When are the rows permutations?

$*$	a_1	a_2	a_3	$\dots$
b	$b*a_1$	$b*a_2$	$\dots$	

perm?

Columns are always perms:

$b*a = b'*a \iff (b*a)*a = (b'*a)*a \iff b = b'$.

$$b*a = b*a'$$

1) Suppose it's a Fox Color puzzle.

$$2a-b = 2a'-b$$

$$\implies 2a = 2a'$$

$$2(a-a') = 0$$

Moral: If $2 \nmid N$ ($\mathbb{Z}/N$ ring) $\implies a = a'$ when $b*a = b*a'$

$\therefore 2 \nmid N \implies$ rows are permutations.

2) $a * b = b a^{-1} b$.

Row problem: $a * b = a * b'$

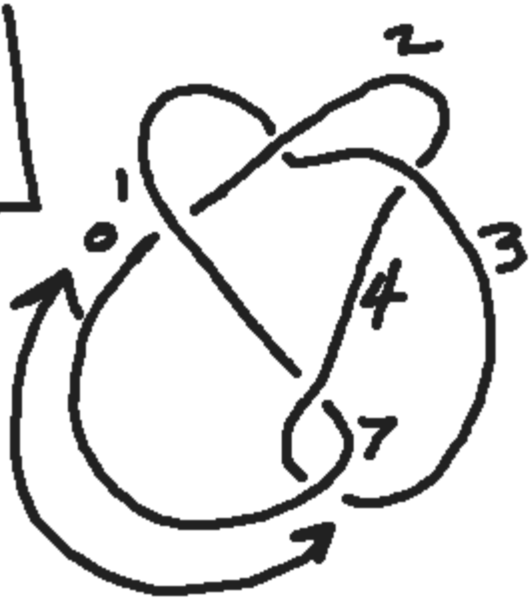
$$b a^{-1} b = b' a^{-1} b'$$

$$a^{-1} = \underline{b^{-1} b'} a^{-1} \underline{b' b^{-1}}$$

$$a^{-1} b b'^{-1} = b^{-1} b' a^{-1}$$

needs thought

5_2



$\mathbb{Z}_7$

$\{0, 1, 2, 3, 4, 5, 6\}$

6_2

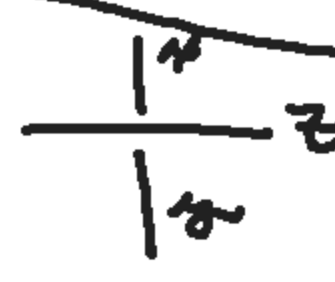
$\mathbb{Z}_{11}$

5 new elts

Conj. (LK & Harary) K alt & modulus $N = p$ prime $\Rightarrow$ all colors on min alt diag of K are distinct!

Pvd in general by Matman & Solis.

1. $2a - b = c$
2. $2c - a = b$
3. $2b - a = c$



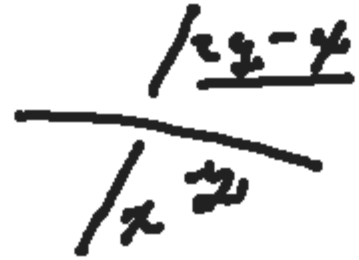
$x + y - 2z = 0$

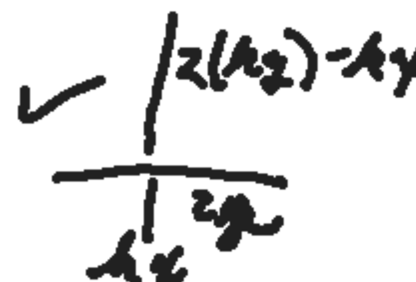
$$M \quad \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 2 & -1 & -1 \\ 2 & -1 & -1 & 2 \\ 3 & -1 & 2 & -1 \end{array}$$

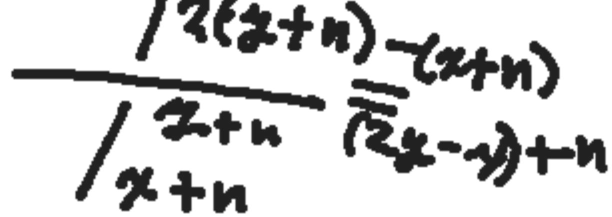
$$\boxed{\begin{matrix} -1 & 2 \\ 2 & -1 \end{matrix}}$$

$\mathbb{Z}_3$

We can choose one edge and call its color zero.







$$M = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \xrightarrow[\det]{} 1-2 = \underline{\underline{-3}}.$$

$a = 0$ by assump

$$M\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ mod N some N.

on computer
$\tilde{M} = \det(M) \cdot M^{-1}$

Given M square, $\exists\ \tilde{M} =$ adjoint of M

s.t. $M\tilde{M} = \det(M)\, \underline{I} \leftarrow$ id matrix.

e.g. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \tilde{M} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\tilde{M}M \equiv 0$ when work over $\mathbb{Z}_N$
$N = \det(M)$.

$M\begin{bmatrix} \text{column} \\ \text{of} \\ \tilde{M} \end{bmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ mod N.

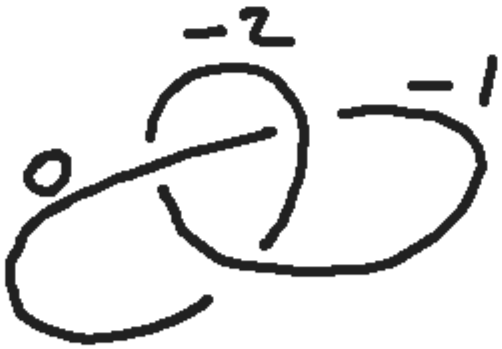
So Any Column of $\tilde{M}$ is a coloring solution.

$$M = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} d & -b \\ -c & e \end{pmatrix}$$

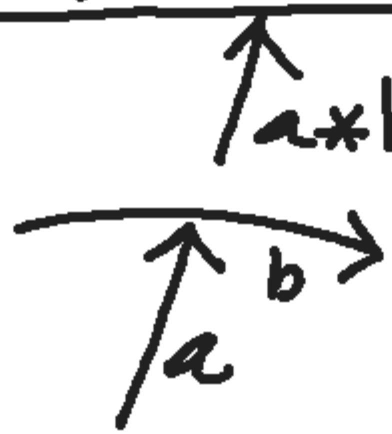
$$\tilde{M} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$$

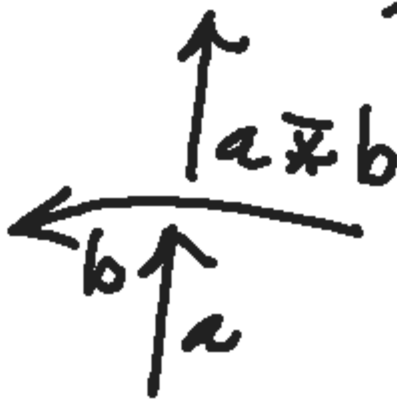


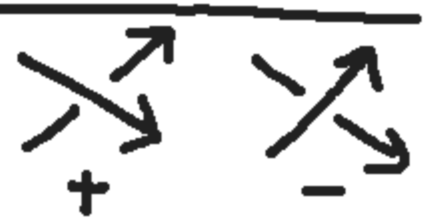
mod 3

Fact. $|\det(M)|$ invar. of the knot.

Oriented Quandles





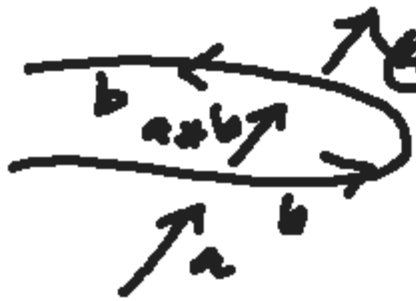


I

$a*a=a$

$a\bar{*}a=a$

II



$(a*b)\bar{*}b = a$ & $(a\bar{*}b)*b=a$

III. $(a*b)*c = (a*c)*(b*c)$

$(a\bar{*}b)\bar{*}c = (a\bar{*}c)\bar{*}(b\bar{*}c)$

__Example.__ G group

define $a * b = b^{-1} a b$

$a \bar{*} b = b a b^{-1}$

---

i) $a * a = a^{-1} a a = a$

$a \bar{*} a = a a a^{-1} = a$ ✓

ii) $(a * b) \bar{*} b = b^{-1} (b a b^{-1}) b = a$

iii) $(a * b) * c = c^{-1} (b^{-1} a b) c$

$(a * c) * (b * c) = (b * c)^{-1} (c^{-1} a c)(b * c) =$

$= (c^{-1} b c)^{-1} (c^{-1} a c)(c^{-1} b c)$

$= (c^{-1} b^{-1} c)(c^{-1} a c)(c^{-1} b c) = c^{-1} b^{-1} a b c$

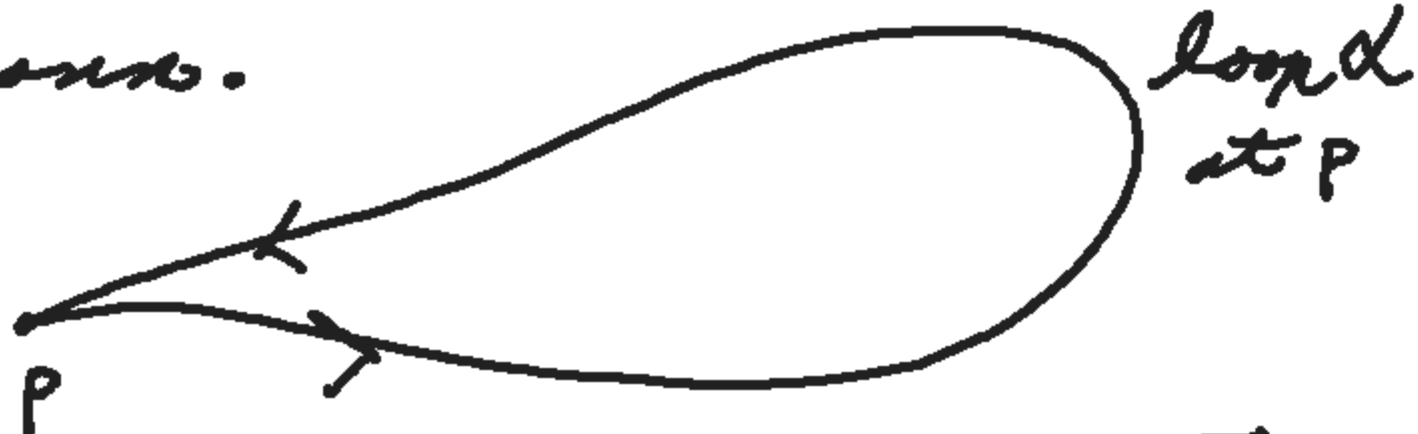
Fundamental Group of a topological space X.

Need a chosen pt $p \in X$ (basept) & assume X is path conn.

$\alpha : [0,1] \to X$

continuous

$\alpha(0) = \alpha(1) = p$



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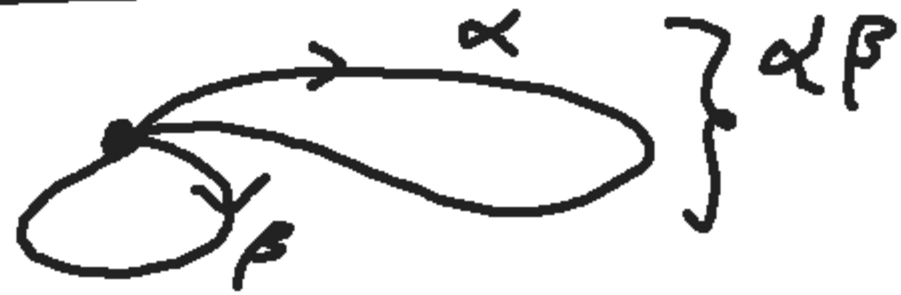
multiply loops:

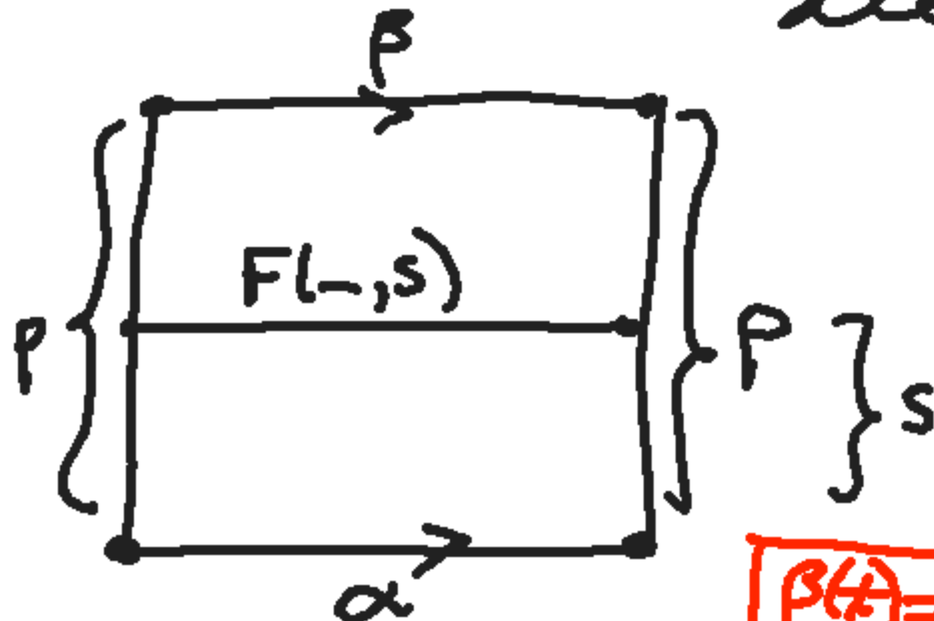


Two loops at p are $\boxed{\alpha, \beta}$ **homotopic** $\boxed{\alpha \sim \beta}$ if $\exists\, F : [0,1] \times [0,1] \to X$ continuous

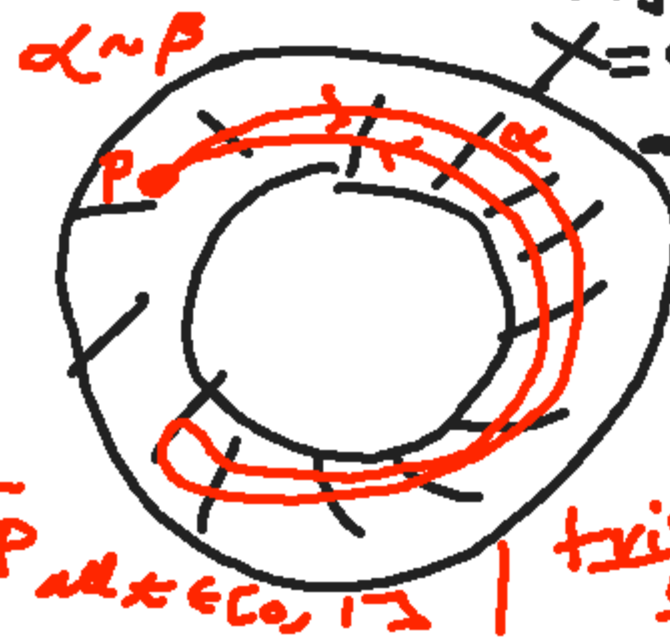
s.t. $F(t,0) = \alpha(t)$

$F(t,1) = \beta(t)$

$+\ F(t,s)$ is a loop at p all $s \in [0,1]$



e.g. $X = S^1 \times [0,1]$ annulus



$\boxed{\beta(t) = p}$ all $t \in [0,1]$

trivial loop

$\mathcal{L}(X,p)$ = all loops at p on X

$= \left\{ \alpha : [0,1] \to X \;\middle|\; \begin{array}{l} \alpha(0) \\ = \alpha(1) \end{array} \right\}$

$\pi_1(X,p) = \mathcal{L}(X,p) / \sim$

$\sim$ = equiv reln homotop of loops.

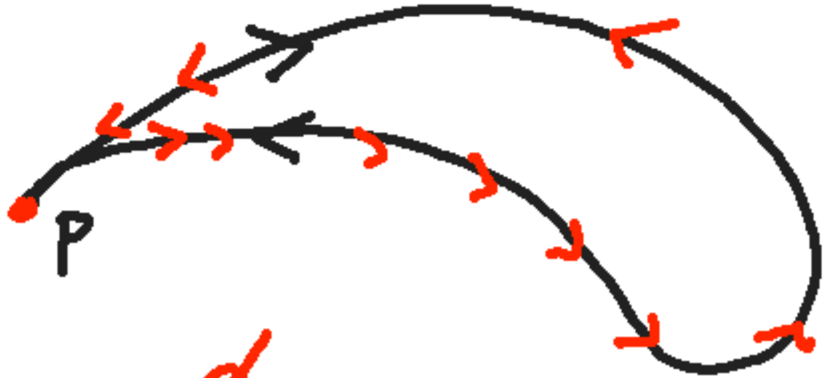
$\pi_1(X,p)$ is
- a group
- an invariant of X.
- $h : X \longrightarrow Y$ conts

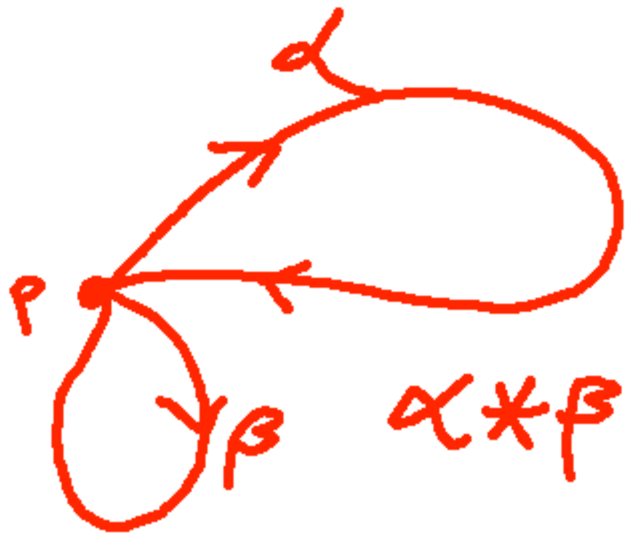
$\Rightarrow \exists\, h_* : \pi_1(X,p) \longrightarrow \pi_1(Y, h(p))$

- h_* isomorphism when h is a homeom.

$\alpha(t)$ $\quad t \in [0,1]$

$\overline{\alpha}(t) = \alpha(1-t)$



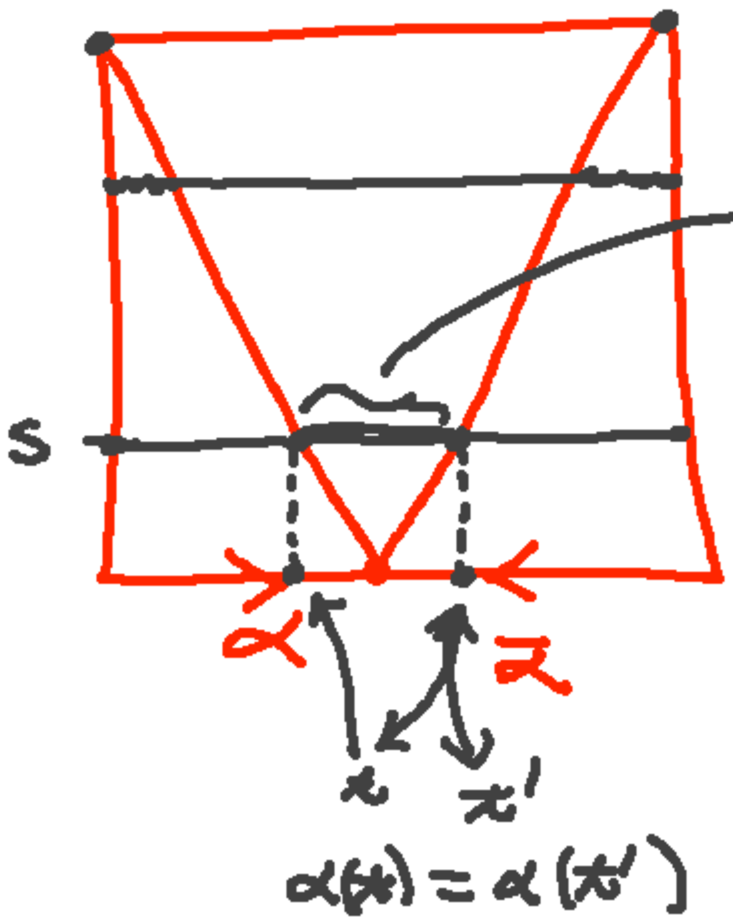


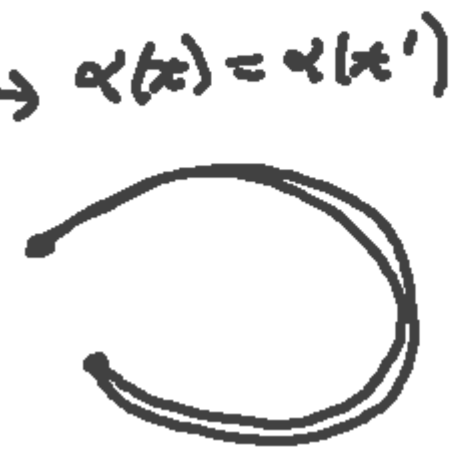
$\alpha * \beta$

$\alpha * \beta \sim \gamma$

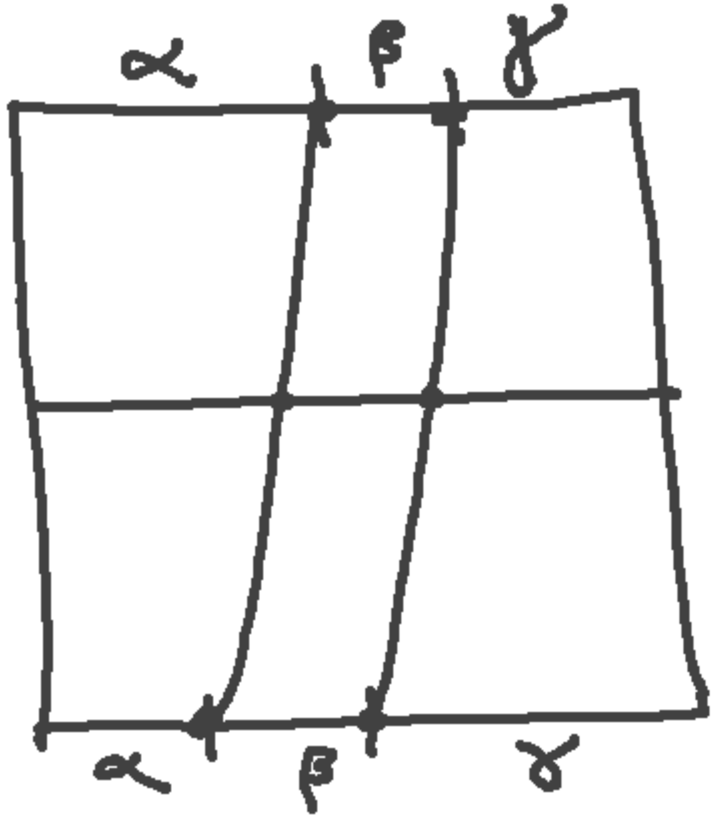
$\alpha\overline{\alpha} \sim$

$\alpha * \overline{\alpha} \sim e$ $\quad \boxed{e(t) = p \ \forall t}$





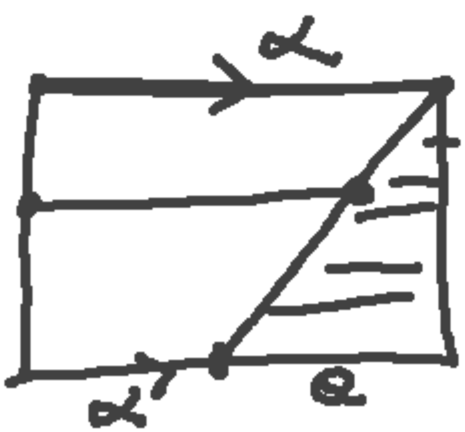
Construct the homotopy $\alpha * \bar{\alpha} \sim e$.



$\alpha * \beta : [0,1] \to X$

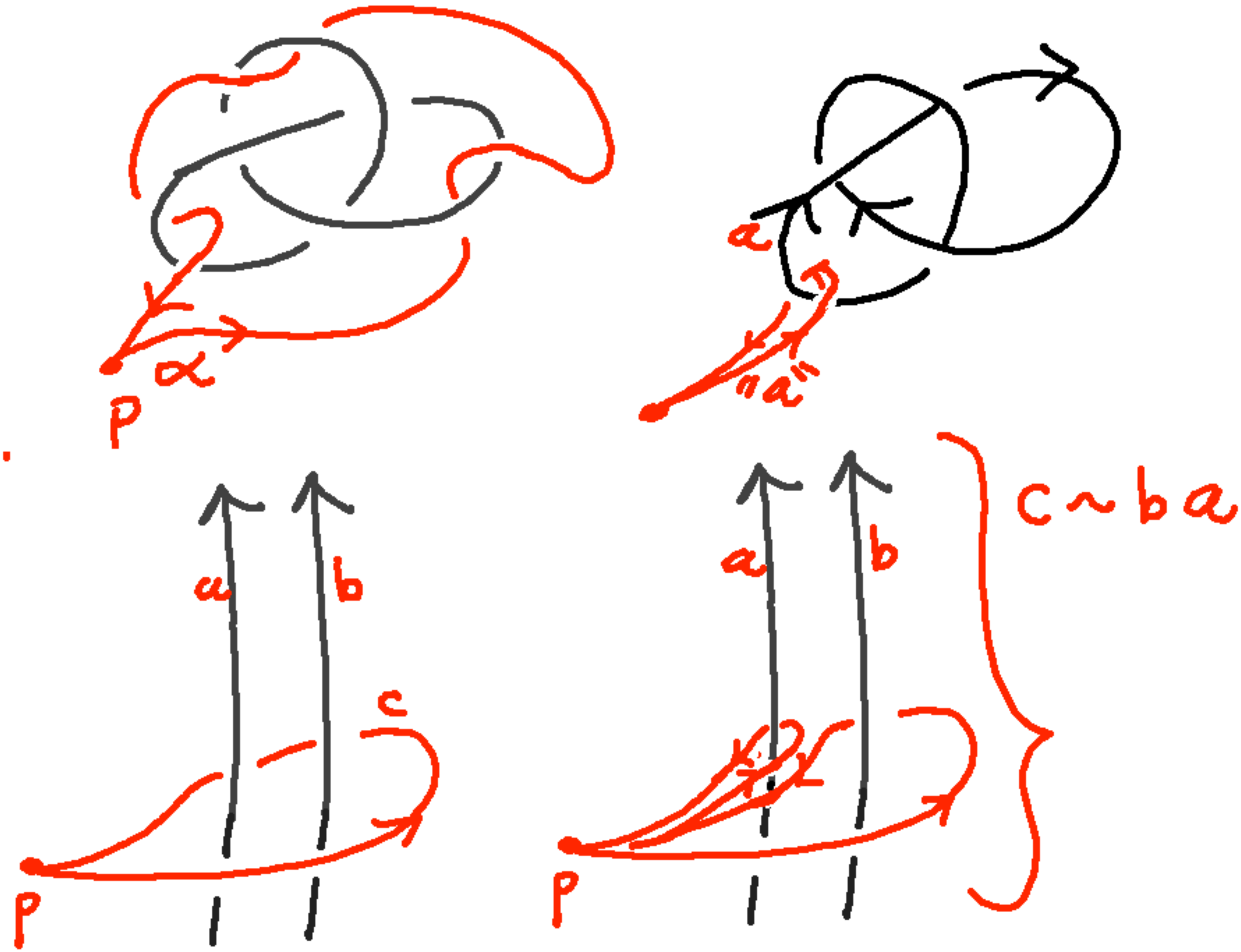
$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$\underline{(\alpha * \beta) * \gamma} \sim \underline{\alpha * (\beta * \gamma)}$



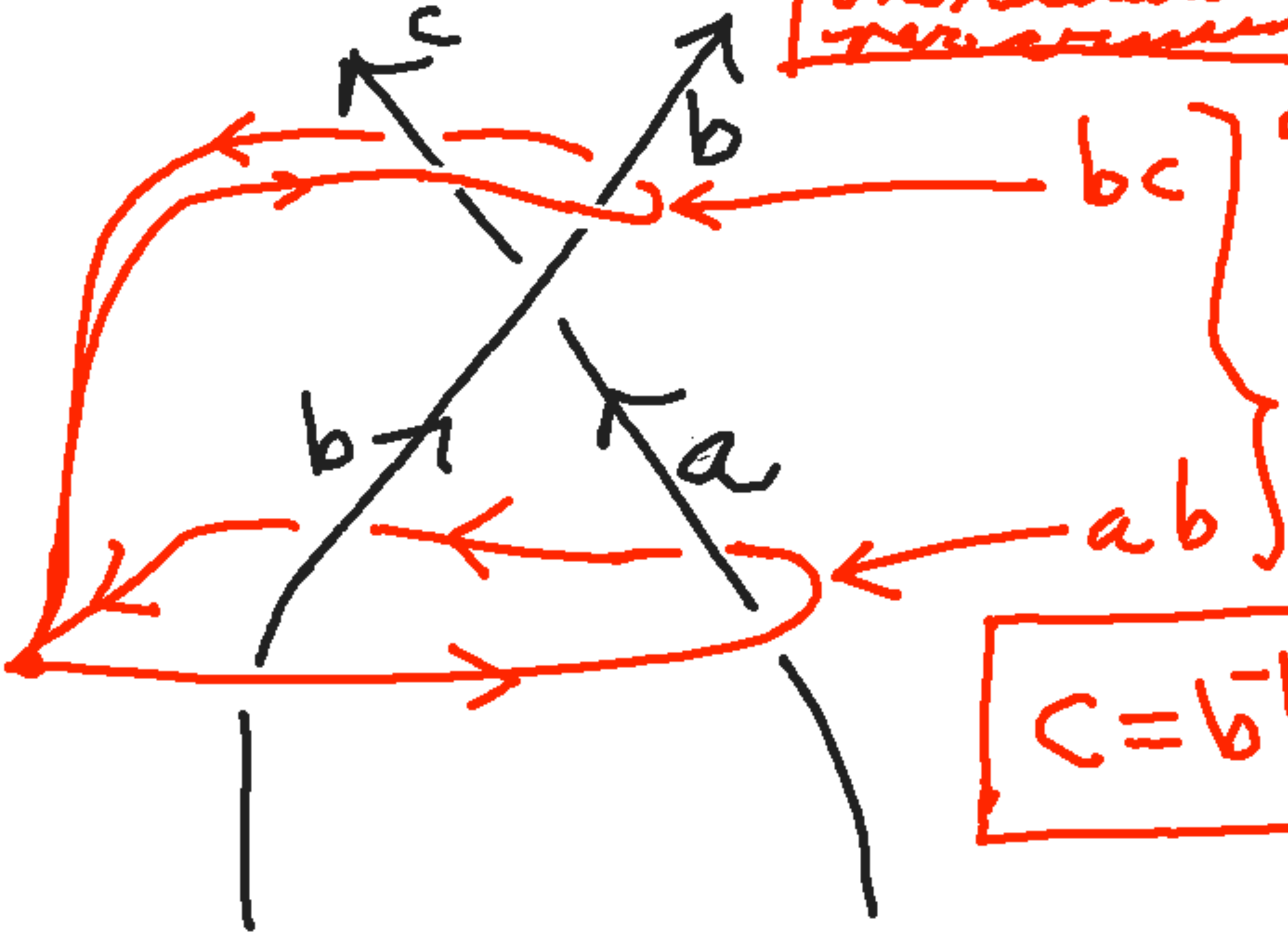
$\forall \; \underline{e * \alpha \sim \alpha}$

$\alpha * e \sim \alpha$


p
α
a
"a"
a
b
c
p
a
b
p
c ~ b a

Fact: $\pi_1(S^3 - K)$ gen by loops $\leftrightarrow$ arcs in a diagram.

+ $\boxed{\text{one relation per crossing}}$



$ab = bc$ in $\pi_1(S^3 - K)$

$$\boxed{c = b^{-1}ab}$$

$$\boxed{c = b^{-1} a b}$$

$$b = c^{-1} a c$$

$$a = b^{-1} c b$$

$$G = \pi_1(S^3 - T) = \langle a, b, c \mid \qquad \rangle .$$

<u>Exer</u>: Show $\underline{G \cong \langle a, b \mid aba = bab \rangle}$