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This year, the U.S.A. was invited to send a team of eight students to Erfurt, in the German Democratic Republic, to participate in the Sixteenth International Mathematical Olympiad. The invitation was accepted and our team was there on July 8 and 9. Thanks are due to the Spencer Foundation for the grant which made it possible for the team to travel to Europe and back.

The Fourth U.S.A. Mathematical Olympiad will take place on Tuesday, May 6, 1975.

DEPARTMENT OF MATHEMATICS, RUTGERS–THE STATE UNIVERSITY, NEWARK, NJ 07102.

INFINITE FAMILIES OF NONTRIVIAL TRIVALENT GRAPHS WHICH ARE NOT TAIT COLORABLE

RUFUS ISAACS

1. Introduction. Among extant edge coloring problems, the 3-coloring of trivalent graphs is prominent because of its relation to the classic 4-color conjecture on maps. The story is splendidly told by Saaty in [6], but we shall follow a self-contained route for the unitiated by supplying the needed definitions and basic results in Part 2. Well-known theorems there are followed by terse, suggested proofs in parentheses.

To prove the classic 4-color conjecture, it suffices to solve these two problems (See Section 2.1, I and II):

(1) To find all uncolorable trivalent graphs or characterize them in some reasonably constructive way.

(2) To prove all such graphs are not planar.

Problem (2) seems tractable enough as indicated by Section 5.2. Thus (1) can be viewed as a version of the 4-color conjecture unencumbered by the topology of the plane. But aside from this application, (1) is a worthy mathematical problem in itself.

The earliest uncolorable 3-graph, the Petersen graph [1], dates from 1891. It is given and discussed here in Section 2.2. For it, V = number of vertices = 10 which is shown to be as small as possible in Section 5.3.

Between then until the present paper, to my knowledge, only three other non-trivial uncolorable 3-graphs have been found. ("Nontrivial" is an essential, but somewhat elusive, qualifier. Our usage of it is explained and defended in Section 5.1.) Their discoverers, dates, V and references are

Blanuša	1946	18	[2]
Descartes	1948	210	[3]
Szekeres	1973	50	[4]

and the graphs themselves will appear in Section 3.1.

This state of affairs merits two antithetical comments:

1. *It is credible.* Uncolorable 3-graphs are extremely rare in the class of all such. The search for the uncolorable I found a fascinating pastime for spare moments over many months. One who so indulges — and I recommend it as a pleasant diversion for any mathematician — will be vividly impressed with the maddening difficulty of finding a 3-graph he cannot color.

2. *It is incredible.* The main result of this paper is the discovery of an opulent infinite set of uncolorable 3-graphs (Part 3) of which the preceding three examples are members without any special conspicuity. I call it the BDS class after the three authors without whose work this class could not have come into being. In this light it is hard to believe that the three stood alone so long and with interludes of about 50 and 30 years.

I am certainly no authority on graph theory, but Professor W. T. Tutte certainly is. He informs me that the first two are the only uncolorable cases he knows since Petersen's (the third appeared subsequently). Everything I have read or asked elsewhere confirms this situation.

Part 4 supplies an infinite sequence of uncolorable 3-graphs, termed $\{J_k\}$, which I believe new and not in the BDS class. There is also in Part 4 one further uncolorable example, the double star graph, which belongs to a class Q. This class, although well-defined, offers only this one new instance at present.

To assess the plenitude of uncolorable 3-graphs, below is their quantity, as derived from the ideas herein, for each V = number of vertices = necessarily even:

$V < 10$	None	(Sect. 5.3)
$V = 10$	1 (Petersen's)	(Sect. 5.3)
$V \geq 18$	More than 1	(Sect. 3.2).

I have not attempted accurate counts in the last cases. Whether uncolorable 3-graphs exist (aside from trivial cases as later defined) for $V = 12, 14, 16$ is still open.

2. GENESIS

2.1 Basics. A *cubic, trivalent,* or *3-graph* G is a connected, finite graph with exactly three edges meeting at each vertex. We also require G to satisfy some further conditions which will be stated and discussed in Section 2.4.

A *Tait coloring* or *3-coloring* of G consists of assigning one of three colors to each edge so that the three edges meeting at a vertex bear distinct colors. In this paper, *graph* or G will mean a 3-graph and coloring (as well as its grammatical variations) a 3-coloring except when otherwise noted; the coloring of a map, for example, will refer to the hues of its countries.

A map, in the classical 4-color conjecture, may, as is well known, be taken so that but three countries meet at each juncture, implying that its edges form a 3-graph. The well-known relation between the two coloring problems we break into two statements.

I. *If a map on a surface of any genus is 4-colorable, then its edge graph G is 3-colorable.*

(Let A, B, C, D be the colors of the countries. Color an edge of G 1 if the adjoining countries are A, B or C, D; 2, if they are A, C or B, D; 3, if they are A, D or B, C.)

The converse is not true. Thus, Heffter's well-known map on the torus, which requires seven colors, has an edge graph which is 3-colorable.

II. *A planar map is 4-colorable if its edge graph is 3-colorable.*

(We use Lemma 2.4.2 and the definitions immediately preceding it. If the Tait set has but one cycle, planarity requires it to have an inside and an outside. The countries inside may be colored alternately A and B; those outside, C and D. When there are more Tait cycles, the plane will be divided into more regions. We bicolor the innermost ones first and, if this is done with suitable choices between the two color pairs, we can work outwardly to complete the coloring.)

We shall persevere in denoting the edge colors by 1, 2 and 3. These symbols shall be used in the spirit that a permutation of them is immaterial. Thus, should we write "(1, 1, 2)", what we really mean is "(x, x, y), where x and y are any two distinct colors."

Let V or $V(G)$ denote the number of vertices of G; E or $E(G)$, its edges. Then there is a positive integer λ such that

$$(2.1) \qquad\qquad V = 2\lambda, \quad E = 3\lambda$$

and if G is colorable, there are λ edges of each color.

(The simple proof is like that of the well-known Euler relation entailing $V - E + F$ (F = number of faces).)

When we speak of the size of a graph (large, small, etc.) we should refer to the magnitude of λ. But we shall follow custom and use the even V instead.

2.2 The Petersen Graph. This primary instance of an uncolorable graph — hereafter denoted by P — is worthy of scrutiny at the outset of our search for more. Figure 2.2 offers three depictions of P. At (a) is the way everyone seems to draw it, but I prefer (b) which I find more wieldy for coloring experiments. At (c) we see P drawn on a torus where it forms the edges of a five-country map requiring five colors. It is not the smallest such map; there is one with V only 8, but its edges are a 3-colorable graph.
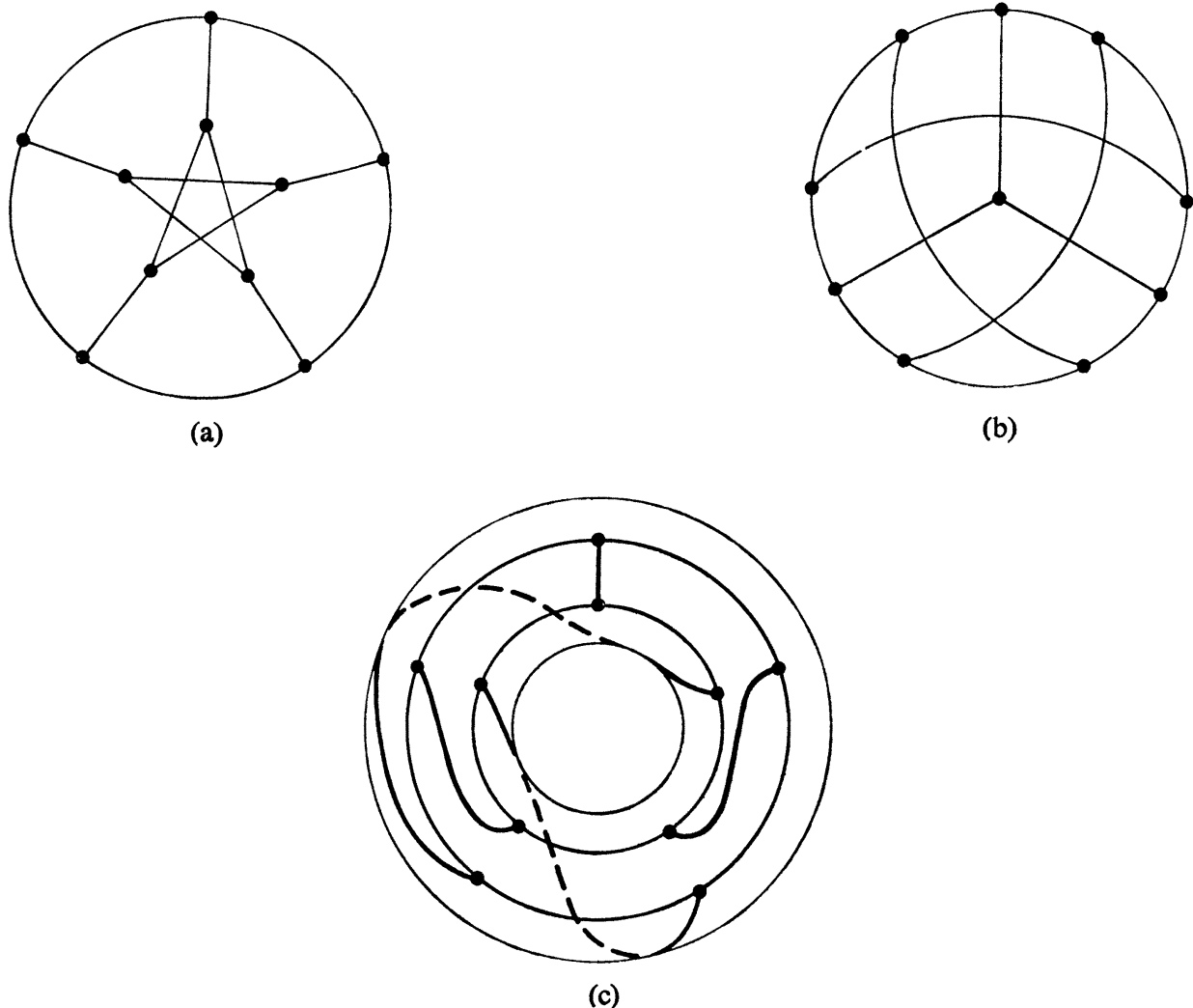

(a)

(b)

(c)

FIG. 2.2

There are several simple ways of showing P to be uncolorable. A formal proof is the case of $k = 3$ of Theorem 4.1.1.

There are further striking advents of P. Whether or not they bear on coloring problems is a tantalizing question.

(Offered by the referee): We obtain P from the graph of edges and vertices of a regular dodecahedron by identifying opposite points.

(Biggs in [8]): The usual $\binom{5}{2} = 10$ pairs from 5 objects can be the vertices of P if an edge between two means "the two pairs are disjoint."

(My observation): The well-known Heawood Graph (see [7], page 61), is a 14-gon with two vertices — i and j under consecutive numeration — also connected when $i - j \equiv 5$ mod 14. If any vertex and its three incident edges are removed, P results.

All edges (and also vertices) of P are alike; more precisely, there is an automorphism of P which takes any edge into any other. Such follows from the preceding models; for example, rotations of the dodecahedron.

2.3 Zones. Let G consist of two connected subgraphs, called *zone bounds* or *bounds* and exactly M other edges $Z_1, \cdots, Z_M$ whose end vertices lie one in each bound. If $M \geqq 2$, then for either zone bound A we require that

$$(2.3) \qquad\qquad V(A) > M - 2.$$

Then the set of Z_i is called a *zone* (or *M-zone*).

Symbols for the latter are Z (or ZM) which may also be used in an adjective or property sense of G to mean "G has a Z (or ZM)."

A zone differs from the familiar cut-set of general graph theory only through the requirement (2.3). "Zone" probably is due to Miss Descartes, although we have extended her definition in [3]. Her term probably has precedence over cut-set.

Now $M - 2$ is the minimal V the bound of an M-zone can have, as Lemma 2.5 will show. For $M = 2$, (2.3) means A is not vacuous; that is, Z_1 and Z_2 cannot be the same edge which enters and then leaves A.

Without (2.3) virtually every G would be ZM for all (sufficiently small) $M > 1$. For example, any three edges incident to a common vertex would comprise a Z3; any edge plus its end vertices would be the zone bound of a Z4.

The *zonality* of a graph G is defined and annotated by

$$\text{Zon}(G) = \min\{M: G \text{ is } ZM\}.$$

2.4 Further graph requirements. Four conditions — indicated collectively by GC — which we shall generally take as part of the definition of a 3-graph, begin with

$$\text{No}ZM \ (M = 1, 2, 3).$$

They mean: G has no Z1, Z2 or Z3. Current papers require NoZ1, "isthmus" being the usual term for a Z1.

Note that NoZM implies that G has no loops, digons, or triangles (that is, closed circuits with 1, 2 or 3 edges) for $M = 1, 2$ or 3 respectively. The literature does not always require NoZ2 and NoZ3, but the much weaker bans on the latter two configurations are often adopted. We return to this matter in Section 5.1.

The general motive for the GC is to avoid trivially uncolorable cases. How the NoZM do this will emerge from Lemma 2.4.4 with amplification in Section 5.1.

In Part 2 we shall allow some violations of the GC in the early stages of certain constructions, but not in the final resulting graphs. For convenience such violating configurations will be referred to as graphs.

Other grounds for sometimes permitting violations is that they are often easily rectified. For example, the reader can readily see for himself how simple it is to purge digons and triangles in coloration problems.

Squares (closed circuits with 4 edges) can also be purged, as the lemma to follow will show. Thus their presence suggests triviality and hence the fourth GC:

NoSq

which means: G has no squares.

But this GC can be taken as optional. It will play no part in our analyses; the reader who wishes to abide uncolorable graphs with squares is free to do so.*

LEMMA 2.4.1. *If G contains a square, S_1 and S_2 be the graphs obtained by deleting each of its two pairs of opposite sides. Then G is colorable if and only if one of S_1 and S_2 is.*

The reader can prove this more easily by exploring possible coloring cases than I can with text.

Note the rectification done. More may be needed on the S_1 and S_2 generated by the first.

A *Tait cycle* of G is a set of an even number of edges of G and their incident vertices which constitute a simple, closed curve. A *Tait set* is a disjoint set of such cycles which contain all the vertices of G.

Very well known is

LEMMA 2.4.2. *A graph G is colorable if and only if it has a Tait set.*

(If G is colored, starting from a vertex v, form the path lying on edges alternately colored 1 and 2. As G is finite there must be a first recurrence of a path vertex. This can happen only at v. If there are vertices not on the Tait cycle just constructed, start again, etc.

Conversely, if G has a Tait set, color its edges alternately 1 and 2, and the others 3.)

A consequence is: *A Z1 graph is uncolorable.*

This motivates (in part (see Sect. 5.1)) NoZ1.

A Tait set with but one Tait cycle is a Hamilton cycle. Thus when G has a Hamilton cycle it is colorable. The converse — although experience suggests that it holds nearly always — is not true; there are many colorable 3-graphs known which have no Hamilton cycles.

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* But the lemma will be used in Part 5 when we are concerned with minimal size graphs.

The following basic lemma appears in both [2] and [3]. It will be used often in the sequel.

LEMMA 2.4.3. *In a ZM of a colored graph G, let n_j be the number of Z_i of color j. Then*

$$n_1 \equiv n_2 \equiv n_3 \equiv M \qquad (\text{mod } 2).$$

Likewise for cut-sets.

Proof. Those Z_i which are colored 1 and 2 must belong to a Tait set which must cross from one zone bound to the other an even number of times. Thus $n_1 + n_2$ is even. This proves the first congruence and also the second. As to the third:

$$n_1 \equiv 3n_1 \equiv \sum_j n_j = M \qquad \text{mod } 2.$$

Likewise for cut sets as (2.3) is not invoked.

An edge lacking one end vertex will be called a *pendant*. Thus, if we sever an edge of G, we create two pendants; if we remove a vertex, we create three. Each resulting figure is an instance of a *graph with* $[M]$ *pendants* or a G_p $[G_pM]$. A G_p of course is not a graph. However, if we take one or several G_p and weld each pendant to another, graphhood is usually restored.

These terms are merely a handy locution. If G is Z and we sever the Z_i, G splits into two G_p which do not differ instrumentally from the zone bounds of Z. In fact, when lecture hall and blackboard is the medium, pendants are dispensable.

Every G_p is assumed to satisfy the GC in the sense that it is such a zone bound of some G; likewise it is connected.

Let G be Z2. Sever Z_1 and Z_2 and weld together the two pendants from each zone bound. We then have, after rectifications if needed, two graphs, G_1 and G_2. For a G which is Z3, sever all three Z_i and weld the pendants from each zone bound to a new vertex. Again call the two resulting graphs G_1 and G_2.

LEMMA 2.4.4. *A G which is Z2 or Z3 is colorable if and only if G_1 and G_2 both are.*

Proof. Let G be colored. If it is Z2, from Lemma 2.4.3, Z_1 and Z_2 will be colored alike and the coloration of G carries over into G_1 and G_2. If G is Z3, the lemma tells us that the three Z_i will be colored 1, 2 and 3. Again the coloring of G persists in G_1 and G_2.

Now let G_1 and G_2, arising from a Z2, be colored. If necessary, permute the colors in one G_i so that the two welded edges of G_1 and G_2 match. The colorings now serve for G. The reasoning for the Z3 case is similar.

Hence some motivation for NoZ2 and NoZ3! If we find an uncolorable G, which is Z2 or Z3, we can perform on it the preceding dissection and know that one of G_1 and G_2 is uncolorable. By discarding the other one, we rectify the GC violation.

More forceful is the converse. Suppose we find an uncolorable graph U. By severing an edge and welding the two arising pendants to an arbitrary A, which is a G_p 2, we obtain an uncolorable graph denoted by $U2A$. Similarly $U3A$ arises from removal of a vertex from U and welding the three pendants to A, now a G_p3.

As A is arbitrary either operation yields an infinity of uncolorable graphs. It seems natural to condemn all but U as being trivial; see Section 5.1.

2.5 Minimal G_p. Such are G_pM of least V.

LEMMA 2.5. *If A is a G_pM with $M \geqq 2$, the minimal possible $V(A)$ is $M - 2$.*

Proof. Let $f(M)$ be the sought minimum. For $M = 2$, we accept A's being the vacuous graph and so $f(2) = 0$. For $M > 2$, this — two pendants thought of as lying on one edge with no end vertices — cannot occur, or A would be not connected.

From $M = 3$, clearly $f(M) = 1$. But for $M > 3$ we cannot have three pendants from the same vertex or again A would not be connected.

For $M > 3$, one of these cases must arise: (1) Each pendant is incident to a distinct vertex; (2) A pair of pendants meet a common vertex v. For (1), $V(A) \geqq M$ and $= M$ when A is an M-gon. For case (2), letting A be minimal, we proceed inductively. The remaining edge from v and the $M - 2$ pendants other than the pair meeting v can be considered a fresh set of $M - 1$ pendants from a G_p, which must be minimal if A is and therefore has $f(M - 1)$ vertices. Then

$$V(A) = 1 + f(M - 1) = 1 + ((M - 1) - 2) = M - 2.$$

As such is less than the M ensuing from case (1), the lemma is proved.

3. THE BDS CLASS OF UNCOLORABLE GRAPHS

3.1. The class. The letters in the title stand for Blanuša, Descartes and Szekeres* whose graphs belong to this class and who inspired its construction.

We shall use U and W to denote uncolorable graphs and A an arbitrary pendant graph which may or may not be colorable. Such will be used as components in constructing a final graph G. Pendants will be bestowed on the U and W component graphs and pairs of pendants will be welded together. There will be various *types* of pendant bestowal which will imply rules for pendant colors should G be colorable.

We have already seen two such constructions: $U2A$ and $U3A$. The types used here will be symbolized by (e) and (v). Two other types will suffice for us:

(e, e). Sever *any* two edges of a component U which meet no common vertex. From each we obtain a pair of pendants. Then each pair must bear the same two unlike colors, for any coloring of G.

Proof. The four pendants lie on a Z4 of G and we can apply Lemma 2.4.3.

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* If, through ignorance, I have omitted other prior discoverers of uncolorable graphs, I beg any writer who adopts my nomenclature to include their initials in the title.

If one pair had matching colors, from the lemma, so must the other. But then the unsevered U would be colorable. Thus we may suppose one pair is colored 1, 2. Again from the lemma, so must the other.

(vev) (or $(v_1 e v_2)$). Here v_1 and v_2 are the end vertices of *any* edge e of a component W of G. Sever the other two edges meeting v_1 and obtain a pendant pair. Obtain another likewise from v_2, discarding e, v_1 and v_2. Then each pair must bear matching colors.

Proof. As before we assume a colored G with a Z4 and may apply Lemma 2.4.3. It tells us that if one pair were colored 1, 2 so would be the other. But then, by letting e be colored 3, the unsevered W would be colored.

We now define our basic operation which we call a dot product. Let U and W be any two graphs. Let four pendants emerge from U of type (e, e) and four from W of type (vev). By $U \cdot W$ we mean a graph obtained by welding the pendants of U to those of W in *any* way as long as pairs — in the sense of definitions of (e, e) and (vev) — weld to pairs.

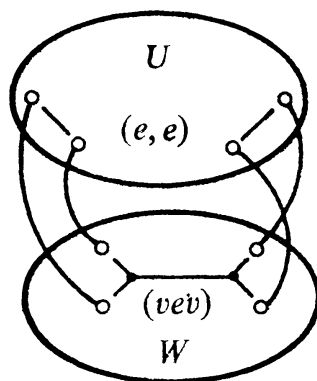Figure 3.1(a) diagrams $U \cdot W$.

FIG. 3.1 (a)

Observe that $U \cdot W$ is a set of graphs, for in the preceding text there are three usages of "any." They can denote choices, hence different $U \cdot W$. (Not always: if $W = P$, all (vev) are alike if we recall the final paragraph of Section 2.2. However, we shall soon have graphs with A components and the choice will be rich.)

THEOREM 3.1. *If U and W are uncolorable, so is $U \cdot W$.*

Proof. Were $U \cdot W$ colorable, it could be G in the proofs of the (e, e) and (vev) color rules and these rules would hold. But they are incompatible.

We are now in a position to place one of the extant examples.

Blanuša's graph is $P \cdot P$.

(Rather one of the $P \cdot P$. But which one does not seem very consequential.)

There are variations on the dot theme. First, we can inductively compound the operation. Infinite sequences of uncolorable graphs can be built in steps: any uncolorable graphs from earlier steps can be used for U or W in the next.

Szekeres' graph, with $V = 50$, is of this kind. It is

$$P \cdot (P \cdot (P \cdot (P \cdot (P \cdot P))))$$

where, in terms of Figure 2.2(a), he takes the five e in (vev) as the five radial edges of the rightmost P in the formula and the (e, e) of the other P are a side of the star and the opposite arc of the circle.

A second variation arises from letting U violate NoZ2 or NoZ3. If the two e of the (e, e) are chosen on opposite zone bounds of the Z2 or Z3 of U, then the GC of $U \cdot W$ will hold.

We could form, for example, $U2A \cdot W$. Another example is diagrammed in Figure 3.1(b). It is

$$A_1 2U3A_2 \cdot W$$
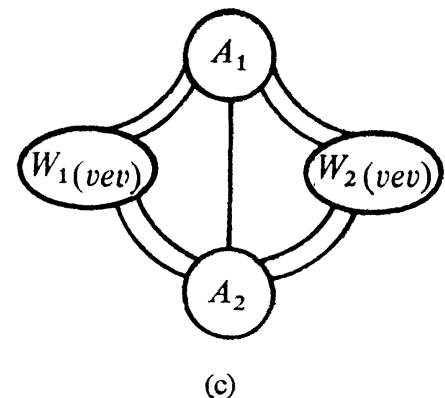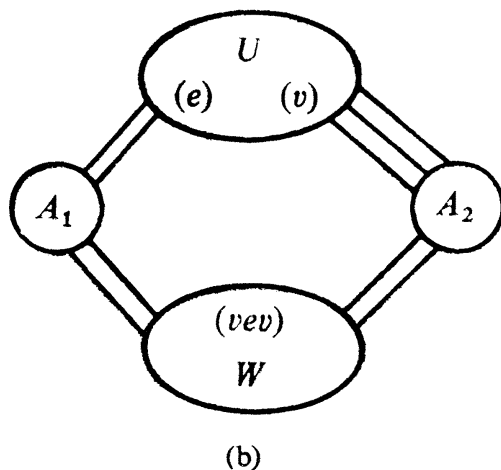
and is, I hope, self-explanatory.

(b)   (c)

FIG. 3.1

Even a NoZ1 is permissible. Were we to form $A_1 1A_2 \cdot W$, it would be Z3 and so unacceptable. But we can restore the GC by a second dot operation, obtaining

$$(A_1 1A_2 \cdot W_1) \cdot W_2$$

which is depicted in Figure 3.1(c).

Finally, a variation leading to large uncolorable graphs. We first form the graph(s)

$$G_2 = A_3 3U_2 3A_4 \cdot (A_1 3U_1 3A_2 \cdot W).$$

(The operational 3's could be replaced by 2's and so throughout) so that the two edges appearing in the two (vev) of W have a common end vertex v_0. The reader who carries out the construction will see that, after the first dot operation, the remaining edge from v_0 has become an edge WA_2, but it can be used in the second (vev) nontheless. The G_2 he will obtain appears in Figure 3.1(d).
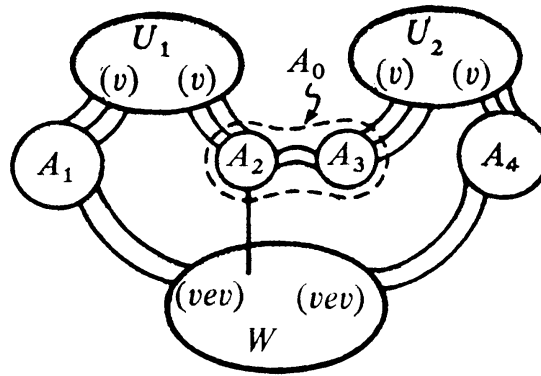
FIG. 3.1 (d)

We now ask: Can A_2 and A_3, connected by two edges, be replaced by a single arbitrary graph A_0, as suggested in the figure? They can. We only need realize that the sole role of the A_i in these constructions is as a vehicle for Lemma 2.4.3. If it holds for A_0, then the two edges $A_2 A_3$ (which is not a Z2 of G_2) can be so colored that it holds for each of A_2 and A_3, while the converse is clear.

We so blend A_2 and A_3 into A_0 and then construct

$$G_3 = A_5 3 U_3 3 A_6 \cdot G_2$$

using for the e of the (vev) in G_2 the third edge incident to v_0, which has by now become the central vertical line in the Figure 3.1(d). We again blend two A_i into one, which now has nine edges emerging from it, three going to each U_j.

We continue this procedure until the edges of W are exhausted. We reach $G_{E(W)}$, from whose diagram W has disappeared. But our final graph is isomorphic to W in the following sense. The (final) A_i correspond to vertices of W and the U_j, to the edges. When a vertex and edge of W are incident, their isomorphs are connected by triples of edges which are of the (v) type in each U_j.

Note that "triples" could as well be "pairs" (of the (e) type at the U_j) which gives a second isomorph of W.

Blanche Descartes' graph is of this (first) isomorphic type with the U_j and W all equal to P and the A_i all "nonagons."

Her derivation (or proof) is quite different from ours. It is not based on repetitions of one operation but proceeds directly (and more simply) to the final graph. She employs reasoning somewhat akin to our approach to Theorem 3.1. We leave to the reader the pleasure of reconstructing her elegant proof; the adaptation to the second kind of isomorph is likewise rewarding.

If our derivation is longer it is because our objective is to unify. The Descartes and dot techniques at first looked very disparate, but unity is obtained by the latter. To recapitulate, it is:

The BDS class consists of results of repeated dot products of given uncolorable graphs. The second operands may violate the NoZM ($M = 1, 2, 3$) provided that connections can be made which restore ultimate obedience to the GC.

3.2 Graph sizes in the BDS class. We inquire as to graphs of what V appear in our class. We can see that $V(P \cdot P) = 18$. This is as small as possible, for Theorem 5.3 will show P to be the smallest uncolorable graph.

If the A_i appearing are allowed to be completely arbitrary (see Sections 3.3 and 5.1) we can use this lemma from which we omit a proof:

LEMMA 3.2. *For a fixed $M \geqq 4$, there exists G_pM with V any even number $\geqq M - 2$, except when NoSq holds, $M = 4$ and $V = 4$ or 6.*

Using the preceding examples we find that there are BDS graphs with $V = $ any even number $\geqq 18$.

For certain V they will be especially numerous due to variegated dot products, using, say, the J_k of Part 4 as U or W components, etc. But we have not attempted any precise counting.

3.3 Zonalities in the BDS Class. Clearly

$$(3.3) \qquad \qquad \mathrm{Zon}\,(U \cdot W) = 4$$

for the four edges UW are a Z4 and U and W fulfill the GC, banning lesser zones. The same zonality might appear to hold for any BDS graph G: if G is constructed from repeated dot operations, we can regard (3.3) as valid for the last. But our conclusion is untrue.

We return to G_2, depicted in Figure 3.1(d). Before A_2 and A_3 were blended, $\mathrm{Zon}\,(G) = 4$ as the two edge pairs WA_4 and A_2A_3 are clearly a Z4. But if A_2 and A_3 are blended, the choice of into what A_0 determines the zonality. We see that $\mathrm{Zon}\,(G_2) = 4$ can require that, by severing two edges of A_0, it splits into two parts of which one meets the three edges A_0U_2 and these only. But there are choices of A_0 that require more than two severings to attain this split. If so, $\mathrm{Zon}\,(G_2) = 5$ (the edges WA_1, WA_0, WA_4 are a Z5) provided that the zonalities of U_1, U_2, and W_2 each $\geqq 4$ (such seems true, for example, if all three are P).

But it is also possible to choose A_0 so that one severing will suffice: the zonality is then 3.

Were the zonality of all BDS 4, it would seem to be useful towards characterizing this class. For example, we would know the uncolorable graphs of Part 4 are a distinct set (see Section 4.3).

Despite the preceding arguments, there does seem something intrinsic about zonality 4 for the BDS class. Perhaps in the future of this theory, something like this will be done:

The BDS graphs are divided into equivalence (or something like) subclasses so that all with the same diagram — in the sense of Figure 3.1 — belong to the same subclass. In each there are canonical members determined by some canonical choice of the A_i. A new zonality of G could be defined as the old zonality of a canonical representative of G's class.

This seems a hopeful possibility for canonical A_i: A_i is a polygon such that all edges leading to the same other component of G emanate from consecutive vertices of A_i. Then two severings of A_i would lead to splits suitable for zonal divisions.

4. AN INFINITE SEQUENCE OF UNCOLORABLE GRAPHS, THE Q CLASS AND THE DOUBLE STAR

4.1 The sequence. These graphs will be denoted by J_k for k an odd integer ≥ 3. The first three, which are typical, appear in Figure 4.1(a).

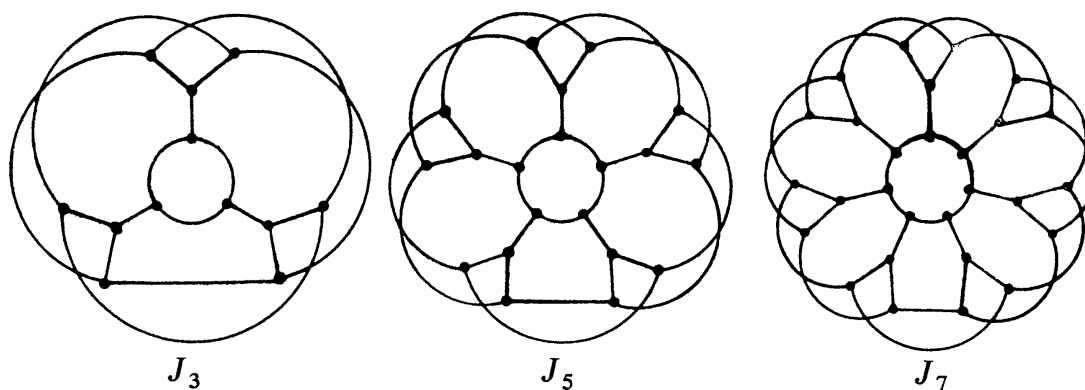
J_3 J_5 J_7

FIG. 4.1 (a)

We see that J_3 is the Petersen graph after rectifying the violation of NoZ3 by replacing the central circle by a single vertex. (Notationally, if T is the triangle, $J_3 = P3T$.)

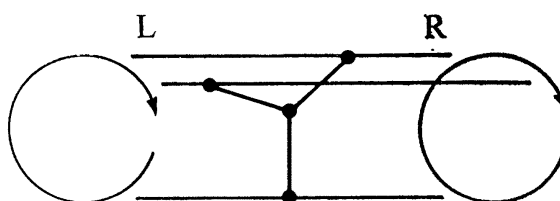THEOREM 4.1.1. *The J_k are uncolorable.*

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FIG. 4.1 (b)

Proof. Let Y be the G_p shown by Figure 4.1(b) which has three pendants extending to the left and three to the right. If Y is colored, how will the right pendants respond to assigned colors on the left? There are three possibilities for this assignment:

(1) $(1,1,1)$ (or all colors on the left alike). This is impossible as one of the three central edges must be colored 1.

(2) $(1,1,2)$ (or just two colors alike). The reader who explores the simple possibilities will find there are two. Both yield for colors on the right $(2,3,3)$ but in two different orders.

Had we started with $(2,3,3)$ on the left we similarly will conclude with $(1,2,2)$, in some order, on the right. Thus, if we form a chain of replicas of Y by welding the right pendants of one to the left of the next, the two kinds of colorings will alternate. We conclude that a closed circuit so made from an odd number of Y cannot be colored in this fashion.

(3) $(1,2,3)$ (no colors alike). Again we beg the reader to explore. There again result two possibilities. Both are $(1,2,3)$, but in both cyclic order, in the sense of the arrows in Figure 4.1(b), is preserved. So it will be if we weld together two Y when both remain as drawn. But if we were to make one pair of the newly welded edges cross, the cyclic ordering would be reversed. Therefore, if we build a closed circuit of Y with an odd number of such crossovers, it could not be colored in this fashion.

Hence a circuit with an odd number k of Y and an odd number of crossovers cannot be colored at all. But such is J_k — in fact, should be taken as its formal definition.

REMARK. Graphs can be deceptive graphically. Thus, when making the three drawings of Figure 4.1(a), my intent was to depict just one crossover in each on its double outer rim. This was to occur at the bottom! Superficially such seems exactly contrary to appearance. I leave this mild paradox to the reader.

Or we can simply say that an odd number of crossovers is to mean that the two outer rims consist of a single circuit.

4.2 The possible Q class and the double star. The class Q of uncolorable graphs is called possible, because as yet I know of only three members. Of these only one is new; I call it the double star graph and it is depicted in Figure 4.2.

Of a G_p5, from Lemma 2.4.3, the pendants must be colored

(1) $1, 1, 1, 2, 3$

in some order. Now suppose the G_p5 has rotational symmetry, so that, from a suitable color ordering of (1), a cyclic permutation will give another. There are only two possibilities, representable either by $C = (1,1,1,2,3)$ or $S = (1,1,2,1,3)$. In other words, either the three matching pendants are consecutive (C) or they are separated in the only way possible (S).

Let H_S $[H_C]$ be the set of all rotationally symmetric G_p5 which are *not* colorable when the colors of the pendants are $S[C]$. We define: $H = H_S \cup H_C$.

The class Q is the set of G arising in

THEOREM 4.2.1. *From each pair* H_1, H_2 *(distinct or not) in H we can construct an uncolorable G.*

Proof. If $H_1 \in H_S$, $H_2 \in H_C$, we form G by welding the pendants of H_1 to those of H_2 so as to preserve their cyclic order. The new edges must be colored according to C or S; either way contradicts the definition of H_1 or of H_2.
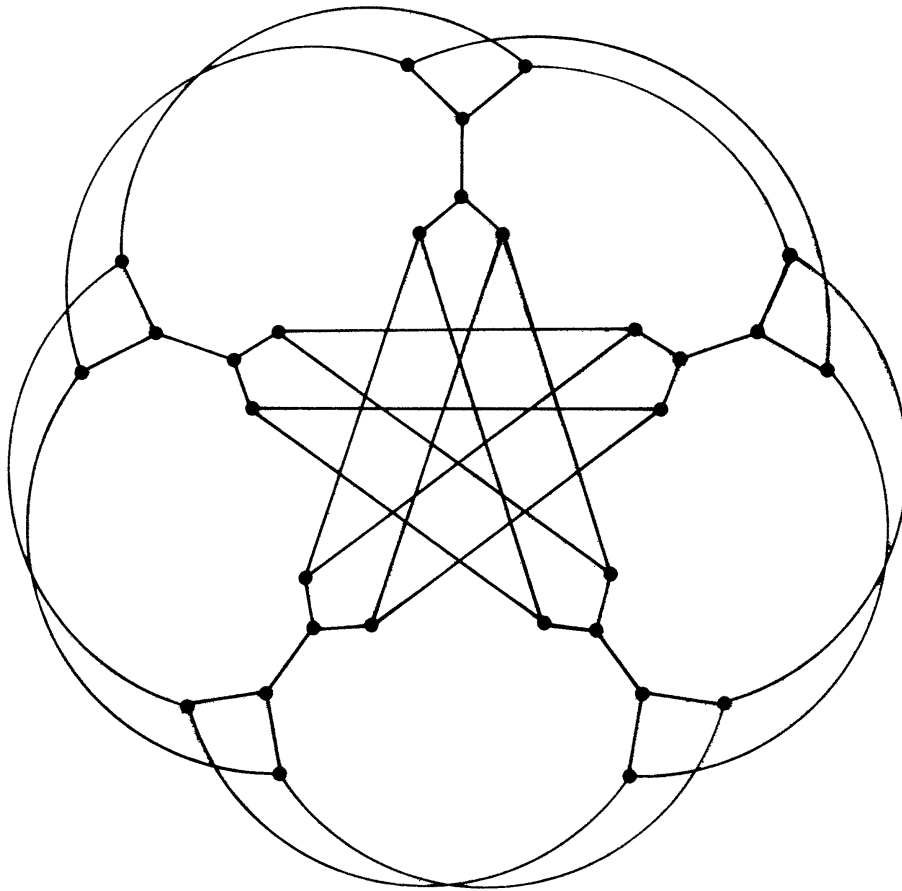
FIG. 4.2

If H_1, H_2 both $\in H_S$ (or H_C), we now weld their pendants so that adjacent ones on H_1 attach to alternate ones on H_2. Then an S (or C) coloring of the pendants of H_1 transfers into a C (or S) coloring on H_2, so that a total coloring is impossible.

$$\text{Notation:} \quad G = \langle H_1, H_2 \rangle.$$

In Figure 4.1(a) sever the five radial edges of J_5, cutting it into two rotationally symmetric G_p5. The inner one, a pentagon, we call V, the outer, we call V^*. Now V can be colored with C holding for the pendants by coloring the sides of pentagon $3, 2, 3, 2, 1$. Then V^* cannot be colored with C holding or, by rewelding our severings, we would have colored J_5. Therefore $V^* \in H_C \subset H$.

Observe that P is two copies of V with adjacent pendants of one welded to alternate ones of the other. Were V also colorable with S holding, then P would be colorable. Therefore $V \in H_S \subset H$.

We have established

$$P = \langle V, V \rangle, \quad J_5 = \langle V, V^* \rangle.$$

The third known member of Q is

$$\langle\, V^{*},\ V^{*}\rangle$$

which is the double star. Thus we have proved

THEOREM 4.2.2. *The double star graph is uncolorable.*

4.3 Nonmembership in the BDS class. The preceding graphs certainly do not appear to belong to the BDS class, although I do not claim a rigorous proof. The evidence is:

(1) We have seen in Section 3.3 that there is an aura of zonality 4 over BDS graphs, while for Part 4 we appear to have

$$\text{Zon}\,(Q) = 5$$

$$\text{Zon}\,(J_k) = 6 \qquad (k > 5).$$

(2) In the BDS class arbitrary A can be inserted into any related pair or triple of edges connecting a U or W. The simplest such nonvoid A is a "rung," that is, a new edge bridging two new vertices inserted in each of two edges (with triangles avoided).

Such a rung cannot be inserted in P without rendering it colorable, as can be learned from exploring all cases. From some sample trials on the J_k, the same appears likely true, but I know of no general proof.

(3) The BDS spring from at least two initial uncolorable graphs. But the J_k and double star do not appear to contain anything of such ilk.

We might note that BDS graphs might be built using a J_k or double star for the various U and W, but U and W themselves are not in BDS.

Also note that we have constructed both BDS graphs as well as the J_k and Q by welding together components, but the underlying ideas seem very different.

5. SOME GENERAL IDEAS

5.1 What means trivial? We have claimed nontriviality in the main title and feel a defense is obligatory. As to the question in the section title, this seems the underlying principle:

If we have found one evasive solution to a problem and others arise from it in an obvious way, we tend to call these others trivial. They can then be banned by adding postulates designed for this purpose.

Thus the answer reduces to that of another question:

What means obvious?

It is, of course, a relative term, depending on our sophistication and that of our mathematical era.

But let us turn to levels of what might be deemed trivia in our current subject.

(1) Graphs with an isthmus or Z1 seem universally banned. There are two reasons. First, if such a graph is planar it cannot be the boundary edges of a map. Second, although there surely are infinitely many $A1A$ graphs, all uncolorable, they are about as obviously so as a 3-graph can be.

(2) We have seen that from one U we can form infinitely many $U2A$ and $U3A$, all uncolorable. Although the literature I have found on Tait coloring is sparse, I think it astonishing that I have not encountered a specific ban on such graphs, aside from the weak special cases of digons and triangles. Nevertheless, I assume the ban is there tacitly. It is hard to imagine anyone seeking uncolorable 3-graphs without becoming aware of the $U2A$ and $U3A$ possibilities. Yet I have read the statement that very few uncolorable G are known. This must be so from either ignorance or a tacit ban; I take the latter as far more likely. Yet "How obvious?" is here a puzzling question.

Note that all this occurs in map coloring too, where the literature is much vaster, although I have read but a bit of it. For example, if U is the edge graph of a map on a torus 7-or-more-colorable and A any planar G_p, then $U2A$ and $U3A$ are edge graphs of other such maps.

(3) If arbitrary A components, as in (2), are to be banned, as we did in the GC, should they not be banned in the BDS class also? Hardly, for then such desirable specimens as Miss Descartes' isomorphs would have to go. We could check a trivial infinitude by only admitting A of some canonical form. But what form? We could, as in Section 3.3, select polygonal A so as to preserve zonality. Minimal G_p is another appealing possibility (but there is more than one minimal G_pM for a given $M \geq 4$).

Probably some reckoning by subclasses, as suggested in Section 3.3, is the future course.

The principle given earlier could be pushed further. The BDS class springs from applications of the dot process. We could become so sophisticated as to call this an "obvious way" and hence exile the entire BDS class.

But such exiling, if pushed far enough, would annihilate all mathematics.

5.2 Planarity. We know that "Every uncolorable 3-graph is not planar" implies the classical 4-color conjecture. Let us accordingly look then at the ascertainment of planarity of 3-graphs.

The well-known theorem of Kuratowski (see, for example, [5]) states that a general graph is not planar if it contains one of two particular graphs. Now one of these two is not trivalent, so we need only be concerned with the other. It is shown in Figure 5.2(a) and is called the *utilities graph* — UG for short. By *G possesses UG* is meant that a graph like UG, except that its edges may be replaced by paths, is a subgraph of G. In other words, if we are allowed to alter Figure 5.2(a) by putting new dots on the interior of its arcs, it can become a depiction of a subgraph of G.

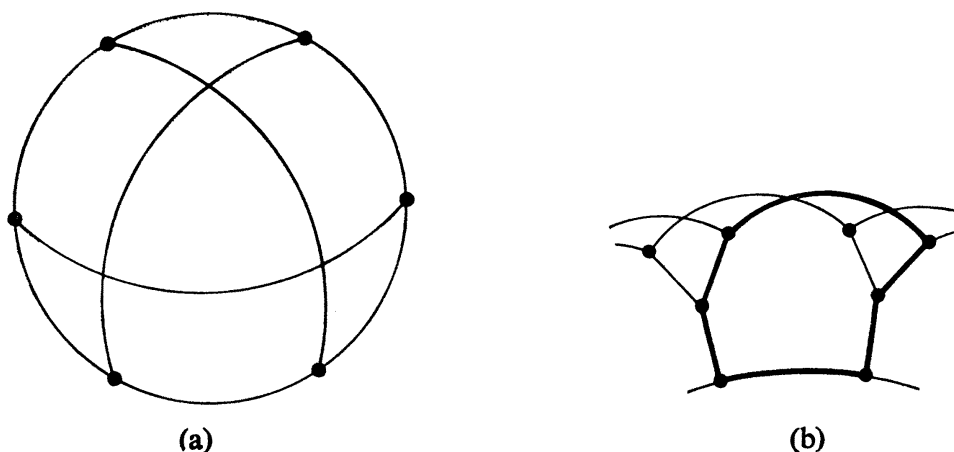(a)   (b)

FIG. 5.2

Thus our tool here is

LEMMA 5.2. *G possesses UG if and only if G is not planar.*

THEOREM 5.2.1. *The Petersen graph is not planar.*

Proof. A glance at Figures 2.2(b) and 5.2(a).

THEOREM 5.2.2. *Let G be in the BDS class. Then G is not planar if one of its original U or W components is not.*

Proof. Let U be such a component so that U possesses a UG. The UG will survive the construction of G. Suppose a pair of connecting edges from U to another component C (a U, W or A) if G were of type (e) at U and that the severed edge e of (e) belonged to the UG. We can replace e by a path in G by starting from one end vertex of e, following the connecting edge to C, then, as C is connected, following a path in it to the other connecting edge and back to the other terminal of e. Other types of connecting edges are handled similarly.

THEOREM 5.2.3. *The J_k are not planar.*

Proof. For the outer circle of the UG we can use the hexagon indicated by the heavy lines in Figure 5.2(b). The remaining paths of the UG are not hard to discover.

I have deliberately refrained from thinking about the planarity of the double star graph so that I can bequeath a lottery ticket to the reader, albeit at long odds. Let him investigate the question. If the answer is yes, he will have the glory of having resolved the 4-color conjecture.

5.3. The Petersen graph is the sole smallest uncolorable graph. The *girth* $\gamma(G)$ of G is the smallest n such that G contains an n-gon. Tutte, in [7] Chapter 8, proves that the smallest graph of girth 5 can only be P. As his work is embedded in a framework of broader results, we give a simplified adaptation here.

If $\gamma(G) = 2, 3$ or 4, G would contain a digon, triangle or square, and so, as shown earlier, could not be smallest uncolorable graph. Thus we assume $\gamma(G) \geq 5$.

Let $\gamma(G) = n$ so that G contains an n-gon H. There is a third edge (not in H) e_i emanating from each vertex v_i of H. Clearly no e_i can join two v_j. Nor can any e_i and e_j terminate at the same vertex (not in H) or e_i, e_j plus the smaller "arc" of H joining v_i, v_j, would lie on an m-gon with $m \leq 2 + [n/2]$ which would be smaller than H when, as here, $n \geq 5$. Thus, $V(G) \geq 2n$. Thus, if $n > 5$, $V(G) > 10$ and G is larger than P.

We now have $\gamma(G) = 5$ with each v_i connected to a new vertex v_i'. If there are to be no further vertices, the five v_i' must, to avoid triangles, etc., belong to a second pentagon. Now the two pentagons must have consecutive vertices of one connected to alternate vertices of the other (again to avoid squares, etc., as the reader can easily see by sketching the various possibilities) and so P, as depicted in Figure 2.2(a) ensues and hence the titular theorem.

5.4 Tutte's conjecture. Throughout our work P seems ubiquitous and Tutte conjectured that this would be so when very few uncolorable graphs were known and to date he seems right.

Our version of his well-known conjecture — which implies the 4-color conjecture — shall be

Every uncolorable graph possesses the Petersen graph.
This form allows us to state

THEOREM 5.4. *A graph in the BDS class satisfies Tutte's conjecture if one of the original V or W does.*

The proof is virtually the same as that of Theorem 5.2.2.
I leave to the reader the hunt for P in the J_k and double star.

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DEPARTMENT OF MATHEMATICAL SCIENCES, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218.