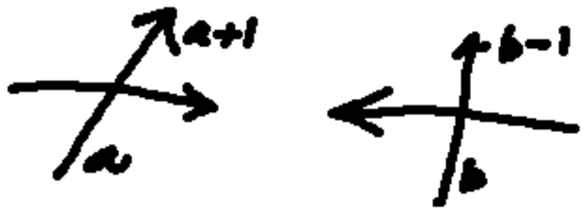
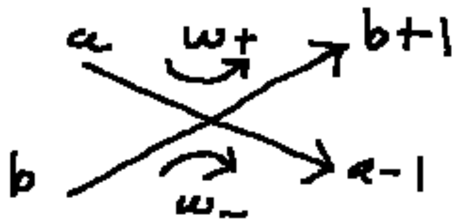
An Invariant of Flat Virtual Knots
by LHK

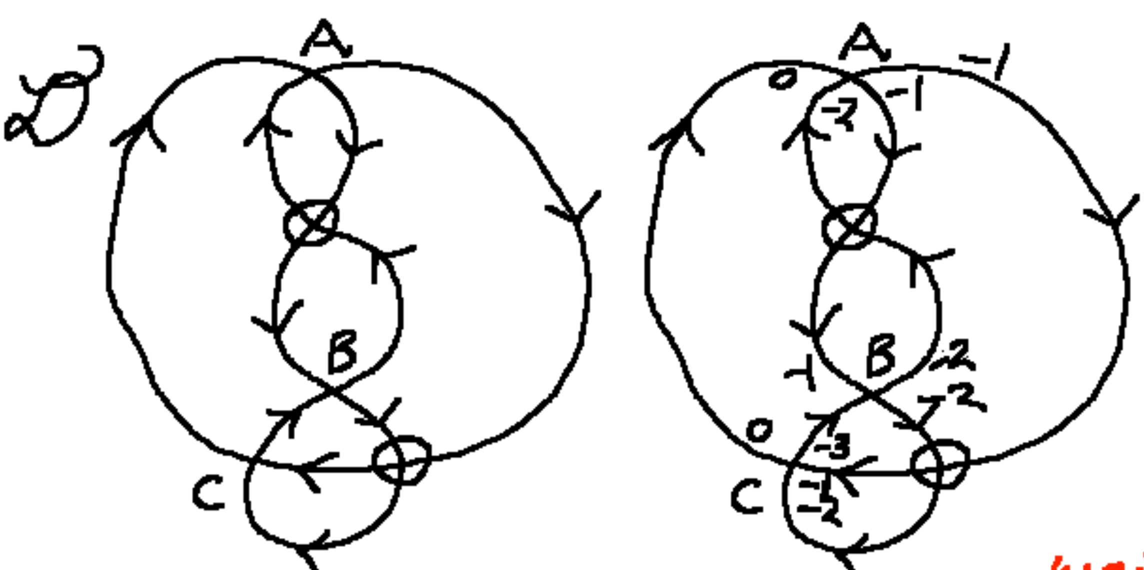
1. 

integer labelling



$$w_+ = a - b - 1$$
$$w_- = b - a + 1 = -w_+$$

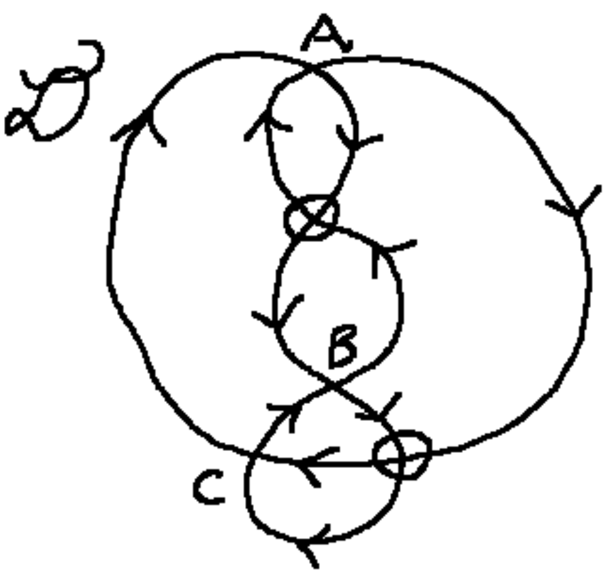
$\mathcal{D}$

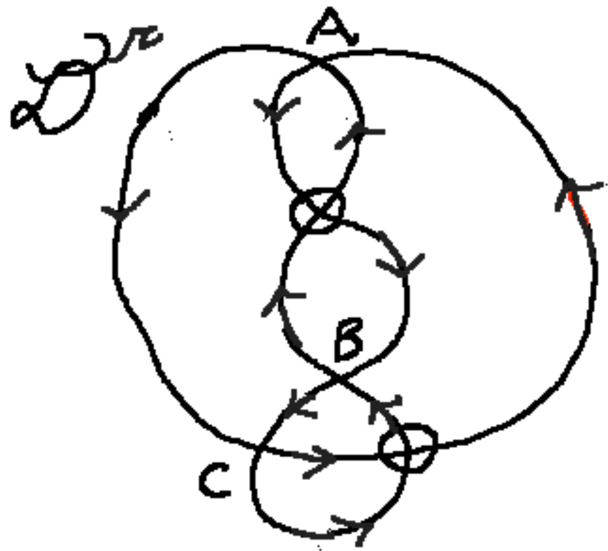


$W(\mathcal{D})$

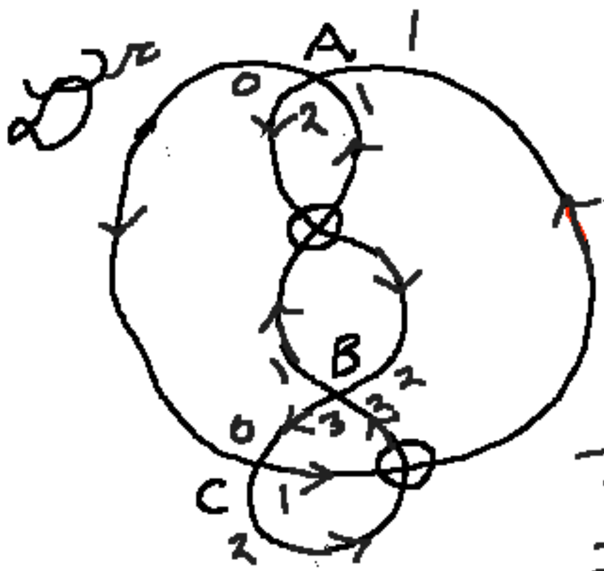
	w_+	w_-
A	1	-1
B	1	-1
C	-2	2

weight table of $\mathcal{D}$





D^r = D with reversed orientation



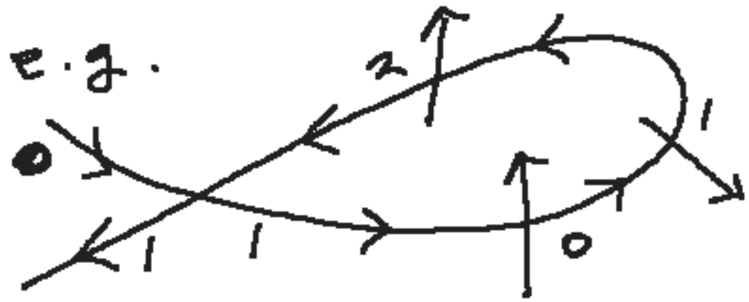
	w_+	w_-
A	-1	1
B	-1	1
C	2	-2

$W(D^r)$

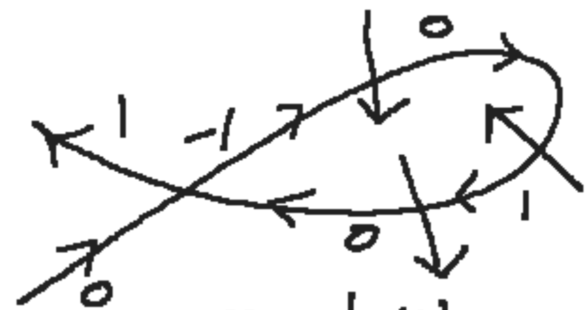
<u>Proposition</u>. $W(D^r) = \widehat{W}(D)$

where $\widehat{W}(D)$ is $W(D)$ with columns w_+ and w_- exchanged.

Proof. e.g.

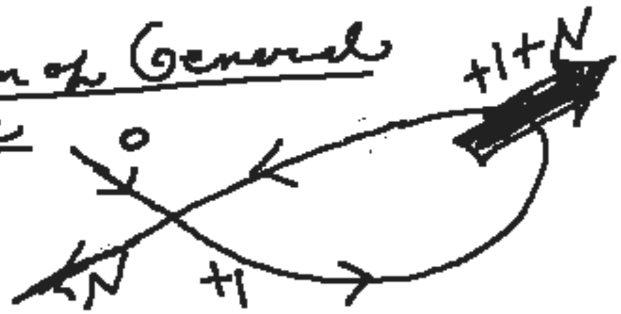


ω_+	ω_-
1	-1

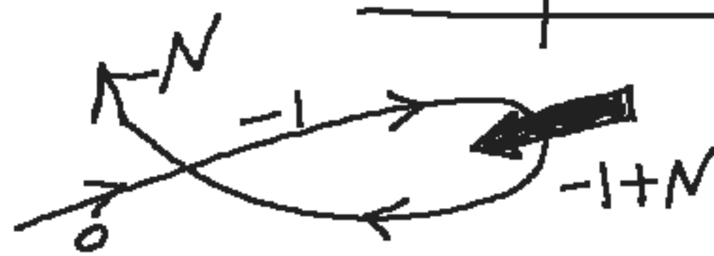


ω_+	ω_-
-1	1

Form of General Case



ω_+	ω_-
N	$-N$

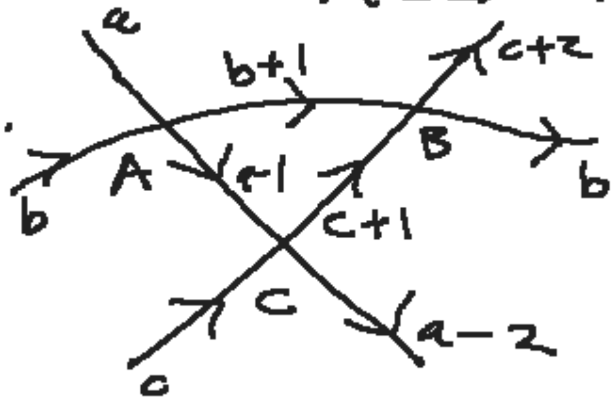


ω_+	ω_-
$-N$	N

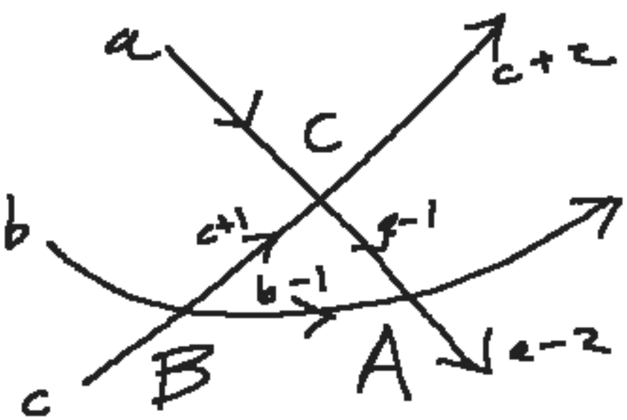
QED

Proposition. $W(D)$ is unchanged by Reidemeister 3 moves.

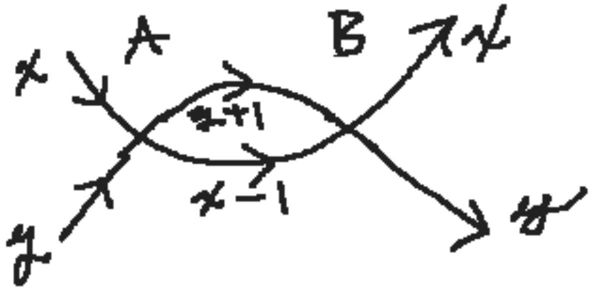
Proof.



	w_+	w_-
A	$a-b-1$	$b-a+1$
B	$b-c-1$	$c-b+1$
C	$a-c-2$	$c-a+2$



QED

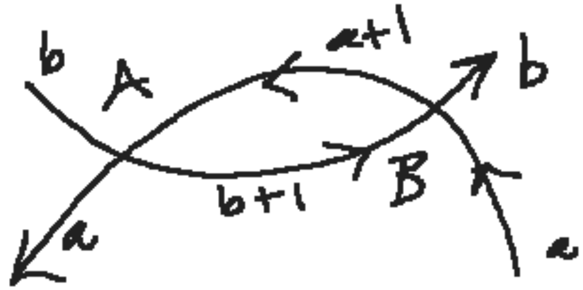


	w_-	w_+
A	$x-y-1$	$y-x+1$
B	$y-x+1$	$x-y-1$

T

A	m	$-m$
B	$-m$	m

Note that $\hat{T} = T$.



	w_+	w_-
A	$a-b$	$b-a$
B	$b-a$	$a-b$

In all cases of R2, $W(D)$ changes by adding or removing a sub-table of the form T above, with $\hat{T} = T$.

	w_+	w_-
A	0	0

S

R1 adds or subtracts a row of zeros $S + \hat{S} = S$.

Proposition. The property $W \neq \widehat{W}$ is invariant under the Reidemeister and detour moves.

Proof. This follows from the previous discussion. //

Proposition. If $W(\mathcal{D}) \neq \widehat{W}(\mathcal{D})$ then $\mathcal{D} \not\cong \mathcal{D}^r$.

Proof. If $W(\mathcal{D}) \neq \widehat{W}(\mathcal{D})$ then $W(\mathcal{D})$ has a row of form $(n, -n)$ $(n \neq 0)$ & no row of form $(-n, n)$.

Equiv under R2 adds and subtracts pairs of rows of form $\begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$.

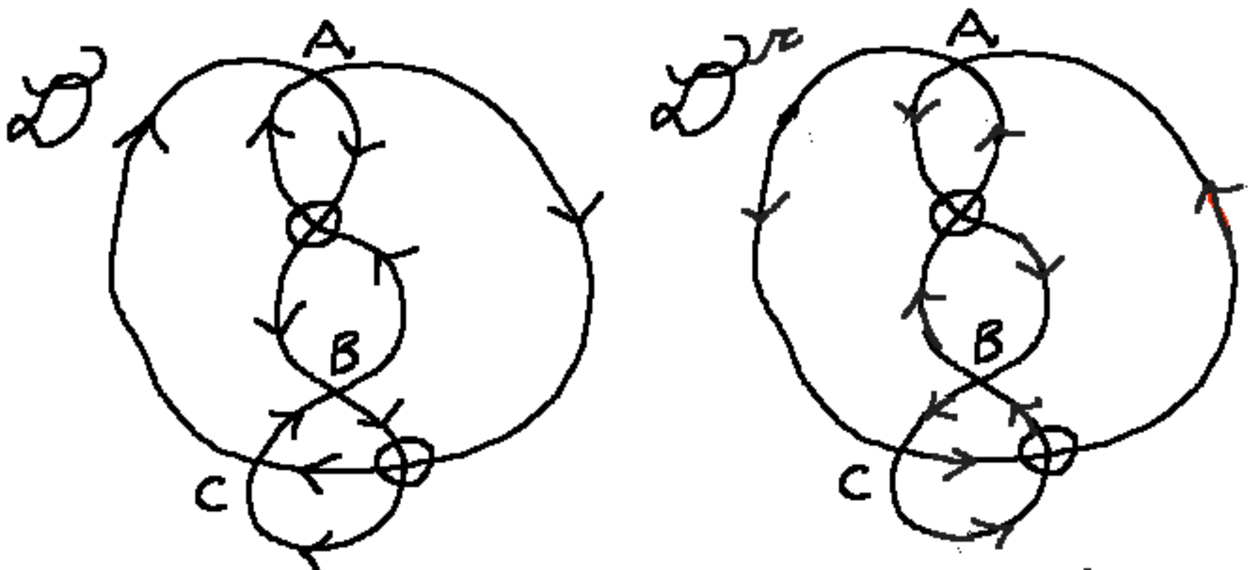
Thus one could go from

$$(n,-n) \rightsquigarrow (n,-n)$$
$$(-n,\ n)$$
$$(n,-n)$$

If it were possible to change $(n,-n) \rightsquigarrow (-n,n)$ by such a process it could only occur by adding + subtracting matrices of the form $\begin{pmatrix} -n & n \\ n & -n \end{pmatrix}$ and this is not possible. So we conclude that if $W(\mathcal{D}) \neq \hat{W}(\mathcal{D})$ then $\mathcal{D}$ is inequivalent to $\mathcal{D}^{r}$.

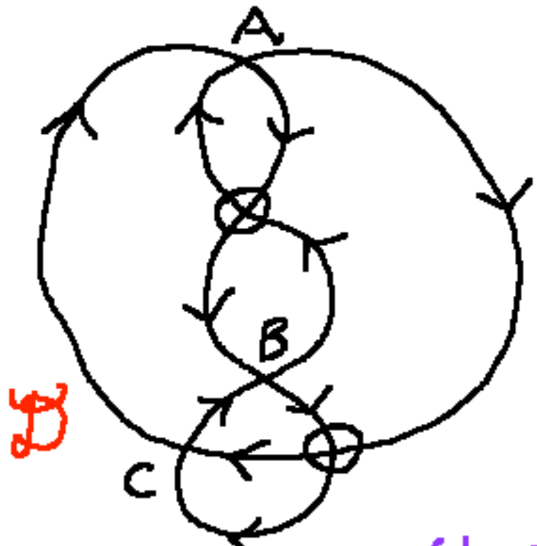
QED

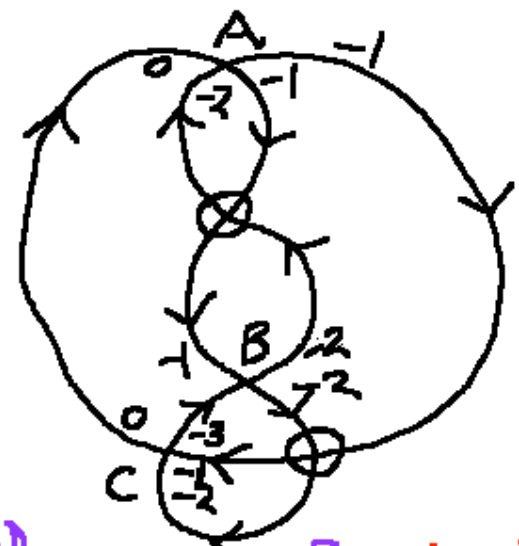
Thus we conclude that our example D below is not equivalent to D^r.



<u>Remark</u>. One can often use the affine index polynomial $P_K(t)$ to show that a virtual knot K is inequiv to K^r, via

$$P_{K^r}(t) = P_K(t^{-1}).$$





	w_+	w_-
A	1	−1
B	1	−1
C	−2	2

weight table of $\mathcal{D}$: $W(\mathcal{D})$

Here $Q_{\mathcal{D}}(t) = 2(t^{1} - t^{-1}) + t^{2} - t^{-2}$.

$$Q_{\mathcal{D}}(t) \underset{\text{defn}}{=} \sum_{c} t^{w_+(c)} - t^{w_-(c)}.$$

It follows immediately from our analysis of the weight table that

1) $Q_{\mathcal{D}}$ is a invariant of flat $\mathcal{D}$.

2) (Note $\begin{array}{|c|c|}\hline n & -n \\ \hline -n & n \\ \hline\end{array}$ contributes ϕ to $Q_{\mathcal{D}}$.

3) $Q_{\mathcal{D}}(t^{-1}) = -Q_{\mathcal{D}}(t)$.

$Q_{\mathcal{D}^r}(t) = Q_{\mathcal{D}}(t^{-1})$.

4) $\therefore$ $Q_{\mathcal{D}}(t) \neq 0 \Rightarrow \mathcal{D} \not\approx \mathcal{D}^r$

We have $Q_{\mathcal{D}}(t) = \sum_{c} \left(t^{w_+(c)} - t^{w_-(c)} \right)$

This contains all the information in $W(\mathcal{D})$ up to R1, R2, R3 for flat virtuals & is a best way to express our results that

$Q_{\mathcal{D}}(t) \neq \phi \Longrightarrow \mathcal{D} \not\simeq \mathcal{D}^{r}$.

The fact that $P_K(t) - P_K(t^{-1})$ is a flat invar was observed by Cheng [arXiv: 1606.01446v1] but not applied to reversibility of flats. Here we work directly in the flat category.

$$Q_{\mathcal{D}}(t) = \sum_{c} t^{w_+(c)} - t^{w_-(c)}$$

is an invariant of flat virtuals.

$Q_{\mathcal{D}^r}(t) = Q_{\mathcal{D}}(t)$.

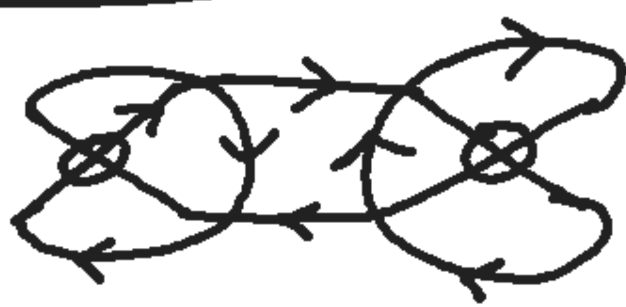
This is a best summary of what now know about detecting reversibility for flat virtuals.

<u>Remark</u>. One can show that $Q_{\mathcal{D}}(t)$ is a concordance invariant for flat virtuals. In this sense <u>irreversibility is preserved under concordance</u>.

Question. $\exists$? D flat irrev + D' flat rev s.t. $D \underset{\text{concordant}}{\sim} D'$?

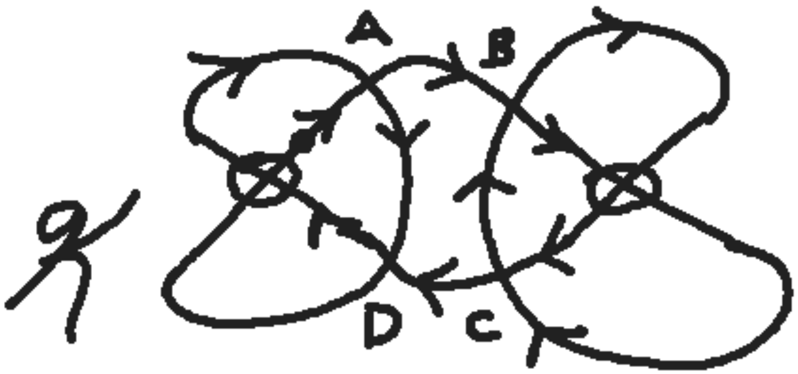
Question. Classify flat virtuals up to concordance.

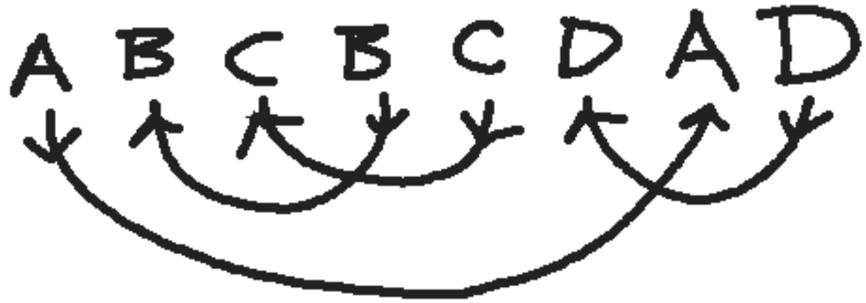
Question. Is Kishino K irrev?

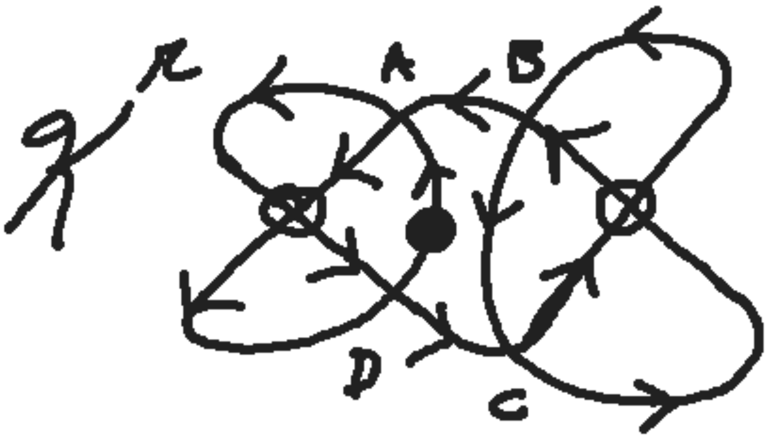


K Note $Q_K(t) = \phi$

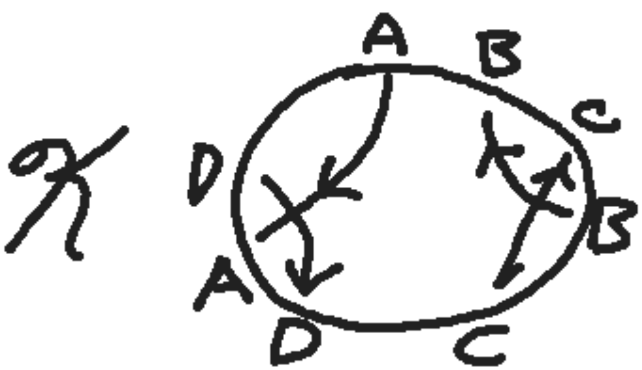
and $K \underset{\text{concord}}{\sim} \bigcirc$ $U \cong U^r$.

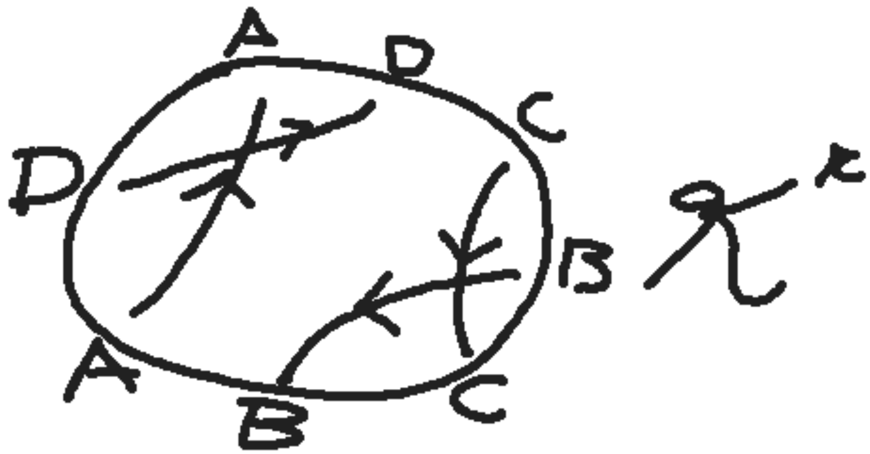

A
B
D
C


A B C B C D A D


A
B
D
C


A D C B C B A D


A
B
C
B
C
D
A
D


A
D
C
B
C
B
A
D

<u>Kadokami</u> [Teruhisa Kadokami, Detecting non-triviality of virtual links, JKTR, Vol. 12, No. 6 (2003), 781–803.]

shows that any flat virtual knot has a minimal genus diagram (in the sense of a surface on which the knot lives as e.g. a virtual knot lives on a torus:

.

<u>and</u> if D and D′ are two min genus flat diagrams then ∃ a sequence of R3 moves (only!) taking D to D′.

Kadokami's Theorem

$\Rightarrow$ If D and D' are min genus diagrams for $D + D'$ then $c(D) = c(D')$ where $c(X) = \#$ of classical crossings in X.

__Question.__ Does $g_s(D)$ min $\Rightarrow c(D)$ minimal?
This may true in general but needs some thought.

__Theorem.__ Suppose $W(D)$ has no zero rows + no sub matrices of form $\begin{bmatrix} n & -n \\ -n & n \end{bmatrix}$. Let $N = \#$ rows$(W(D))$.
$\Rightarrow c(D)$ is minimal for D's flat type.

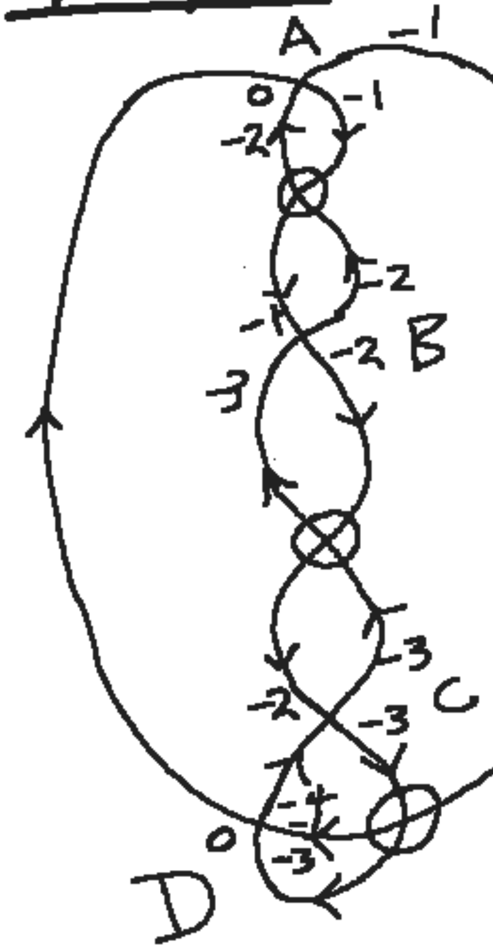
Proof of Theorem. As we have observed if D satis the hypotheses of Thm, then no combination of R1, R2 can reduce N (and R3 always leaves N fixed). This proves the Theorem. //

Thm. If W(D) has no zero rows and no R2 submatrices then $Q_D(t) \neq 0$ and N = sum of the positive coeffs of $Q_D(t)$. Then $N = \min c(D')$ over all $D' \sim D$.

and $D \not\simeq D^r$.

Question: Do our hypotheses ⇒ D has minimal genus?

Example.



$\mathcal{K}$

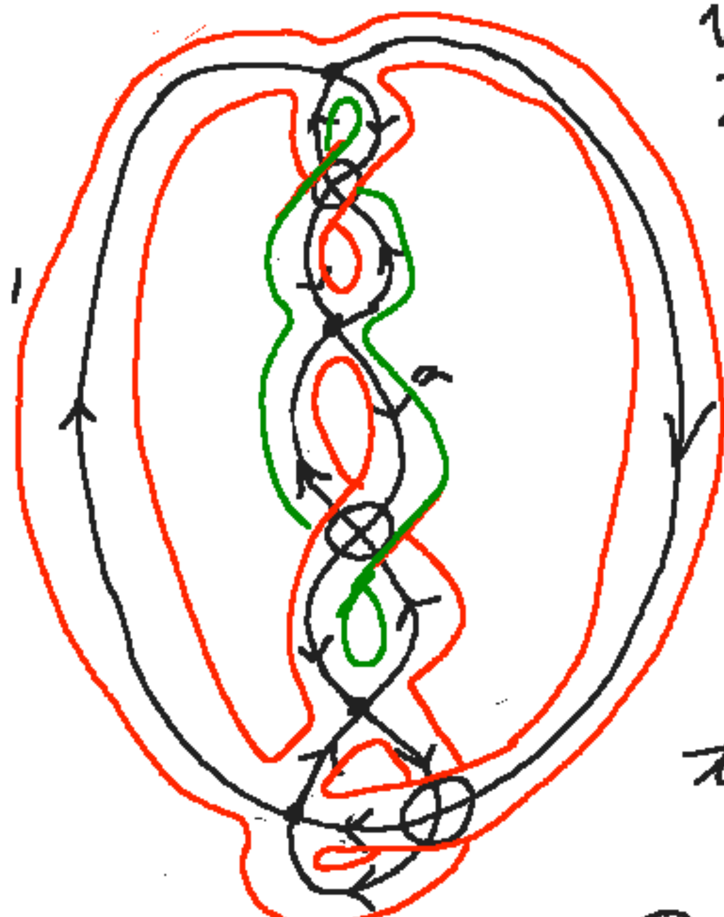
	w_+	w_-
A	1	-1
B	1	-1
C	1	-1
D	-3	3

$\Rightarrow c(\mathcal{K}) = 4 = \underline{\min c}$

$\mathcal{K} \neq \mathcal{K}^r$

$$\boxed{Q_{\mathcal{K}}(t) = 3t + t^{-3} - t^{3} - 3t^{-1}}$$

check genus on next page.



$v = 4$

$\lambda = 2$

$$g = \frac{v-\lambda}{2} + 1$$

$$= \frac{4-2}{2} + 1$$

$$g = 2$$

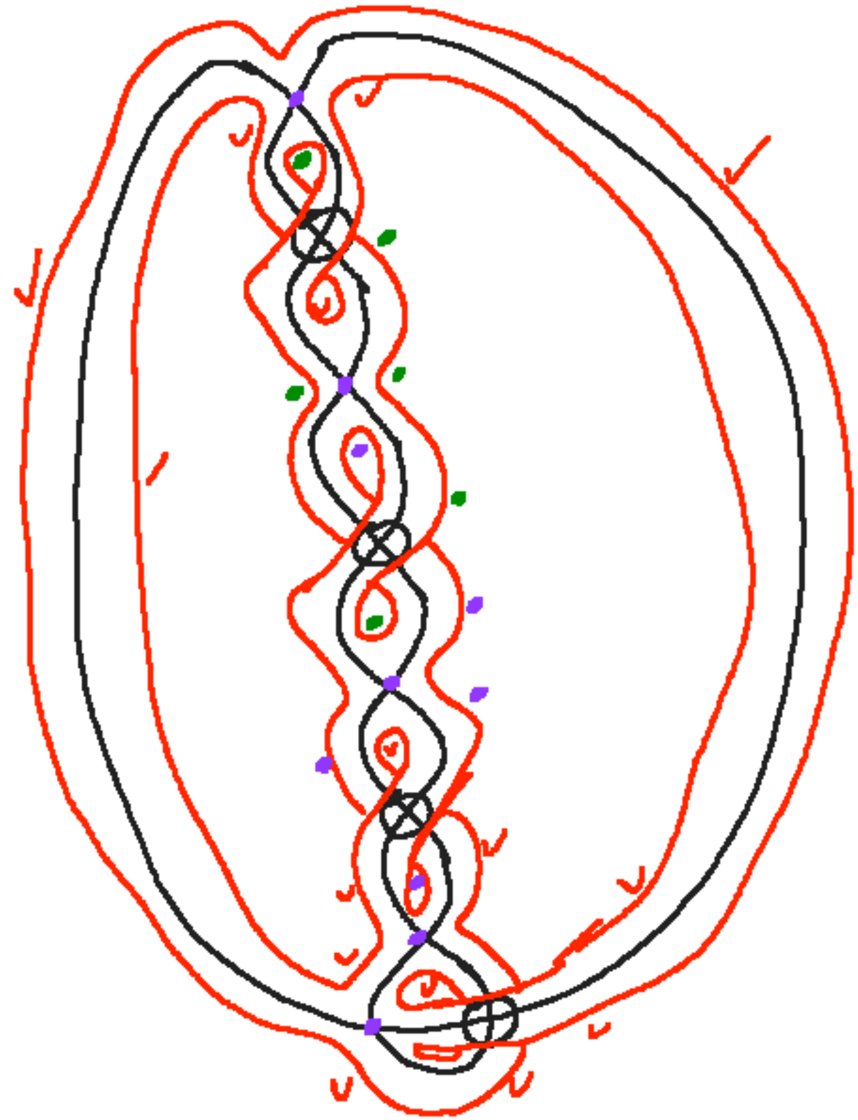
So we have many examples of min crossing number ~~for flat~~ virtuals.

Note that this example generalizes to K_n where $c(K_n) = n+1$

$n = 3, 5, 7, 9, \dots$

$$Q_{K_n}(t) = nt - nt^{-1} + t^{-n} - t^{n}$$

with $K_n \not\approx K_n^{r}$ & $c(K_n)$ min.



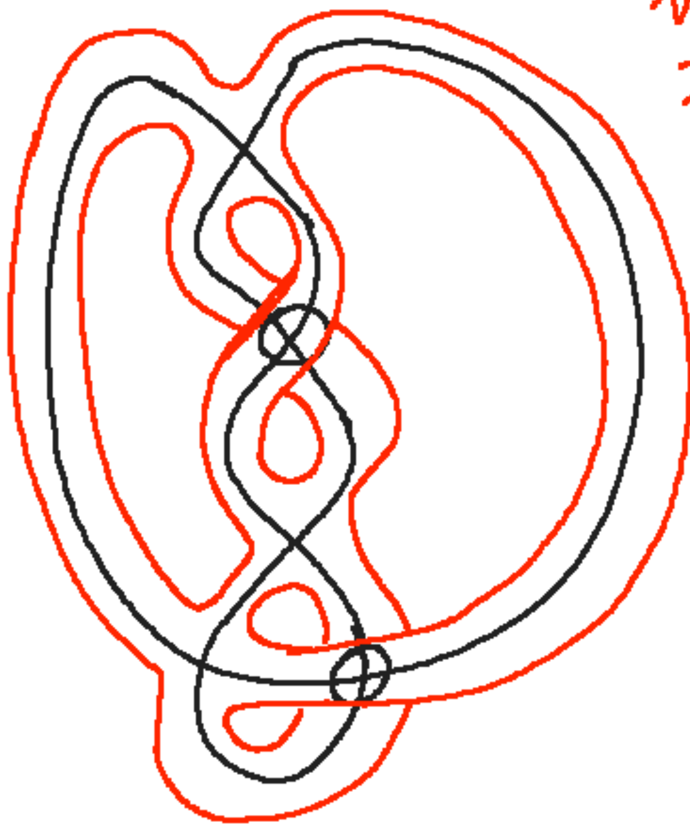
$v = 5$

$\lambda = 3$

$g = \frac{5-3}{2} + 1$

$g = 2$

Probably all these examples have genus 2.



$v = 3$
$\lambda = 1$
$g = \frac{3-1}{1} + 1$
$g = 2$

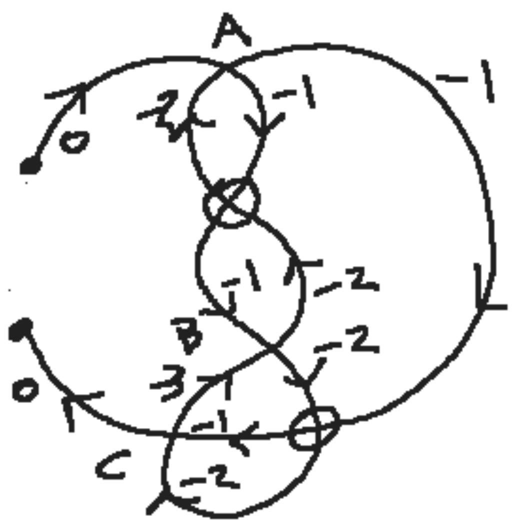
But we do not, at this writing, know any other invariants other than $W(D)$ that can distinguish flat virtual knots from their reversals.

For virtual knotoids there is a technique related to the affine index polynomial.

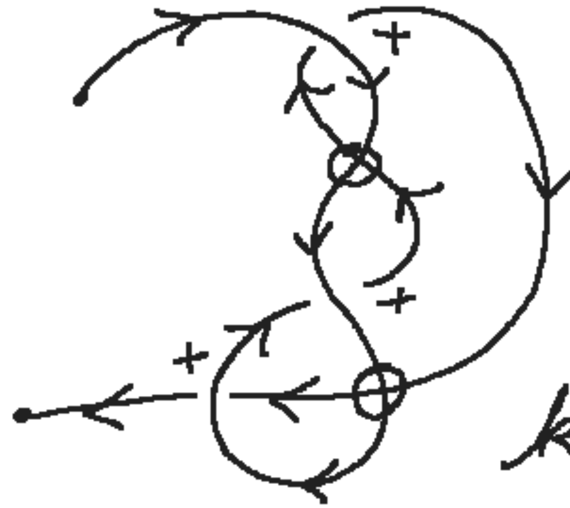
$$\begin{array}{ccc} \text{Flat Knotoids} & \hookrightarrow & \text{Virtual Knotoids} \\ D & \longmapsto & \mathrm{Desc}(D) \end{array}$$

where $\mathrm{Desc}(D)$ = the descending diagram associated with D.
(over before under)

D



$\xrightarrow{\text{Desc}}$



k

	w_+	w_-
A	$+1$	-1
B	$+1$	-1
C	-2	2

$$P_k = 2t + t^{-2} - 3$$

$$P_{k^r} = 2t^{-1} + t^{2} - 3$$

$$\Rightarrow k \not\simeq k^r$$

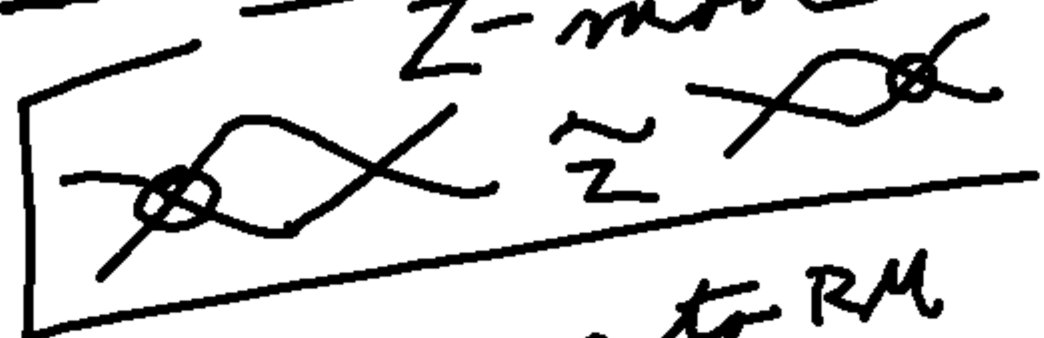
$\Rightarrow$ The flat knotoid D is not $\simeq$ to D^r.

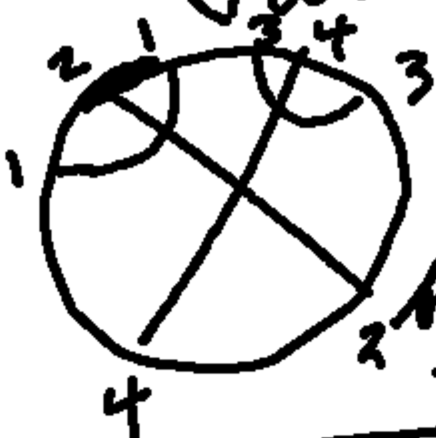
In this case, our arguments apply directly.

Virtuals

- Flat Virtuals

Free Knots ≡ Flat Virtuals / Z-move
(Turaev)





free chord diags up to RM on c.d.

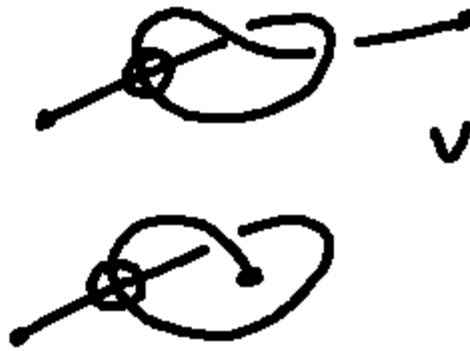
12134324

Knotoid (Turaev)

k

proper Knotoid

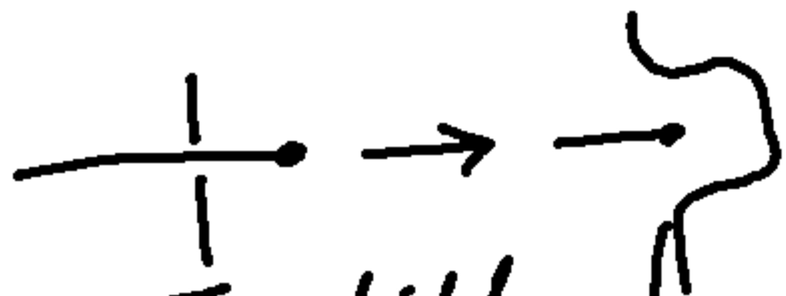
1-1 tangles



virtual knotoids

I all cases RM's are not allowed to go over the ends!

$k \underset{R2}{\sim}$



Forbidden!

Knotoids $\longrightarrow$ Virtual Closure

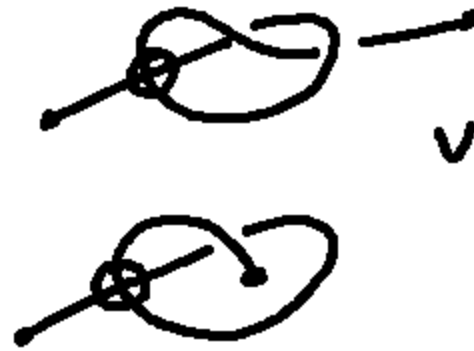
Knotoid (Turaev)

1-1 tangles

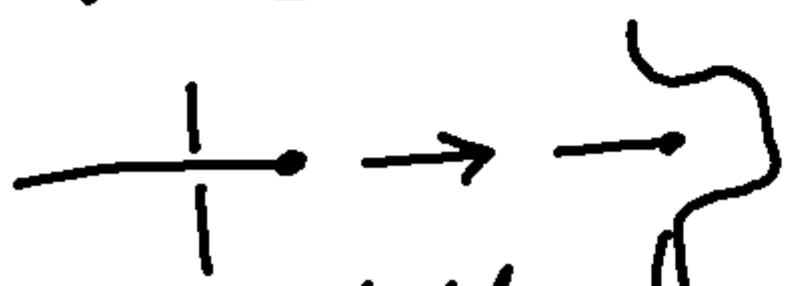
k

proper Knotoid

I all cases RM's are not allowed to go over the ends!



virtual knotoids

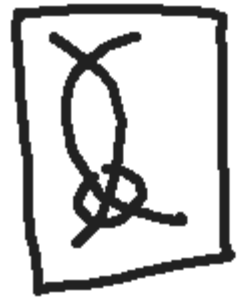


Forbidden!

$k \underset{R2}{\sim}$

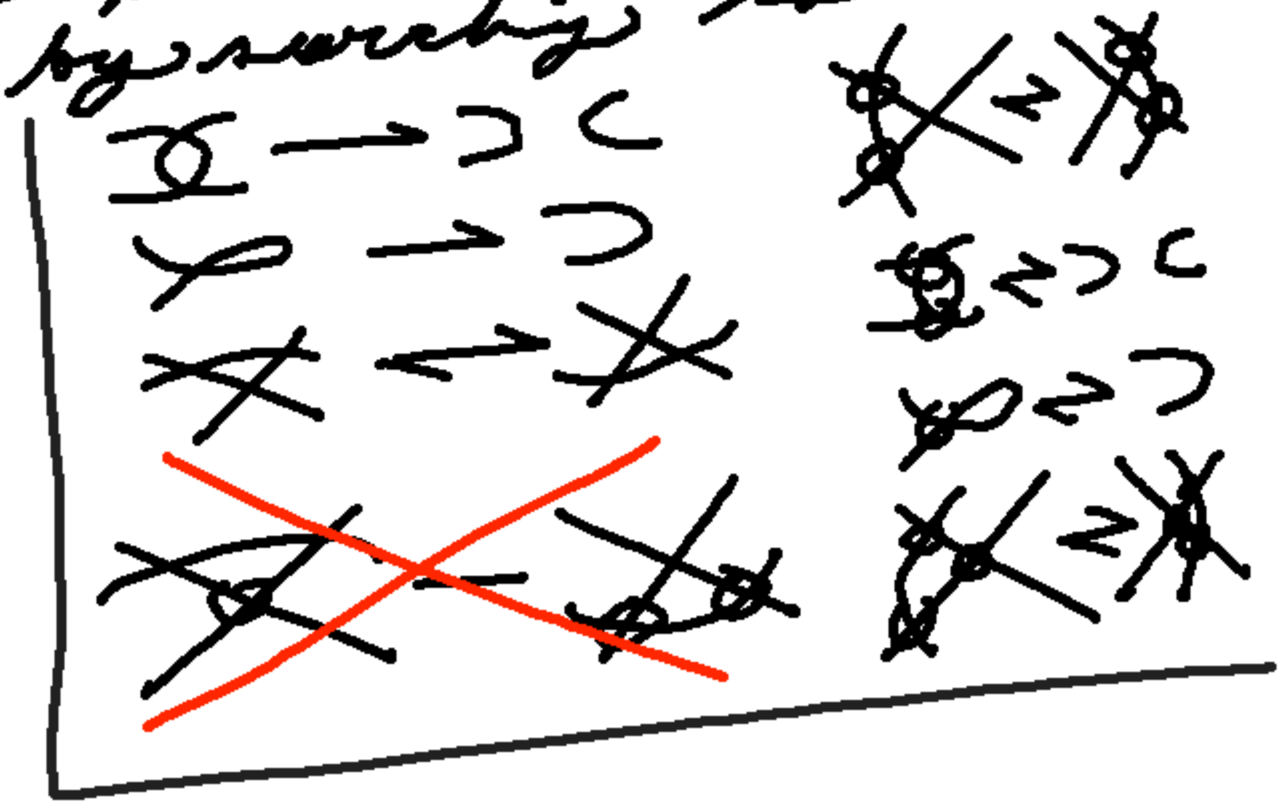
Knotoids $\longrightarrow$ Virtual Closure

Kadokami



Another way to formulate a consequence of Kadokami is that you can decide whether a flat virtual diagram is trivial by searching to reduce it:

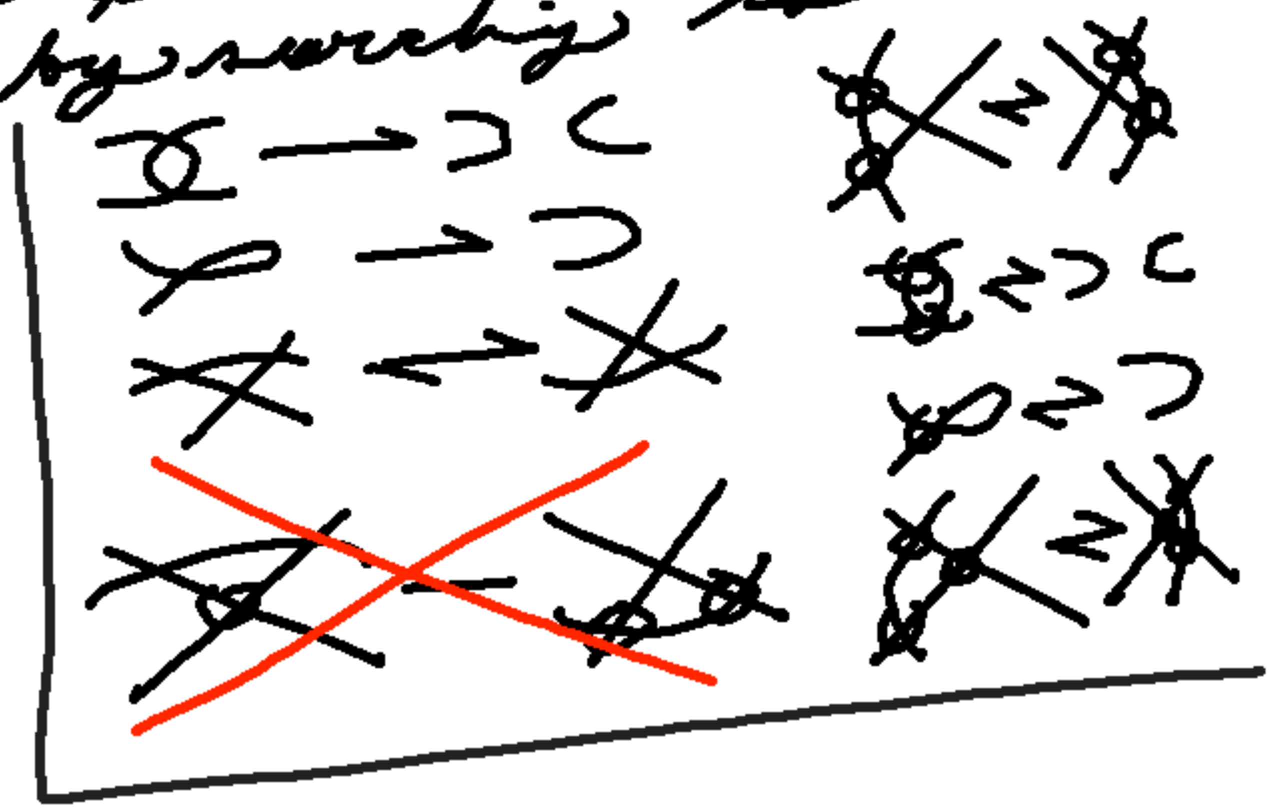
irred



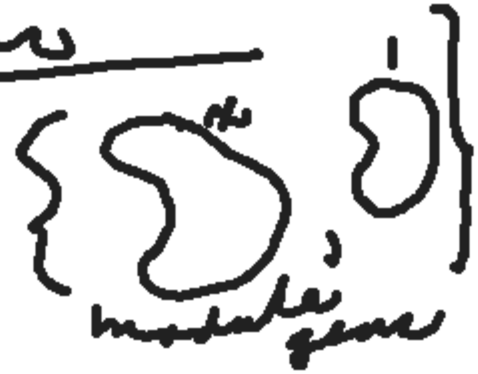
Kadokami

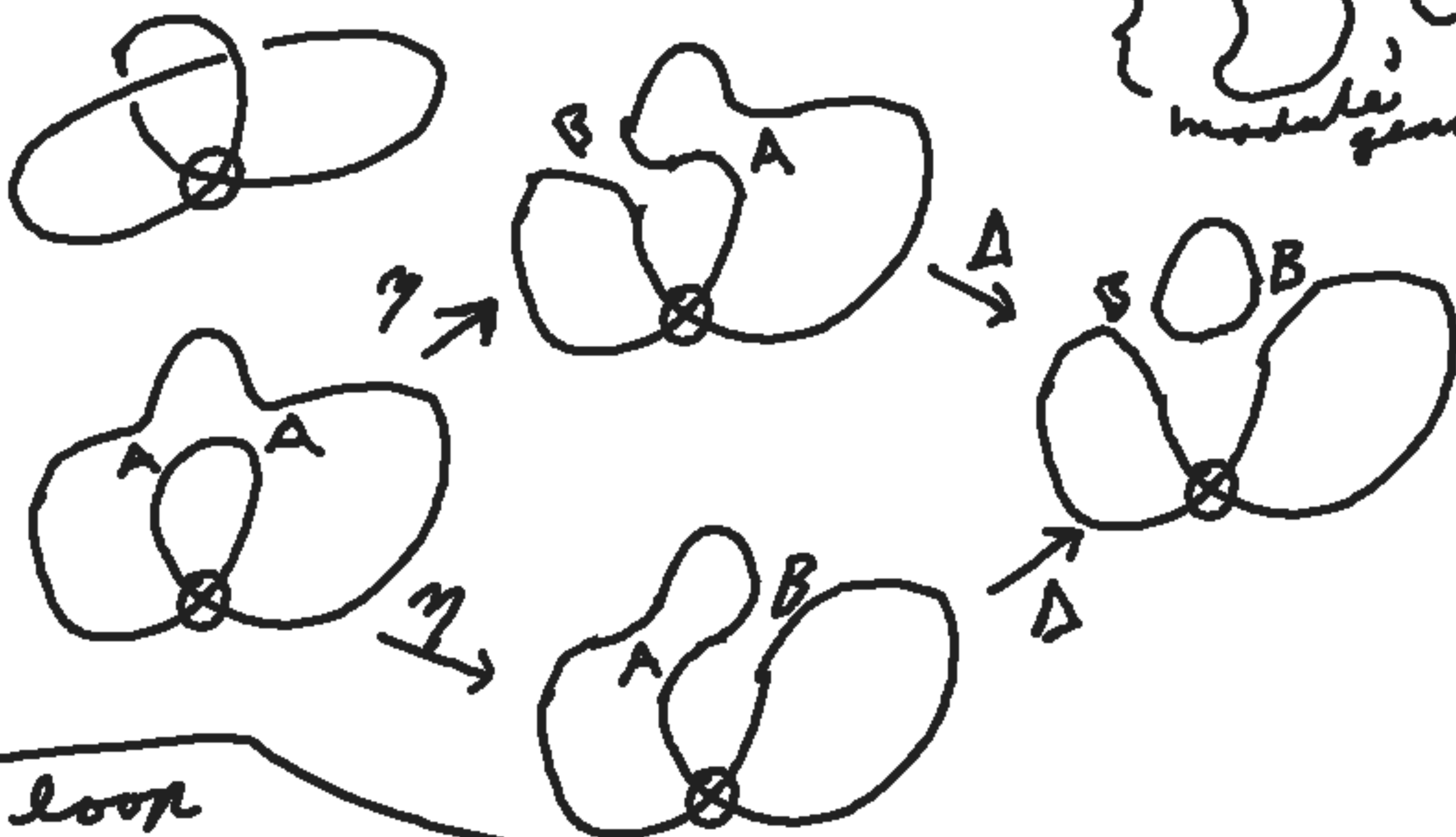
Another way to formulate a consequence of Kadokami is that you can decide whether a flat virtual diagram is trivial by searching to reduce it:

irred

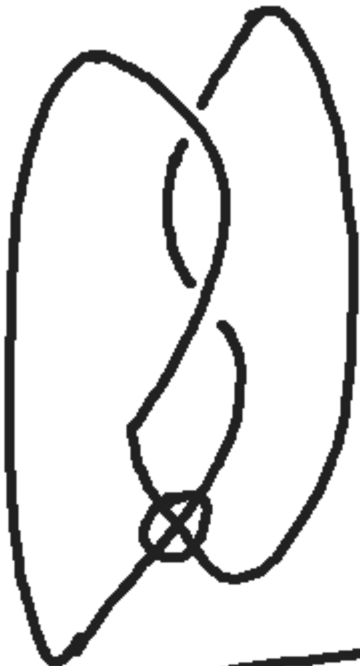


Extending Khovanov Homology to Virtual Knots + Links

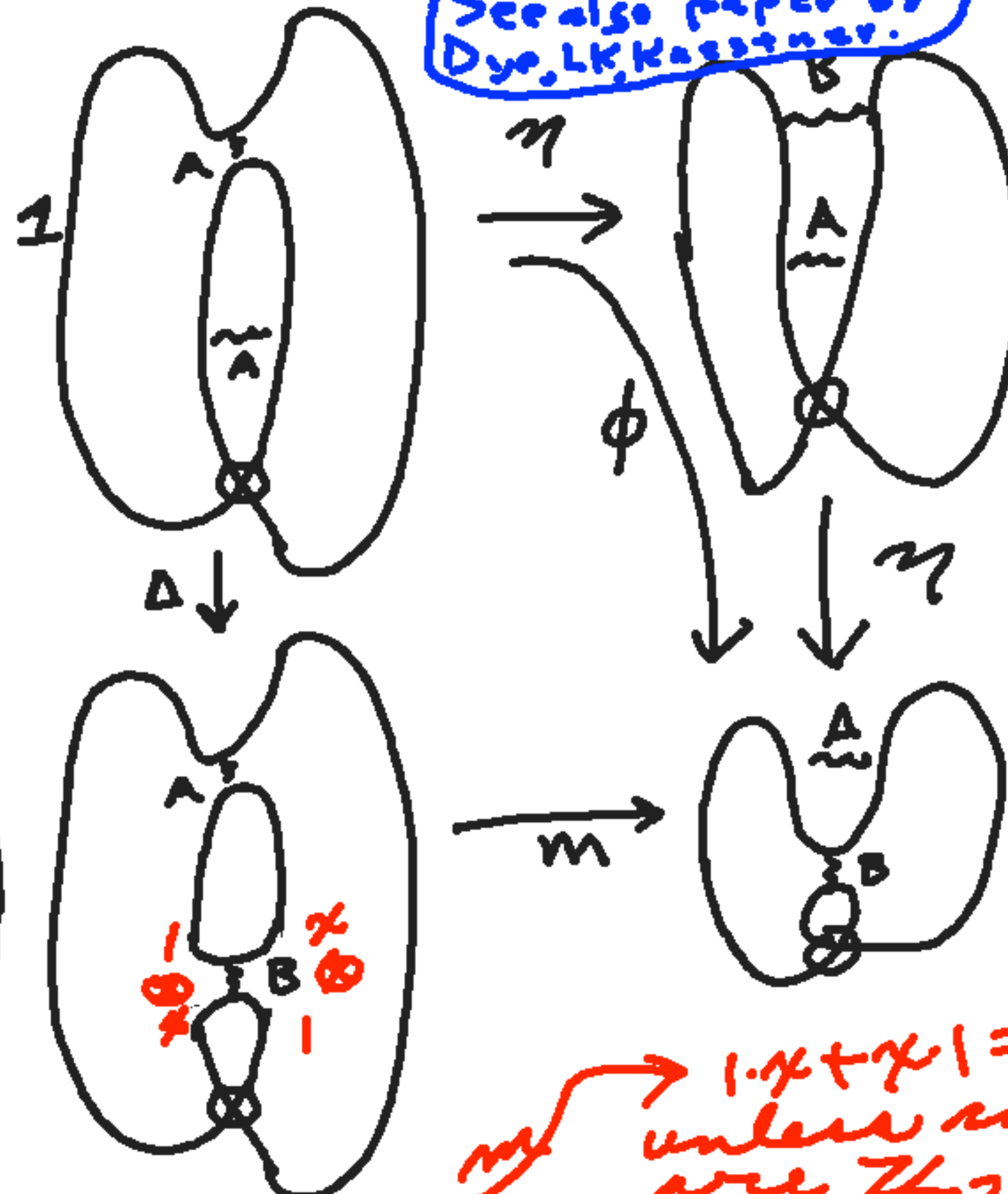




$$1 \text{ loop} \xrightarrow{\eta} 1 \text{ loop}$$



See also paper by Dye, LK, Kaestner.



$Kh_{\sigma}^{*}(K)$ ok for K virtual with mod 2 coeffs.

Manturov + [illegible]

Viro formulated $Kh_{\sigma}^{*}(\ ,\mathbb{Z})$ using Chord (signed) Diagrams

We will choose that $\eta = 0$.

See Tubenhauer for another choice

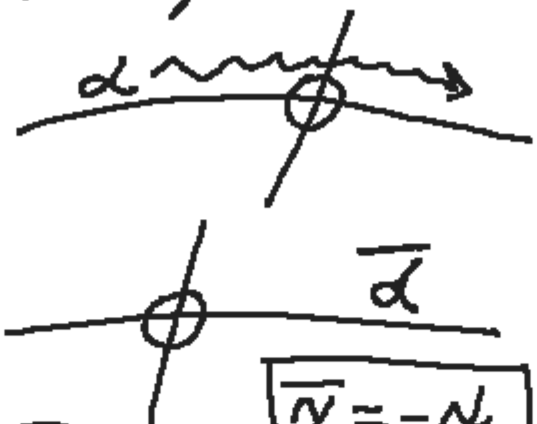
+ See Rushworth

$\Delta(1) = 1\otimes x + x\otimes 1$

m

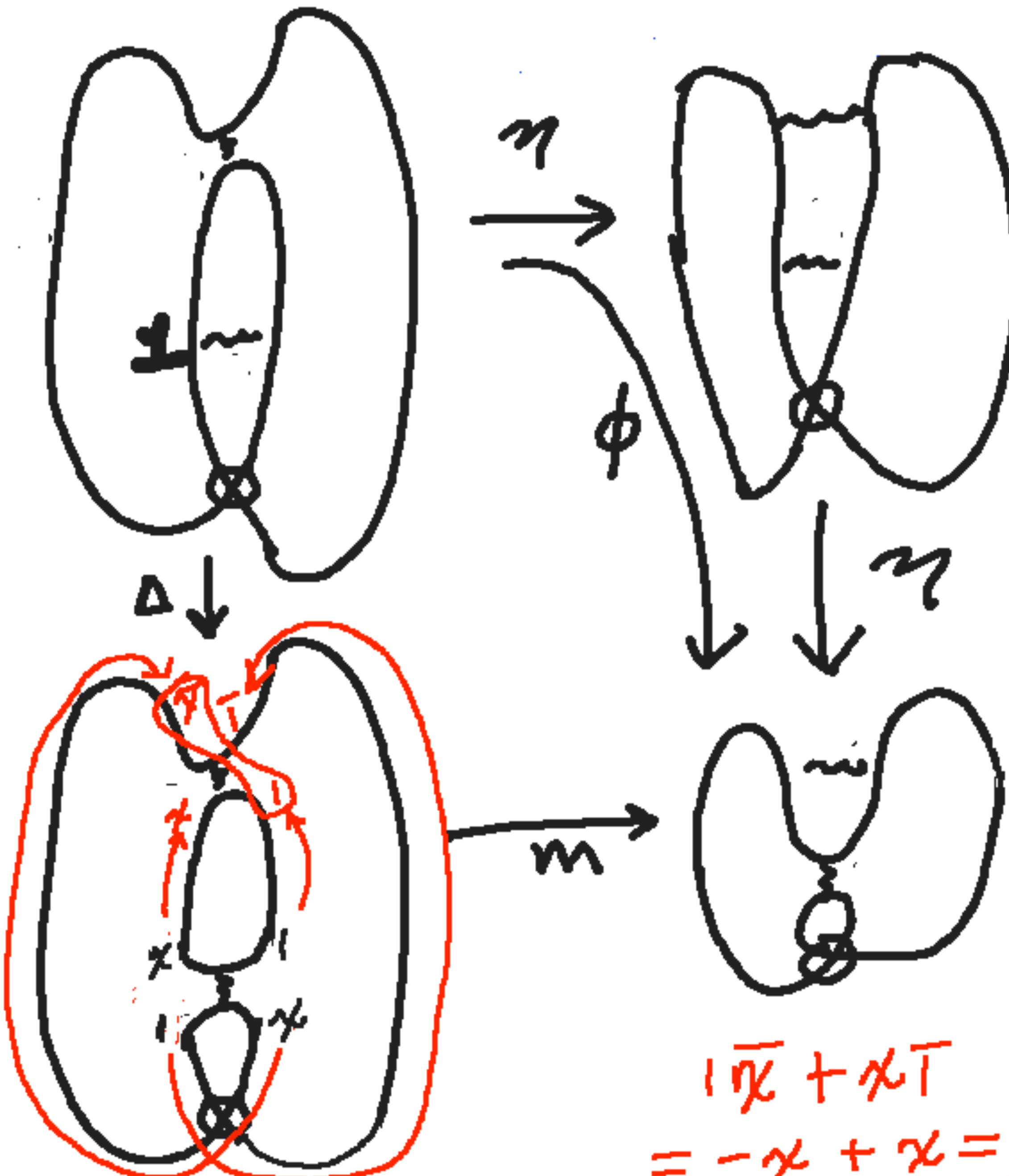
$1\cdot x + x\cdot 1 = 2x \neq 0$ unless coeffs are $\mathbb{Z}_2$.

use local coeffs so that the algebra on a state circle has location. and we'll TRY the following



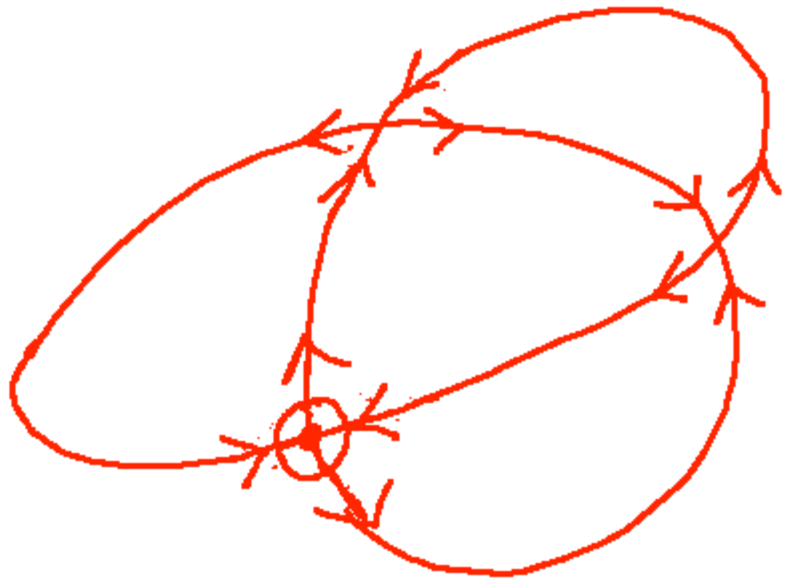
$\bar{\bar{\alpha}} = \alpha$

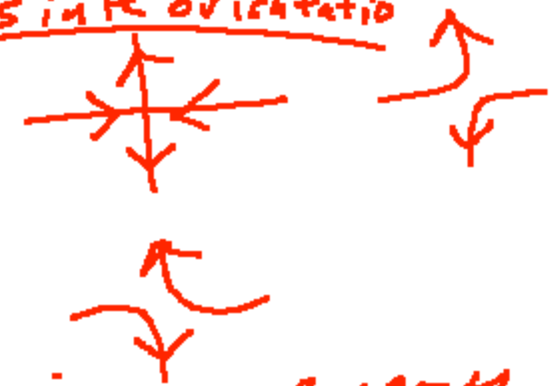
$\bar{x} = -x$
$\bar{1} = 1$



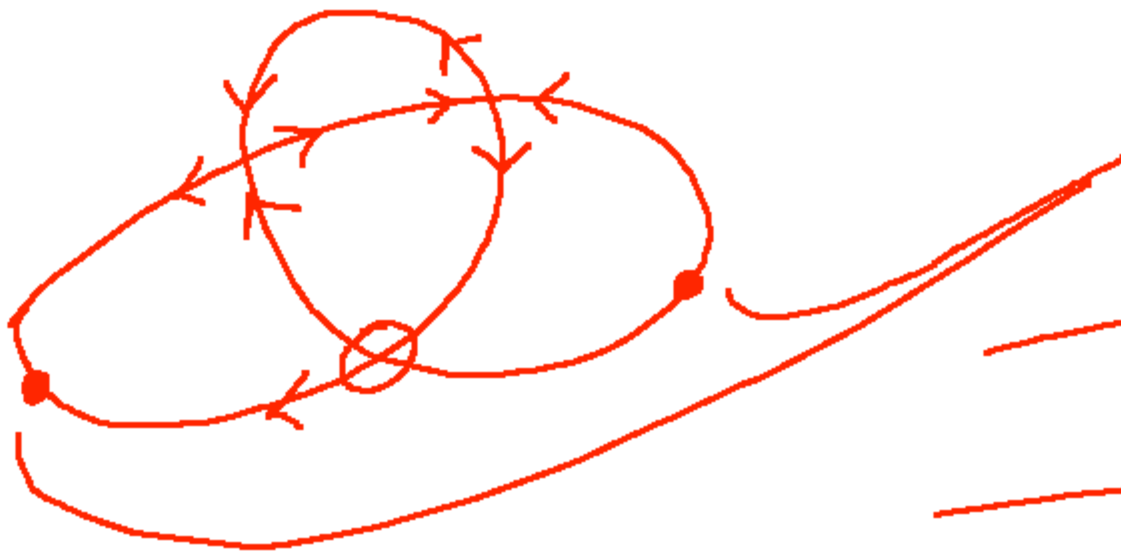
$1\bar{x} + x\bar{1}$
$= -x + x = 0$ ✓

ss source sink orientatio



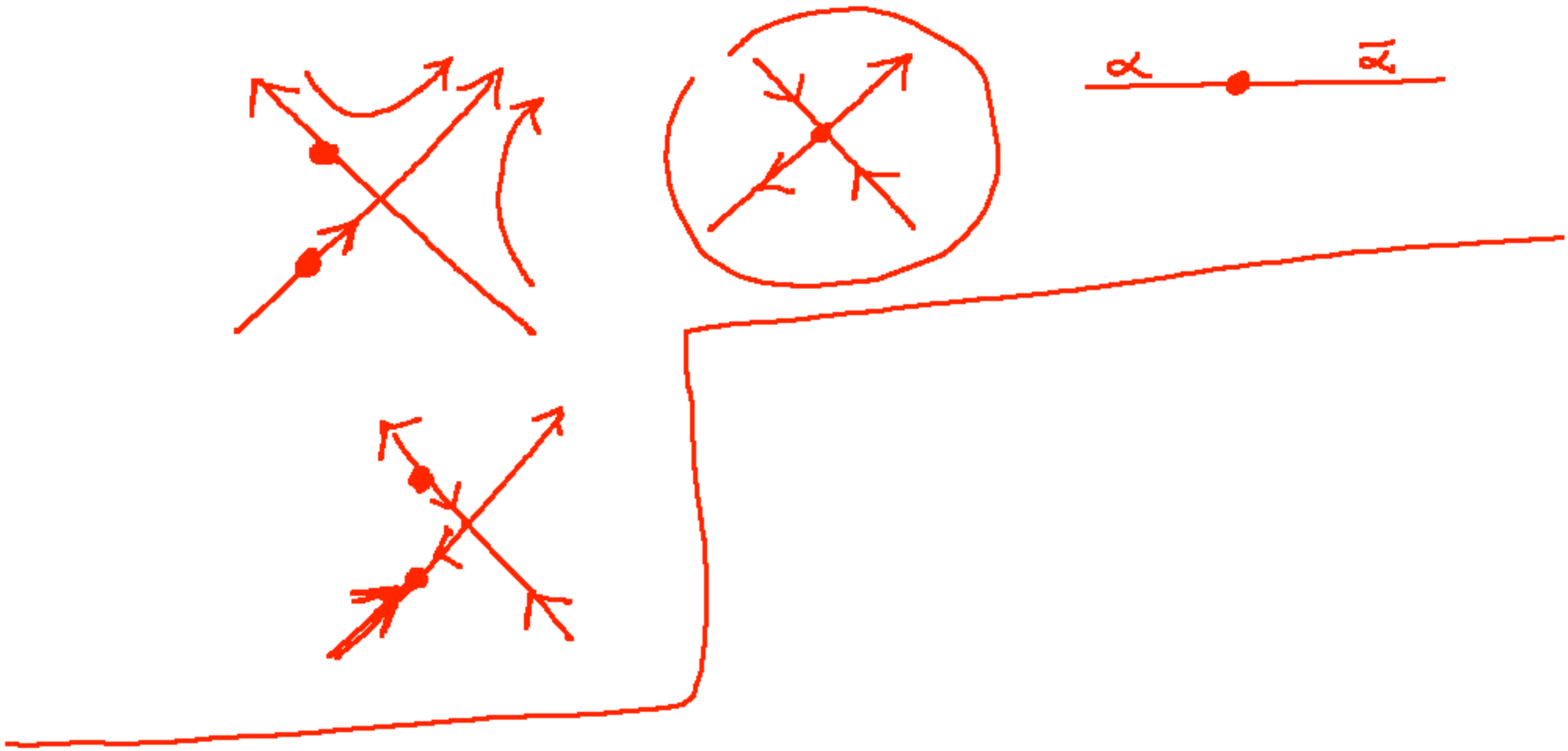


exercise: every classical diag. has ss orientation.



cut points

or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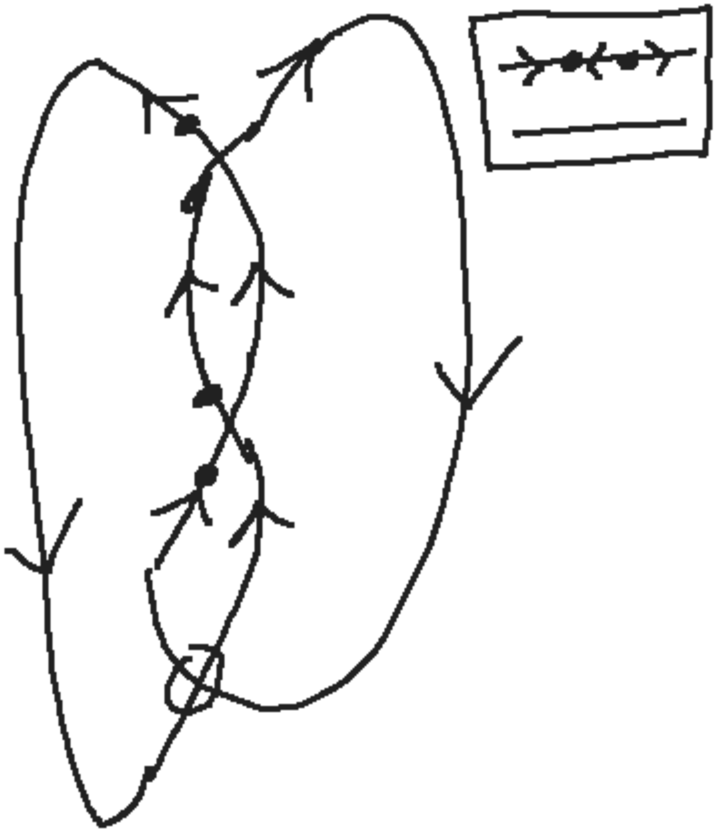
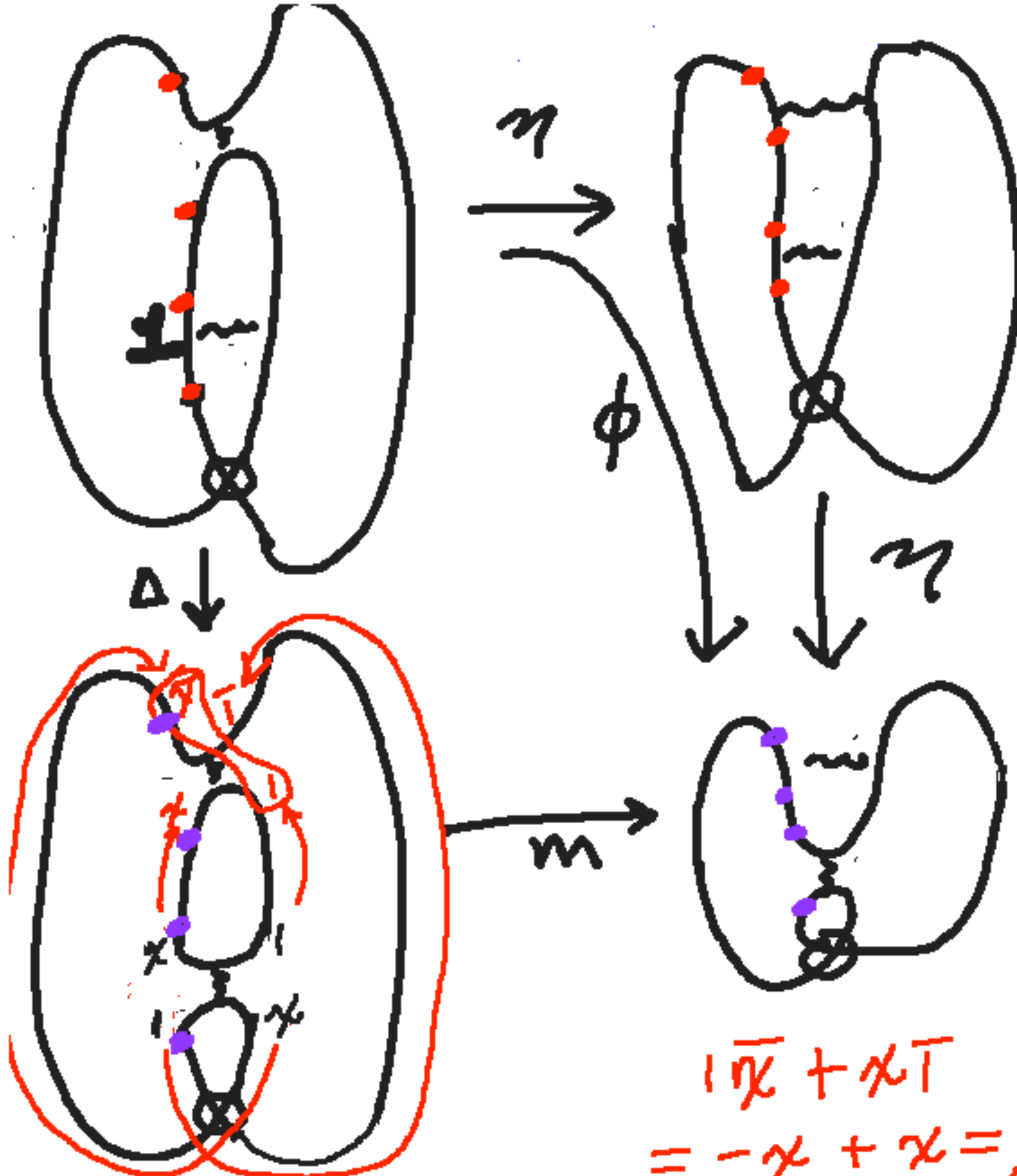
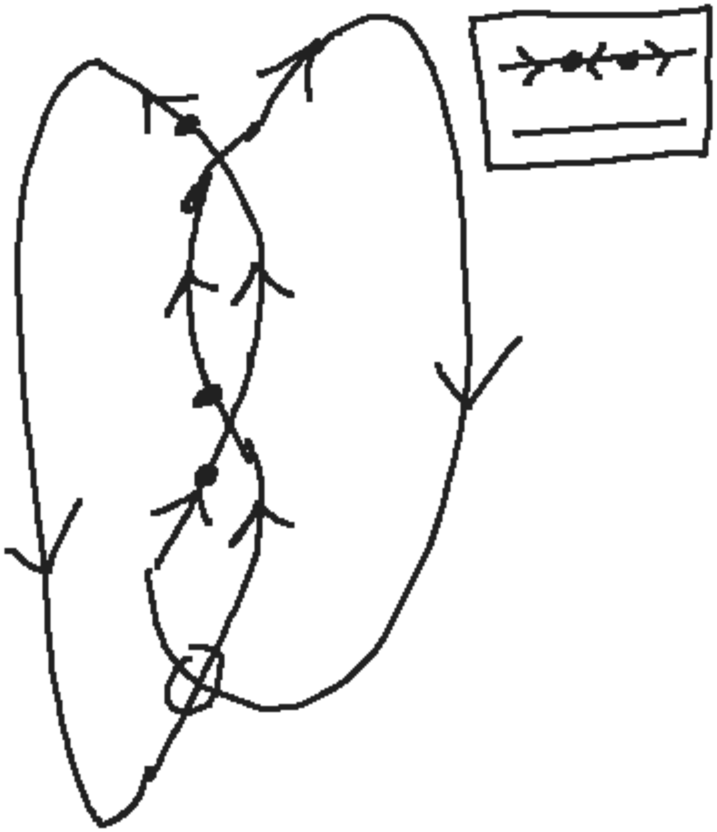
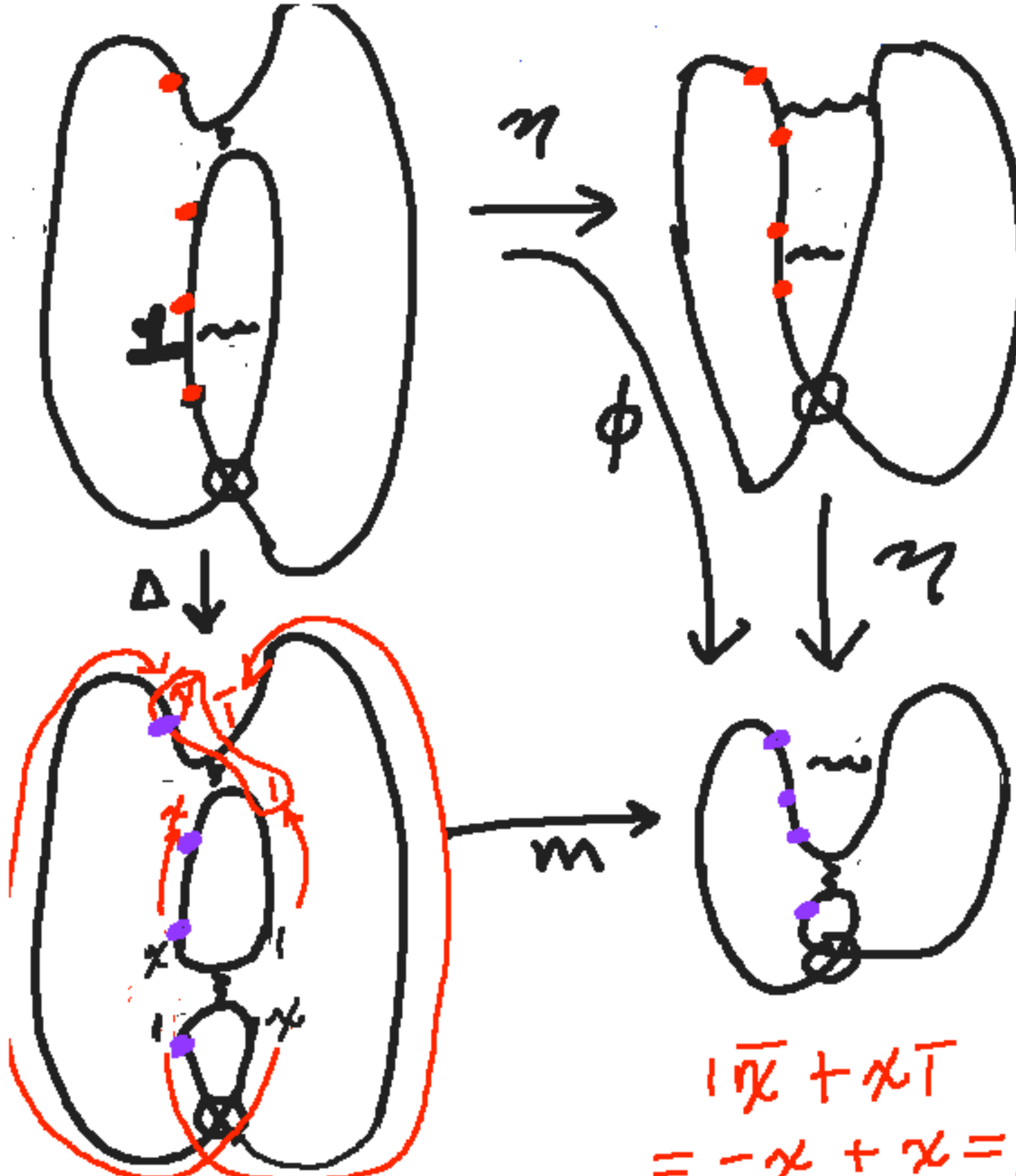




$$1\bar{x} + x\bar{1} = -x + x = 0$$





$$1\bar{x} + x\bar{1} = -x + x = 0$$

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leads to the need to order the whole complex so that $\boxed{x \otimes y \longleftrightarrow -y \otimes x}$, and the squares are then designed to anti commute & get chain complex.