S_4

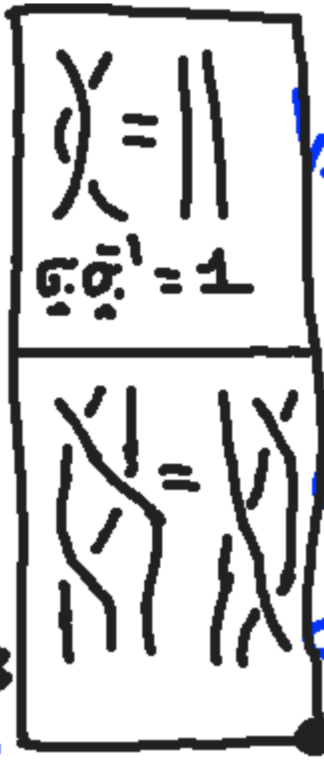
note



$T_1 T_3 = T_3 T_1$

Braiding we have .

$b \in B_3 \longmapsto \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$

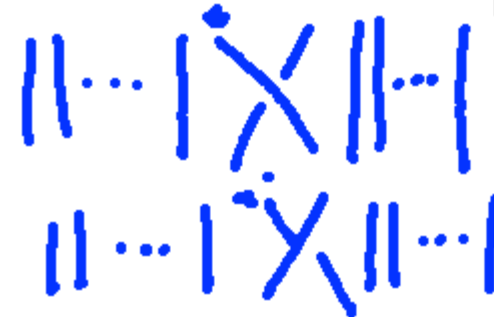


$b \in B_3$

3 strand braid group:

$\sigma_i =$ $|| \cdots | \times || \cdots |$

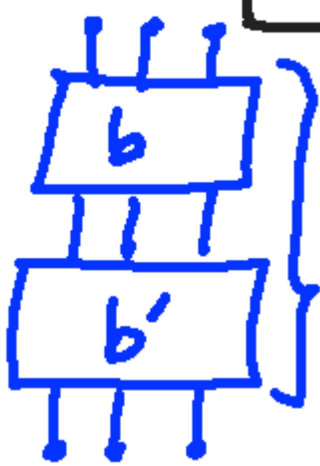
$\sigma_i^{-1} =$ $|| \cdots | \times || \cdots |$



$B_n \twoheadrightarrow S_n$ we present below

$\sigma_i \longmapsto T_i$

n strand B_n



B_n is a group under bb' thin mult.

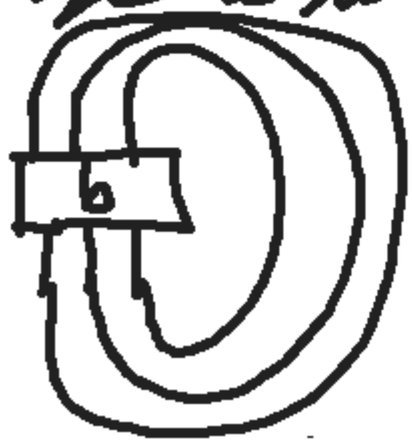
$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i,\ |i-j|>1 \right\rangle$$

presentation of the braid group B_n

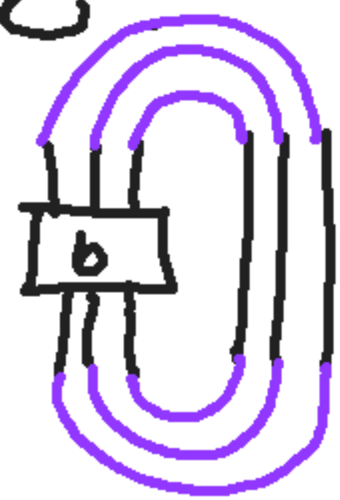
(Emil Artin)

Alexander Theorem

Every link is $\cong$ (isotopic) to the closure of a braid



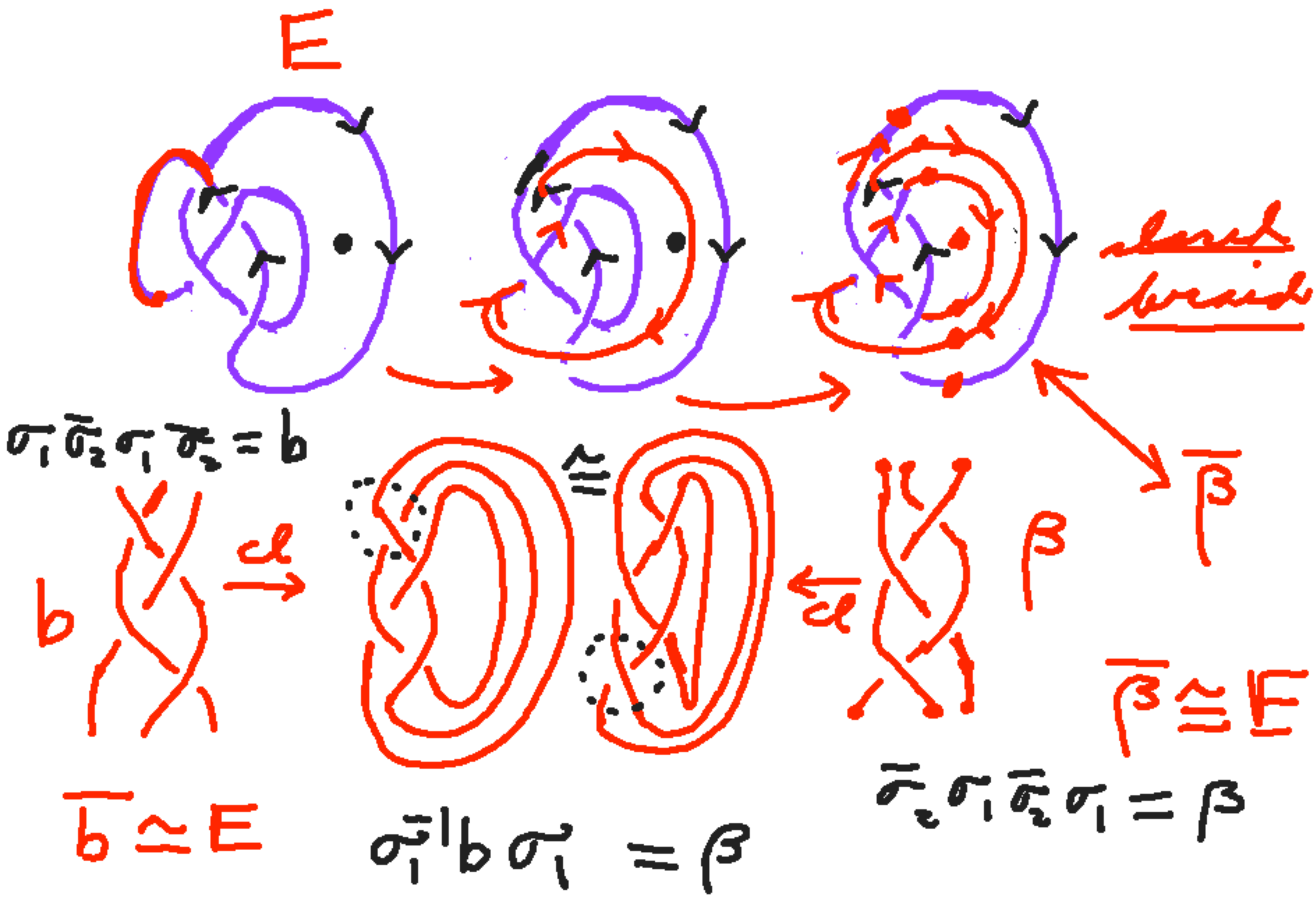
$\bar{b}$



Alexander proved it by choosing an axis & then improving the knot with respect to the axis.

$\cong$

figure eight knot


E
$\sigma_1 \bar{\sigma}_2 \sigma_1 \bar{\sigma}_2 = b$
b
≅
β
$\bar{\beta}$
$\bar{b} \simeq E$
$\sigma_1^{-1} b \sigma_1 = \beta$
$\bar{\sigma}_2 \sigma_1 \bar{\sigma}_2 \sigma_1 = \beta$
$\bar{\beta} \cong E$

Markov Theorem

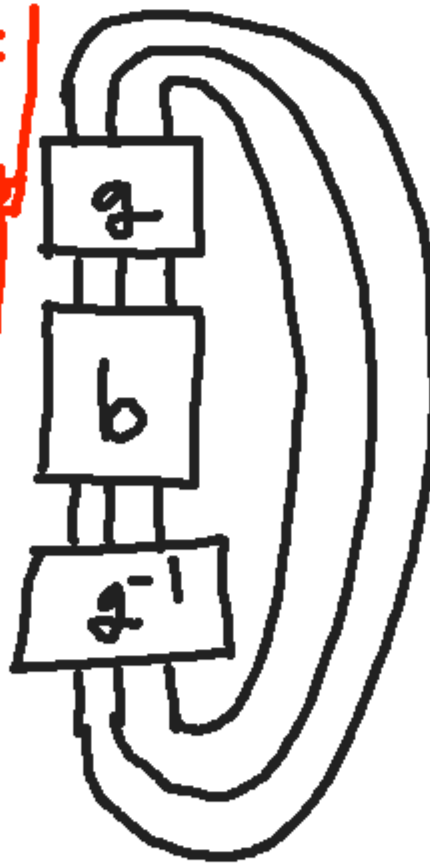
$b, b' \in B_n, B_m$

$\bar{b} \cong \bar{b'} \Leftrightarrow b \sim b'$ Markov Moves

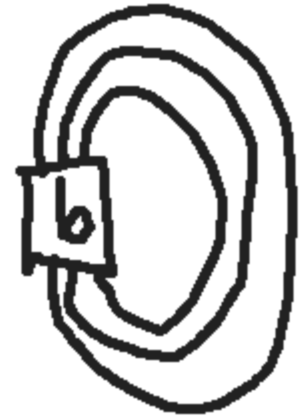
conjugate

$gbg^{-1} =$

$\overline{gbg^{-1}} \cong \bar{b}$



$\cong$



γ

$\in B_2$

$\bar{\gamma}$

Stabil

Markov Move

γ'

$\in B_3$

$\gamma' = \sigma_1^3 \sigma_2 \in B_3$

$\bar{\gamma'}$

$\underset{(R1)}{\cong} \bar{\gamma}$

$G = \pi_1 (S^3 -$  $) = (a, b \mid aba = bab)$

T ≅ B_3 three strand braids

! nice coincidence.

Related to the fact that $T = \text{Variety}(z^2 + w^3) \cap S^3$

$S^3 = \{ (z, w) \mid |z|^2 + |w|^2 = 1 \}$

$\text{Var}(z^2 + w^3 = 0)$ ↑ cx vars.

surface in $\mathbb{C} \times \mathbb{C} \cong \mathbb{R}^3$

Bracket State Sum

$\langle K \rangle$ polyn in A, B, d commute.

↑ knot or link diagram.

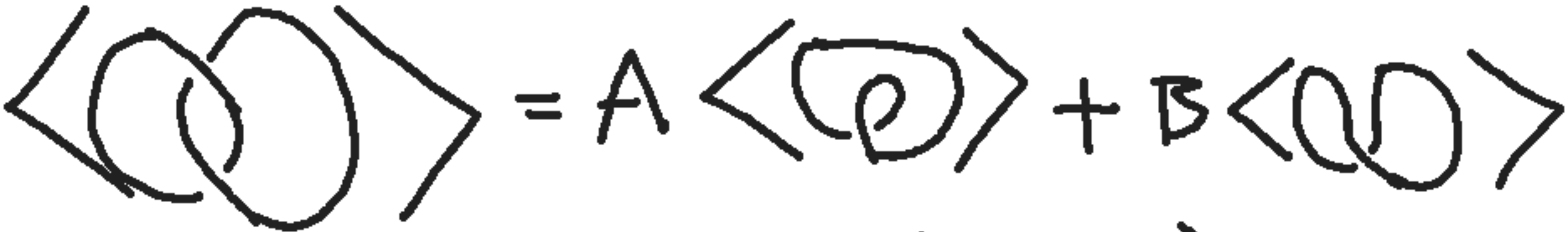
$\langle O\, K \rangle = d \langle K \rangle$

$\langle O \rangle = 1$

This is well-defined.

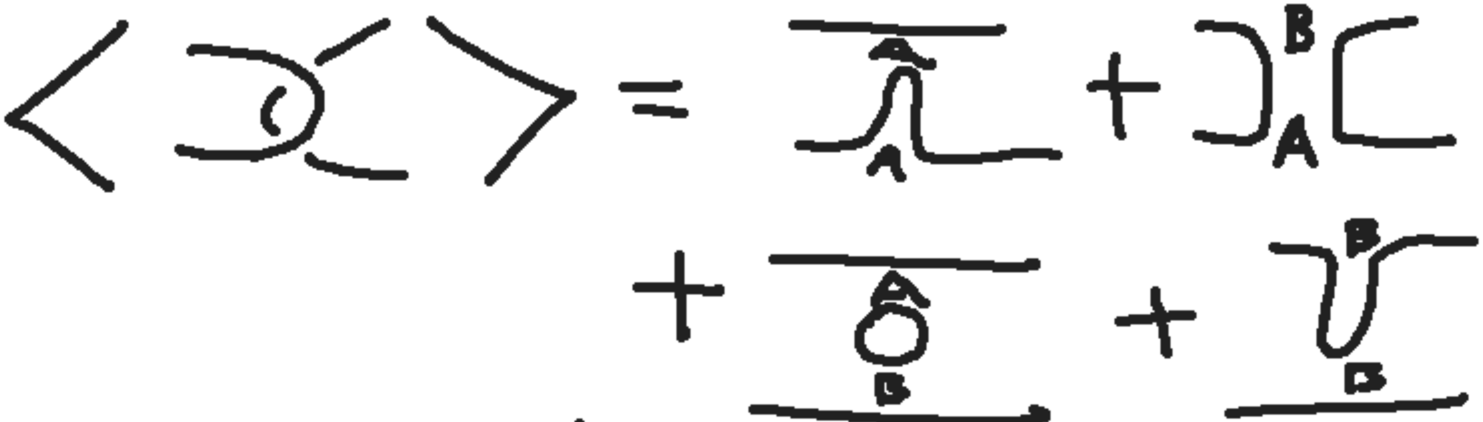
$= A + Bd \langle O \rangle$

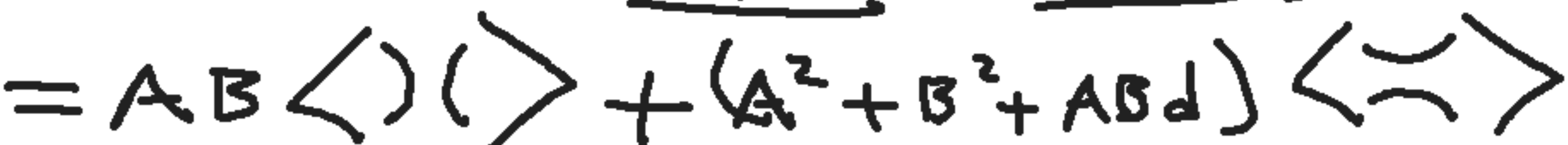
$= A + Bd$



$= A(Ad + B) + B(A + Bd)$

$= (A^2 + B^2)d + 2AB$



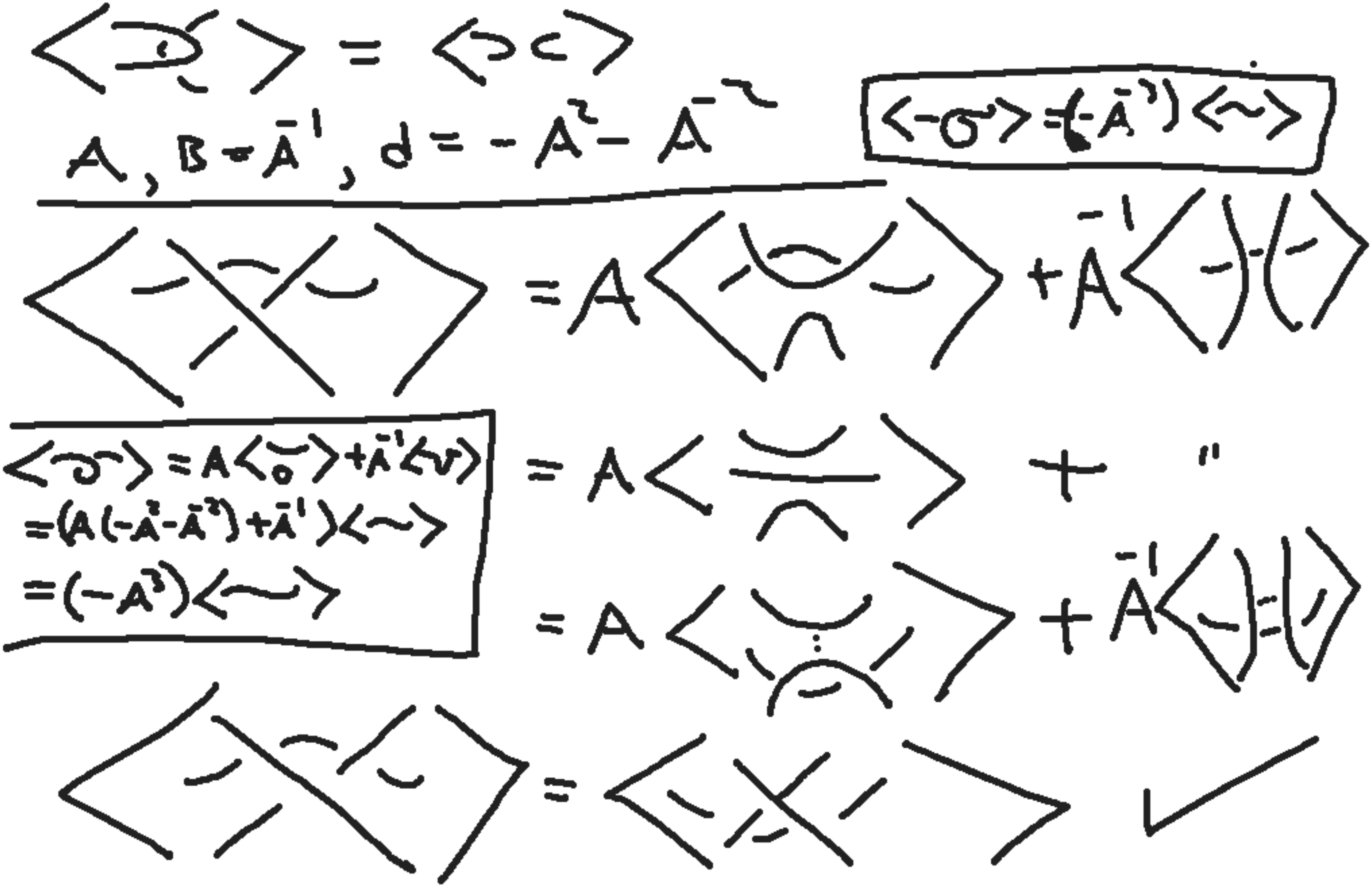


Take $B = A^{-1}$

$A^2 + A^{-2} + d = 0$

$*$

Then (*)


A, B = A^{-1}, d = -A^2 - A^{-2}
= (A(-A^2 - A^{-2}) + A^{-1})
= (-A^3)