Classically, two oriented knots K, K'
are __cobordant__ iff $K \cup K' = \partial(\text{Surface } S)$

gen by

$S \hookrightarrow \mathbb{R}^3 \times I \subset S^3 \times I$

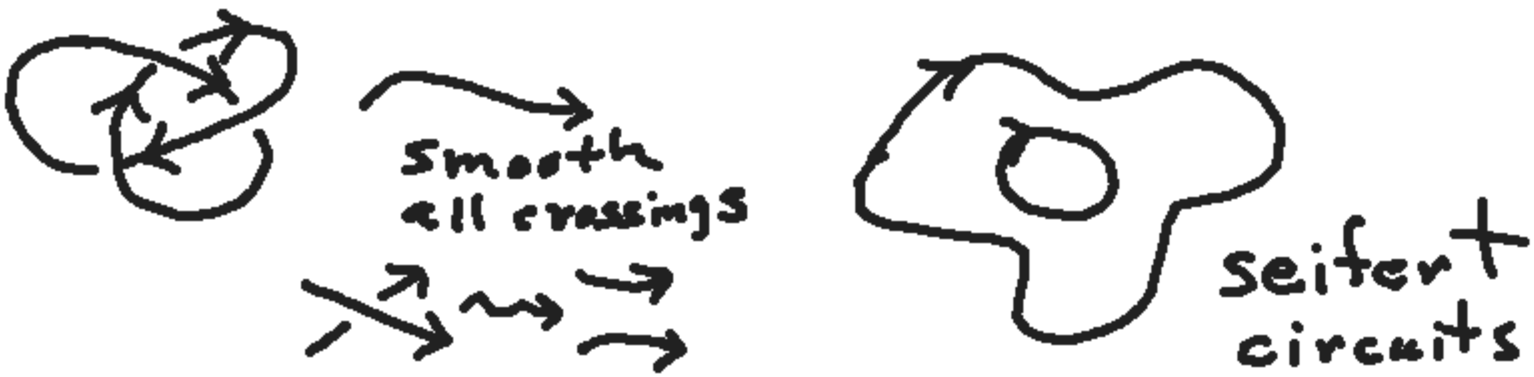
$K \subset \mathbb{R}^3 \times 0$
$K' \subset \mathbb{R}^3 \times 1$

For virtuals we make same defn via saddle, births + deaths.

S of genus g

If $\exists\, S$ genus $g = 0$, say __concordant__.

Seifert's algorithm for making a surface S, $\partial S = K$, K classical.



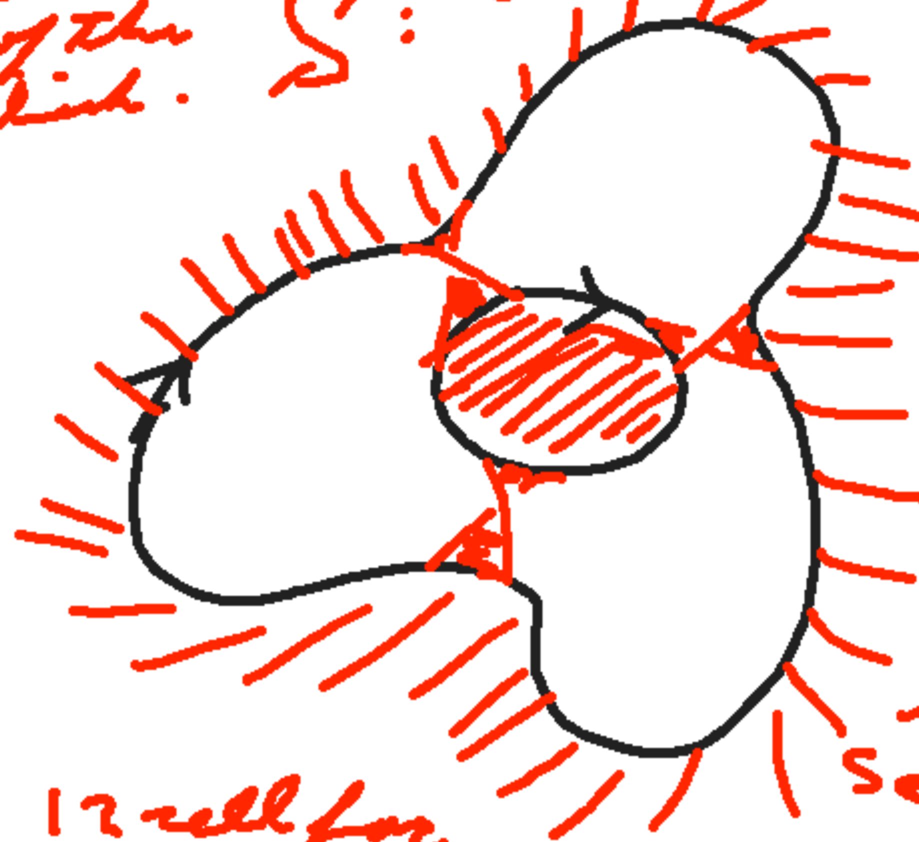
1) bound each Seifert Circuit with a disk

2) add bits of twisted strip.

$v = \#$ crossings

$l = \#$ components of the link.

$S : \partial S = K$



$s = \#$ Seifert Circuits

$v = 3$
$s = 2$
$l = 1$

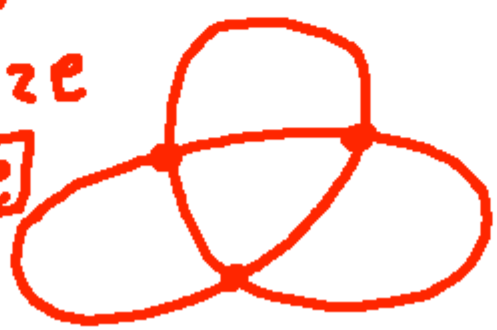
$\frac{3-2-1}{2} + 1 = 1$ ✓

Seifert's Surface

genus?

$+v = 2e$

$\boxed{2v = e}$



1 2-cell for each Seifert circuit.

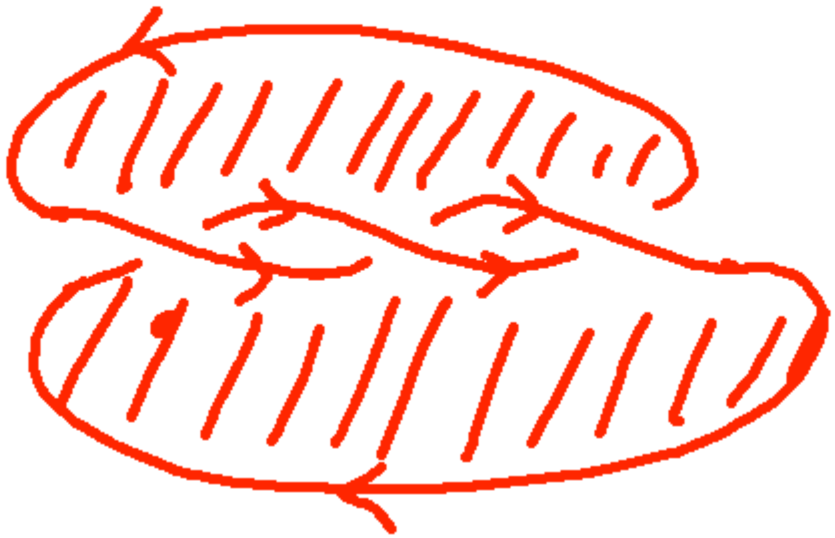
$v - e + s + l = 2 - 2g$

$-v + s + l = 2 - 2g$

$2g = v - s - l + 2$

$$\boxed{g = \frac{v - s - l}{2} + 1}$$

The Seifert surface is orientable.



Some surfaces spanning the link are not orientable.

F

$\partial F = T$ but $F \cong$ Möbius band.

Generally, a choice of checkerboard surface may or may not be orientable.

not orientable

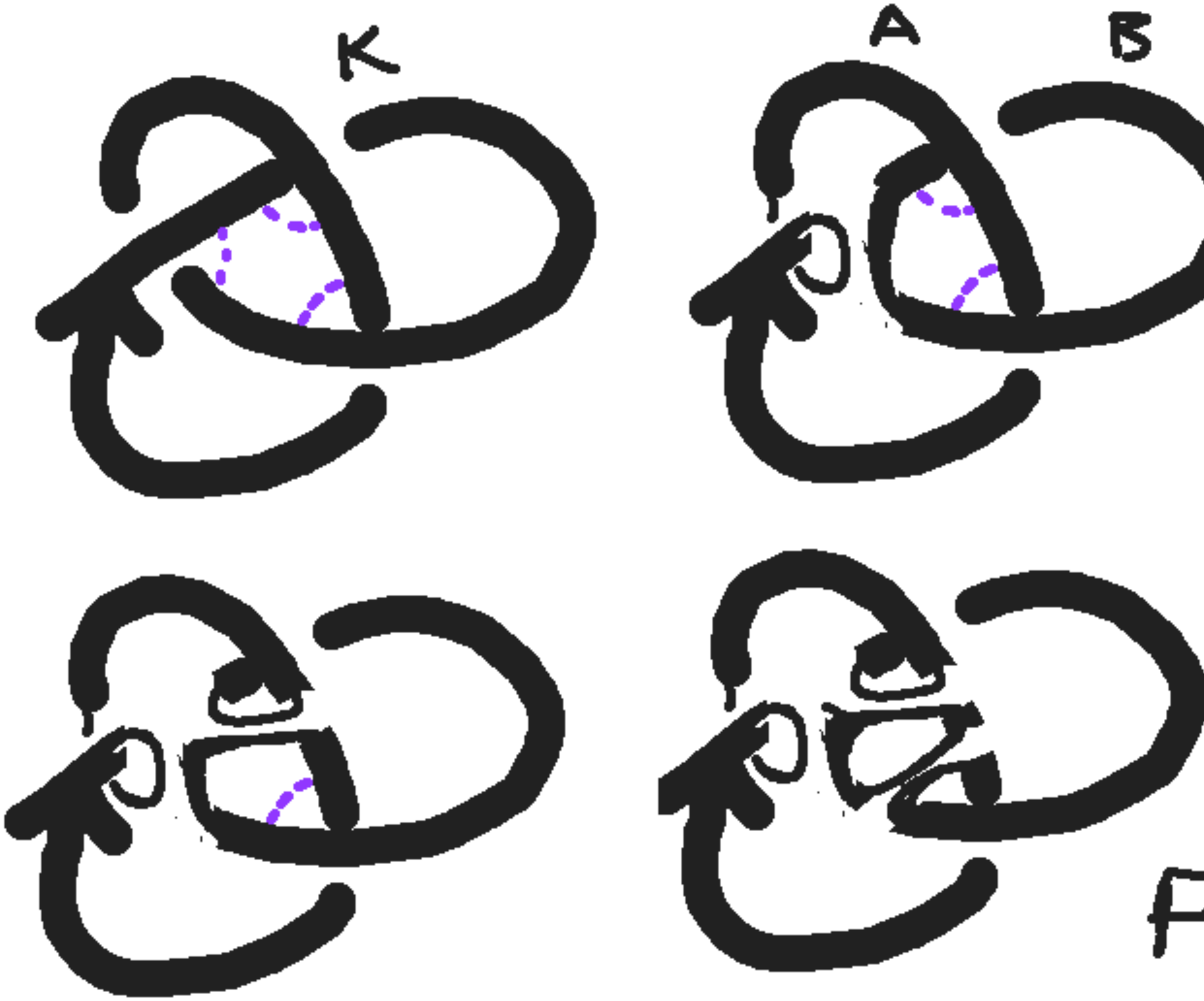
K

can do saddle

$\downarrow$ cob

$\sim$

$\longrightarrow$ death



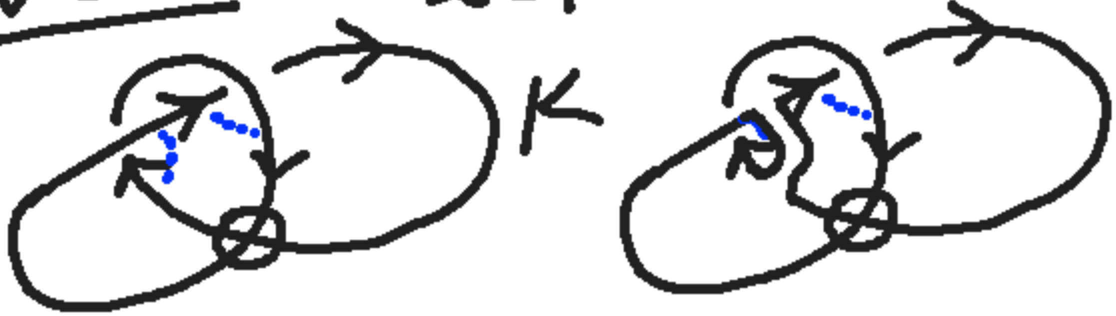
$F \subset S^3 \times I$

Exercises:

1) $g(F) = g(\text{Seif surface})$
2) F = result of 'pushing' Seifert Surface into $S^3 \times I$.

This cobordism can always be done for virtuals.

$v=2$ $\quad$ $\ell=1$



$s=1$

$$g=\frac{v-s-\ell}{2}+1$$

$$=\frac{2-1-1}{2}+1=1$$

$\sim$  $\sim$  $\xrightarrow[\text{death}]{}$

If K is a virtual diagram
v classical nodes, ℓ components
then $K = \partial F$, $g(F) = \frac{v - s - \ell}{2} + 1$
where F is a cobordism surface
for K (obtained by
- virtual moves
- saddles
- births + deaths)

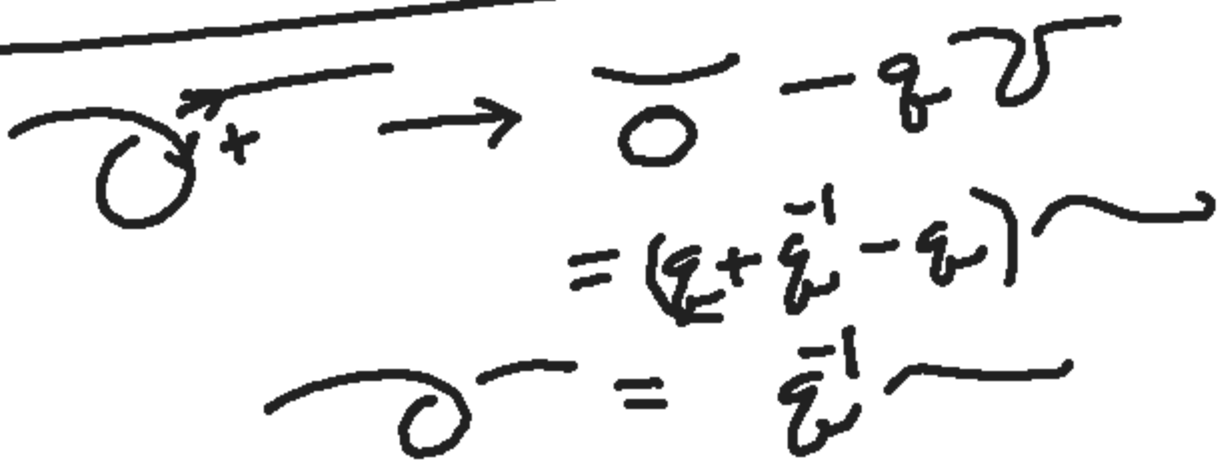
We say $K = \partial F$, $F \subset$ "Virtual Four Space".

<u>Question</u>. Given K, what is the least genus g for an F
$\partial F = K$, $F \subset$ Virt 4 Space?

$$\boxed{P_K = \sum_c \mathrm{sgn}(c)\, t^{w(c) - wr(K)} \qquad w(c) = w_{\mathrm{sgn}(c)}}$$

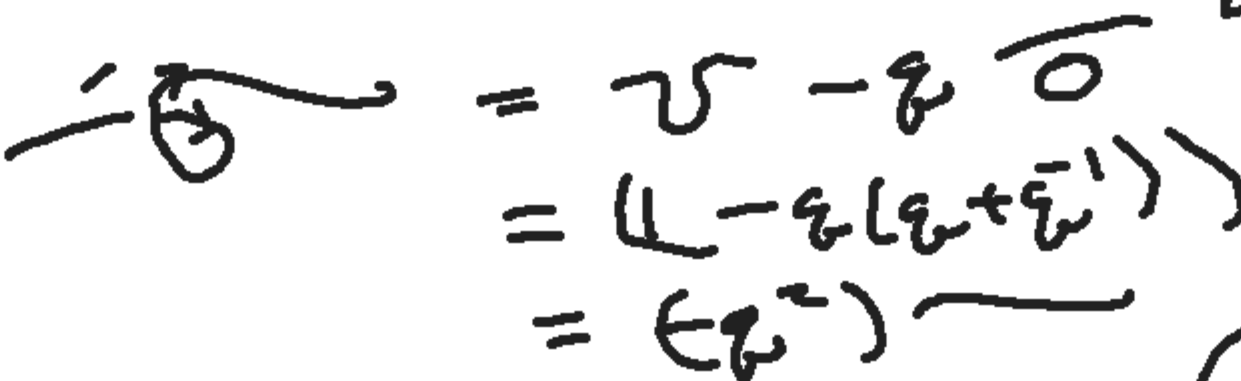
Invar under virtual moves.

$$\langle O \rangle = q + q^{-1}$$



$$= (q + q^{-1} - q)$$

$$= q^{-1}$$



$$= (1 - q(q + q^{-1}))$$

$$= (-q^2)$$

$$J_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle K \rangle$$

normalized invar. Jones poly

$J_K = (-1)^{n_-} q^{n_+ - 2n_-} \langle K \rangle$