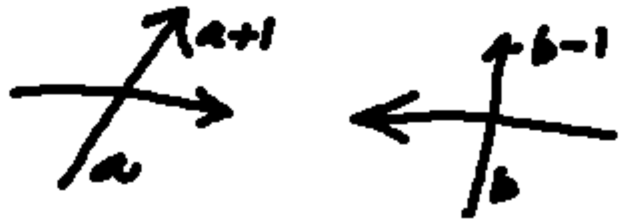
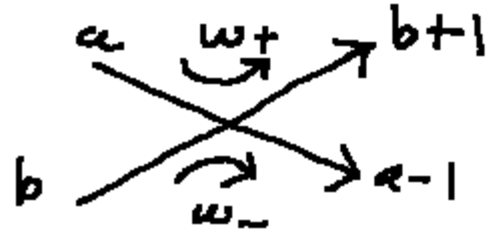
An Invariant of Flat Virtual Knots

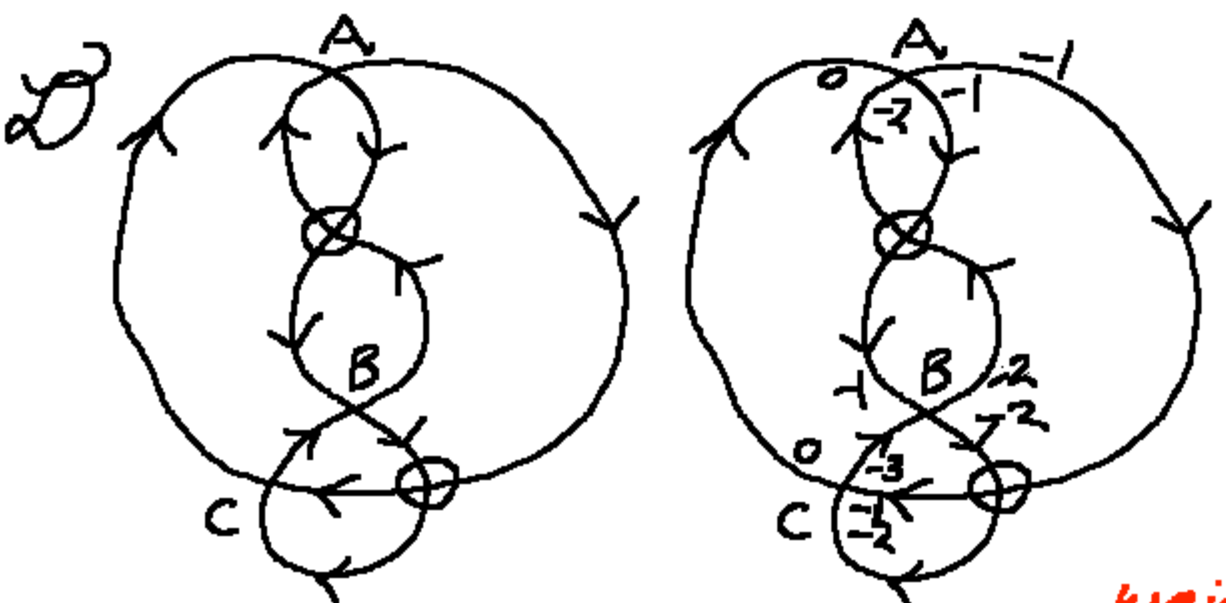
by LHK

1. 

integer labelling



$$\left.\begin{aligned} w_+ &= a-b-1 \\ w_- &= b-a+1 = -w_+ \end{aligned}\right.$$

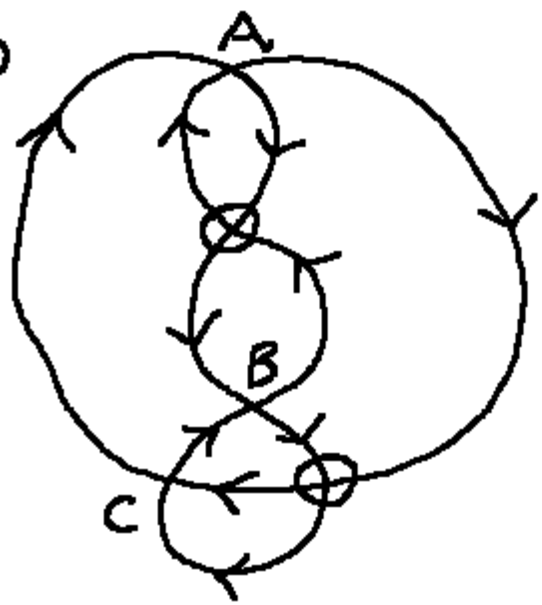


$W(\mathcal{D})$

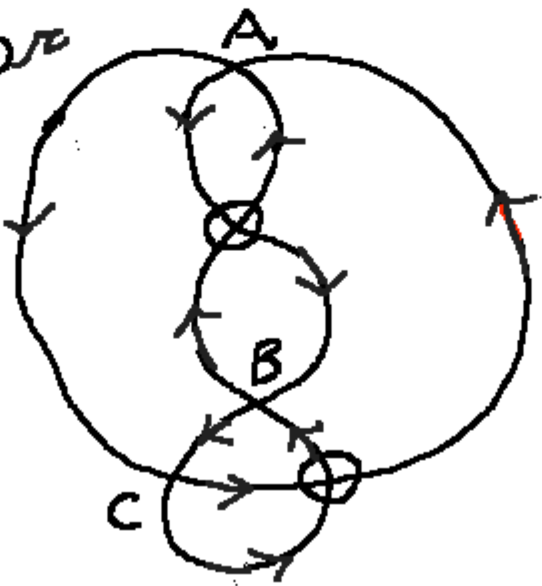
	w_+	w_-
A	1	-1
B	1	-1
C	-2	2

weight table of $\mathcal{D}$

$\mathcal{D}$

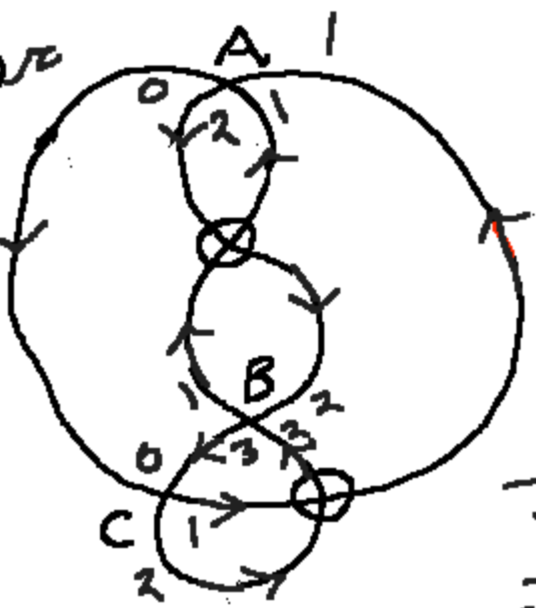


$\mathcal{D}^r$



$\mathcal{D}^r = \mathcal{D}$ with reversed orientation

$\mathcal{D}^r$



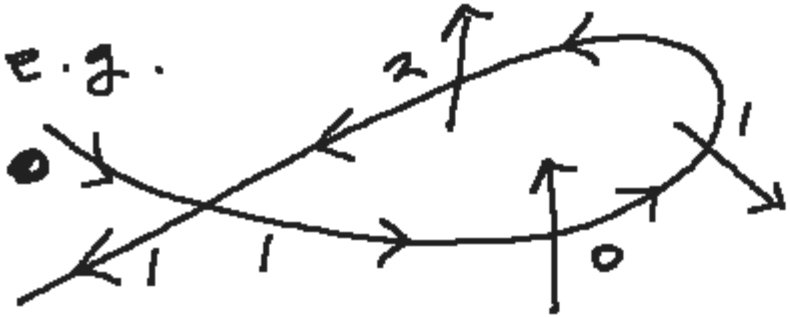
	ω_+	ω_-
A	-1	1
B	-1	1
C	2	-2

$W(D^r)$

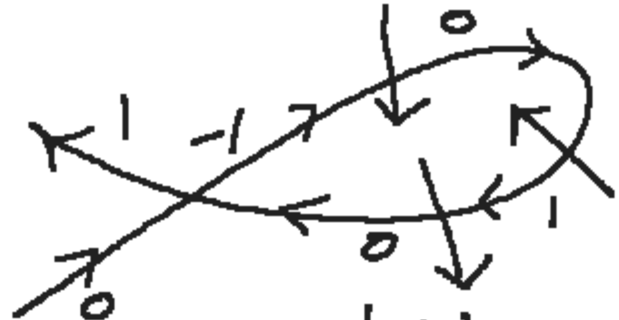
<u>Proposition.</u> $W(D^r) = \widehat{W}(D)$

where $\widehat{W}(D)$ is $W(D)$ with columns ω_+ and ω_- exchanged.

Proof. e.g.

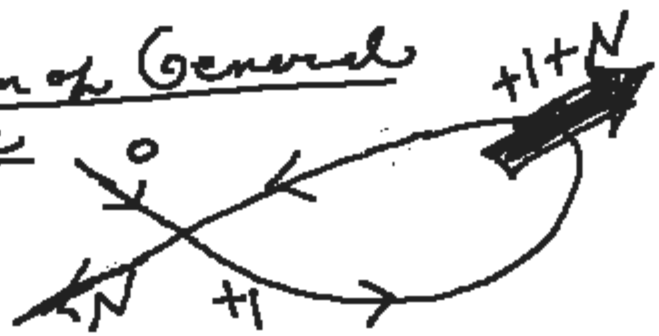


w_+	w_-
1	-1

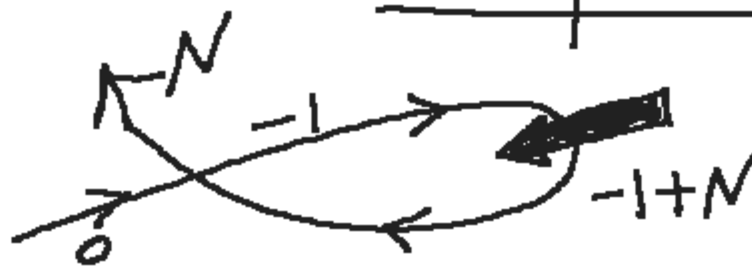


w_+	w_-
-1	1

Form of General Case



w_+	w_-
N	$-N$

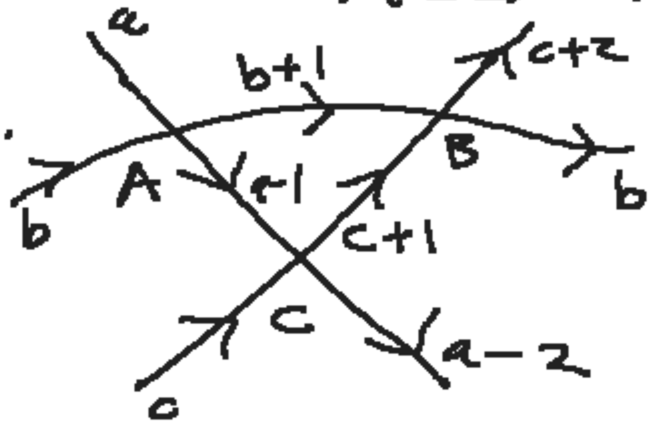


w_+	w_-
$-N$	N

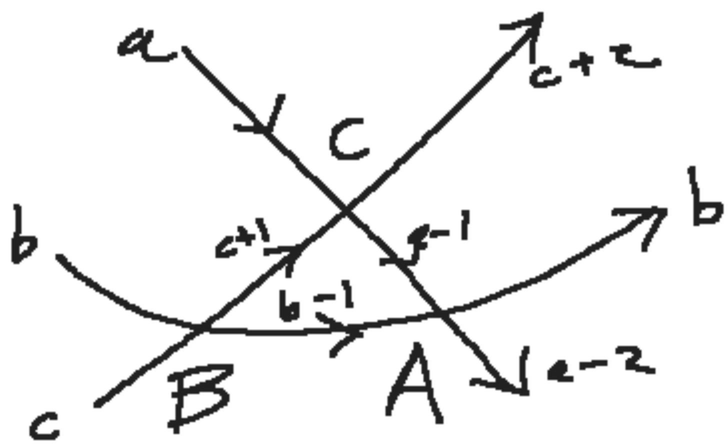
QED

<u>Proposition</u>. $W(D)$ is unchanged by Reidemeister 3 moves.

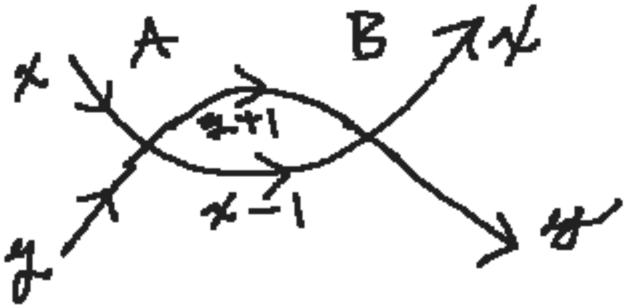
<u>Proof</u>.



	w_+	w_-
A	$a-b-1$	$b-a+1$
B	$b-c-1$	$c-b+1$
C	$a-c-2$	$c-a+2$



<u>QED</u>

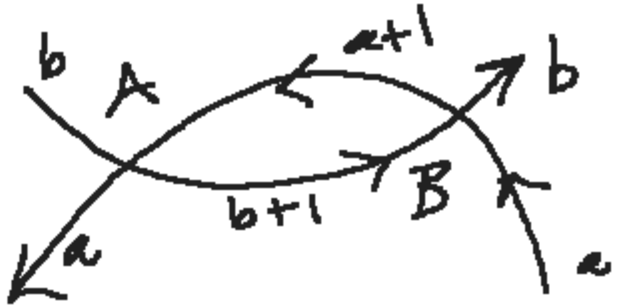


	w_-	w_+
A	$x-y-1$	$y-x+1$
B	$y-x+1$	$x-y-1$

A	m	$-m$
B	$-m$	m

T

Note that $\hat{T} = T$.



	w_+	w_-
A	$a-b$	$b-a$
B	$b-a$	$a-b$

In all cases of R2, $W(D)$ changes by adding or removing a sub-table of the form T above, with $\hat{T} = T$.

	w_+	w_-
A	0	0

S

R1 adds or subtracts a row of zeros $S + \hat{S} = S$.

Proposition. The property $W \neq \widehat{W}$ is invariant under the Reidemeister and detour moves.

Proof. This follows from the previous discussion. //

Proposition. If $W(\mathcal{D}) \neq \widehat{W}(\mathcal{D})$ then $\mathcal{D} \not\approx \mathcal{D}^r$.

Proof. If $W(\mathcal{D}) \neq \widehat{W}(\mathcal{D})$ then $W(\mathcal{D})$ has a row of form $(n,-n)$ $(n \neq 0)$ & no row of form $(-n,n)$.

Equiv under R2 adds and subtracts pairs of rows of form $\begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$.

Thus one could go from

$$(n,-n) \rightsquigarrow (n,-n)$$

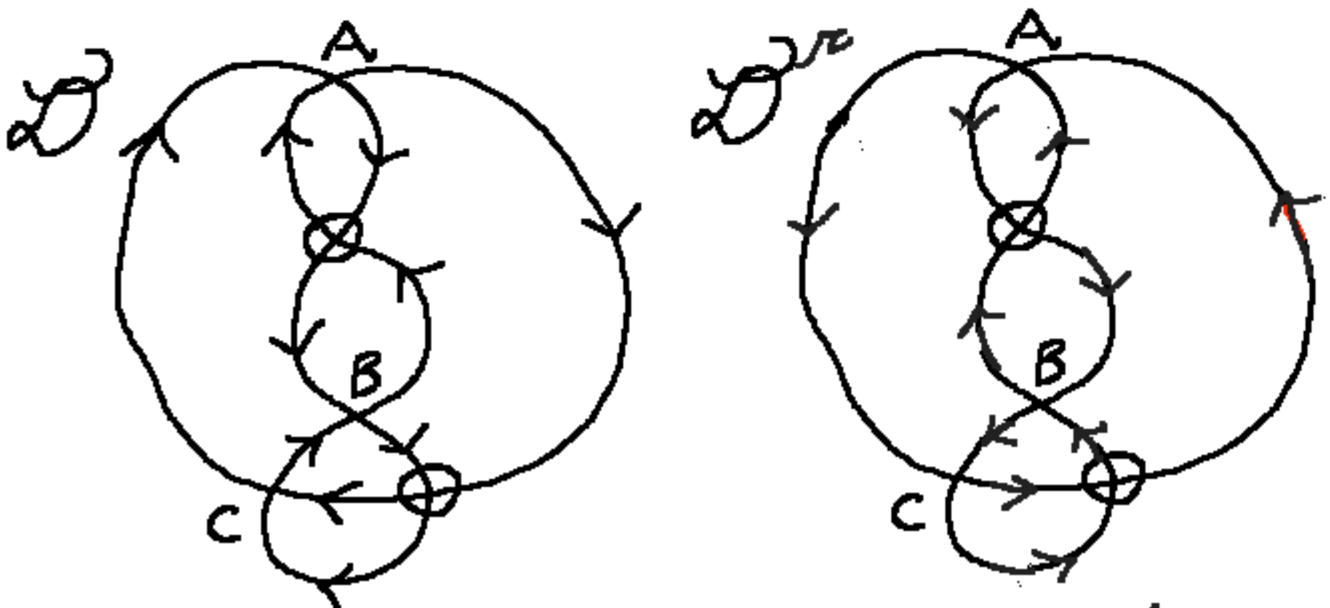
$$(-n,\ n)$$

$$(n,-n)$$

If it were possible to change $(n,-n) \rightsquigarrow (-n,n)$ by such a process it could only occur by adding + subtracting matrices of the form $\begin{pmatrix} -n & n \\ n & -n \end{pmatrix}$ and this is not possible. So we conclude that if $W(\mathcal{D}) \neq \hat{W}(\mathcal{D})$ then $\mathcal{D}$ is inequivalent to $\mathcal{D}^r$.

QED

Thus we conclude that our example $\mathcal{D}$ below is not equivalent to $\mathcal{D}^r$.



<u>Remark</u>. One can often use the affine index polynomial $P_K(t)$ to show that a virtual knot K is inequiv to K^r, via

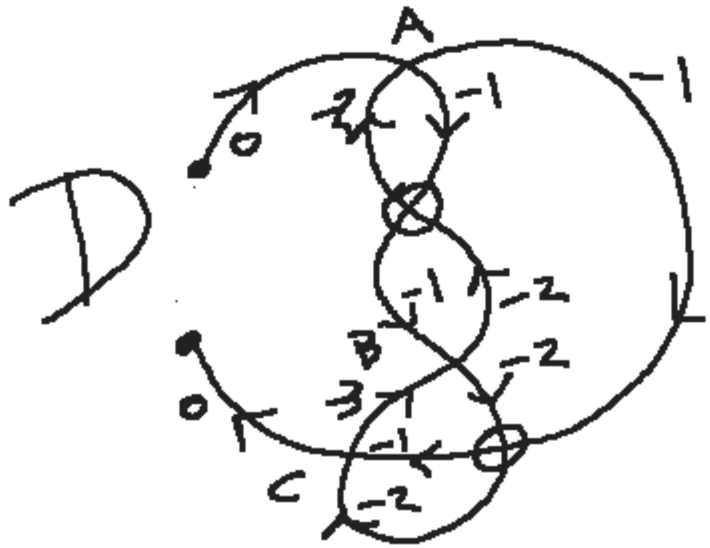
$$P_{K^r}(t) = P_K(t^{-1}).$$

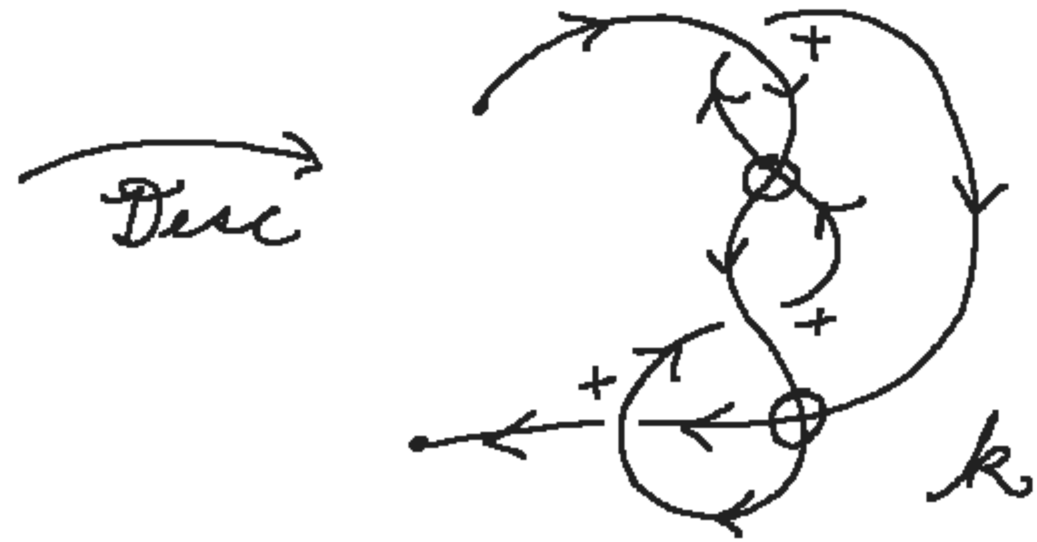
But we do not, at this writing, know any other invariants other than $W(D)$ that can distinguish flat virtual knots from their reversals.

For virtual knotoids there is a technique related to the affine index polynomial.

$$
\begin{array}{ccc}
\text{Flat Knotoids} & \hookrightarrow & \text{Virtual Knotoids} \\
D & \longmapsto & Desc(D)
\end{array}
$$

where $Desc(D)$ = the <u>descending diagram</u> associated with D.
(over before under)





	$W+$	$W-$
A	$+1$	-1
B	$+1$	-1
C	-2	2

$$P_k = 2t + t^{-2} - 3$$

$$P_{k^r} = 2t^{-1} + t^{2} - 3$$

$$\Rightarrow k \not\simeq k^r$$

$\Rightarrow$ The flat <u>knotoid</u> D is not $\simeq$ to D^r.

In this case, our arguments apply directly.