On 3-free links

V. O. Manturov, D. A. Fedoseev, S. Kim.

Bauman Moscow State Technical University,
Moscow State University,
Moscow Institute of Physics and Technology

Zoom Conference on Physical Knotting, Vortices and Surgery in Nature

About ten years ago, the first named author introduced the theory of free knots (previously conjectured by Turaev to be trivial) and found that this theory — a very rough simplification of the theory of virtual links — admitted new types of invariants never seen before: the invariants of links are valued in *pictures*, more precisely, in *linear combinations of knot diagrams*.

For some classes of links, we have the formula

$$[K] = K,$$

where K in the LHS is our favourite link diagram (which is subject to various Reidemeister-like moves), and K in the RHS is the same diagram but seen as the rigid object.

The construction of the the “rigid” diagram from a given diagram of the link is a sort of “state-sum subdiagram summation” which for our special good diagrams can consist just of one term.

This means that if for some diagram K' we have equivalence $K' \sim K$ which yields $[K'] = K$, then it, in turn, yields:

K' contains K,

or more generally,

if a diagram K is complicated enough then any diagram equivalent to it contains it as a "smoothing".

After many years of trying, many problems in combinatorial group theory and algebraic topology solved and a book [1] written, V.O. Manturov saw that this approach does not work immediately for classical knots.
In fact, the reason is that the approach when we look at "nodes" being "double" classical crossings is not the best one for classical knots.
It is much better to look at "triple" crossings.

Below, we construct a map from equivalence classes of closed braids to *3-free knots and links* (elder brothers of free knots and links). We also consider the map from links up to link-homotopy to 3-free links. After that we discuss what to do with 3-free links: how to map them to usual free links, how to construct their invariants similar to invariants of free links, etc.

Basic definitions

Definition 2.1

A *regular 6-graph* is a disjoint union of regular 6-valent graphs (possibly with loops and multiple edges) and circles. Here we call the circles *cyclic edges of the 6-graph*.

Definition 2.2

A *framed 6-graph* is a regular 6-graph such that for each 6-valent vertex the 6 half-edges incident to this vertex are divided into 3 pairs of *formally opposite*.

Let us call two edges e_0, e_1 of a framed 6-graph *equivalent* if there exists a sequence of edges $e_0 = b_1, b_2, \dots, b_n = e_1$ such that for each i the edges b_i and b_{i+1} are opposite. The equivalence class of edges is called a *unicursal component* of the graph. A cyclic edge also is a unicursal component.

Basic definitions (cont.)

Definition 2.3

An *oriented framed 6-graph* is a framed 6-graph such that each of unicursal component is oriented.

The last definition yields that at each 6-valent vertex there are three incoming half-edges and three outgoing half-edges.

3-free diagrams

Definition 2.4

A *3-free diagram* is an oriented framed regular 6-graph such that at each vertex three incoming half-edges are ordered.

In the same way *regular 6-graphs with ends* and *3-diagrams with ends* may be defined. In that case we allow the graphs to have 1-valent vertices.

Remark 2.5

When drawing a diagram on a plane we always assume the ordering to be inherited from the plane: the leftmost component is the first, the middle one goes after it, and the rightmost one is the last).

Moves on framed 6-graphs

We consider the following set of moves on regular 6-graphs:

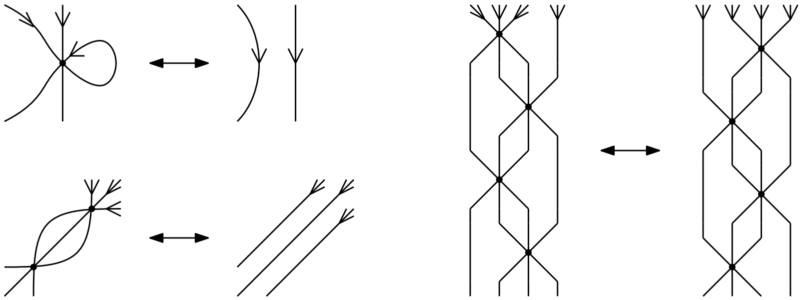

Figure: 1. 3-free moves

3-free links

Definition 2.6

A *3-free link* (resp., *3-free link with ends*) is an equivalence class of 3-free diagrams (resp., 3-free diagrams with ends) modulo the three 3-free moves, see Fig. 1.

Note that these moves do not change the number of unicursal components of a diagram.

Definition 2.7

A *3-free knot* (resp., *3-free knot with ends*) is 3-free link (resp., with ends) with one unicursal component.

Mapping conjugacy classes of closed braids to 3-free links

Our first goal is to construct a mapping from the set of conjugacy classes of closed braids to 3-free links (with ends). Note that in case we only have to deal with the second and third Reidemeister moves, and conjugations of the braid. Implicitly (in algebraic framework) it was done by Manturov and Nikonov in 2015 [4].

Consider a closed braid K. We may assume that the diagram of K lies in some annulus A on a plane Π. Fix a line l orthogonal to that plane and intersecting it inside the inner circle of A.
Consider the family of halfplanes whose boundary is the line l. Let us naturally parametrise this family by an angle $\varphi \in [0, 2\pi]$ and denote the family by $\hat{\Pi} = \{\Pi_\varphi\}$. By a small deformation of K we may assume that the intersection of K and a plane Π_φ is a finite set of points. Let us say that for two angles $\varphi_1, \varphi_2, |\varphi_1 - \varphi_2| < \varepsilon$, a set of points $A_1 = K \cap \Pi_{\varphi_1}$ is *after* the set of points $A_2 = K \cap \Pi_{\varphi_2}$ if $\varphi_2 > \varphi_1$ (angles 0 and 2π are considered equal).

Mapping conjugacy classes of closed braids to 3-free links (cont.)

Now consider the moduli space M of pairs of points on K lying on the same straight ray $m \subset \Pi_\varphi \in \hat{\Pi}$ (the origin of the ray may lie on the boundary of Π_φ). M is a 1-dimensional manifold with boundary.

We orient components of this manifold in the following way. For each $x \in M$ consider the corresponding $x_1, x_2 \in K$. They lie on some plane Π_φ. It naturally defines two halfspaces of the ambient space $\mathbb{R}^3$. Consider two tangent vectors to K in those points. If their endpoints lie in the same halfspace defined by the plane Π_φ, we orient the tangent vector at x downwards, otherwise — upwards. If one of the vectors is horizontal, we define orientation to preserve continuity.

Mapping conjugacy classes of closed braids to 3-free links (cont.)

Now we need to identify some triples of points of the space M. To be precise, we consider such triples of points $(a, b), (a, c), (b, c)$ that the points a, b, c lie on the same straight line $m \in \hat{\Pi}$. We identify them, and define partial ordering so that the component containing the point (a, c) lies *between* the other two components. The ordering of the triple (that is, $(a, b), (a, c), (b, c)$ or $(b, c), (a, c), (a, b)$) is defined as follows.
Consider the points a', b', c' of the closed braid K which lie on the same plane a little *after* than the line we are considering. Let $u = \pm 1$ be the sign of the frame $(a', b'), (a'c')$. Now we define $s(a, b, c)$ as $u(-1)^{\uparrow}$, where $\uparrow$ is the number of the points a, b, c where the braid closure is oriented to the "earlier" halfspace ("go up"). Finally, we set the order $(a, b), (a, c), (b, c)$ if $s(a, b, c) > 0$, and $(b, c), (a, c), (a, b)$ otherwise, see Fig 2, 3.

This way we have obtained a diagram of a 3-free link, which we denote by $f(K)$.

Ordering of the edges in a 6-valent vertex

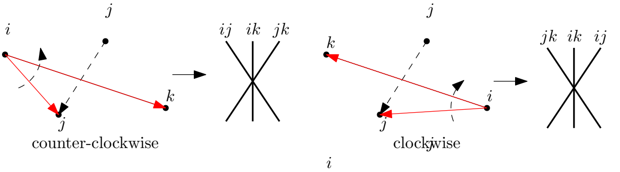

Figure: 2. If even number of the strands i, j, k go up

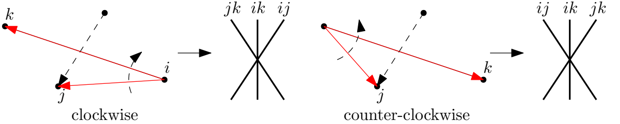

Figure: 3. If odd number of the strands i, j, k go up

The main theorem

Theorem 3.1

The mapping f is a correct mapping from the set of conjugacy classes of closed braids into the set of 3-free links with ends.

This theorem holds for the following reason. We need to check what happens when the knot goes through a codimension 1 singular position.
There are two types of such singularities:

1. small deformation of the closed braid K destroys the ray on which three points lie. This situation gives the second move F_2 on 3-diagrams;
2. closed braid goes through a configuration where *four* points lie on the same ray. This situation gives the third (tetrahedral) move F_3.

The main theorem (cont.)

Essentially, that means that the closures of two braids with (Reidemeister) equivalent diagrams yield equivalent 3-free link diagrams.

Finally, we need to understand what happens when we deal with closures of two braids which differ by conjugation (a non-Reidemeister equivalence of braids). But note that the closure of a conjugated braid may be transformed into the closure of the original braid via transformations, appearing as Reidemeister moves on the diagram of the closed braid, and such transformations produce only the singularities described above.

Intermission: the problematic move

In general, the strategy of “reading” a diagram along a certain family of straight lines and recording the moment when three points lie on a line, is effective in giving 3-free diagrams. The problem is, this mapping is not always well defined.

The problematic situation arises when two maxima (or two minima) of our object interchange their order with respect to the direction of the “reading”. This transformation obviously does not change the original object, but the 3-free diagram suffers the saddle move, which is not in the set of 3-free link moves. Therefore, to construct well defined mappings, we need to use knot theories where such transformation does not appear. The above-discussed case of closed braids and radial lines is an example of such theory. Note that if we considered *horizontal lines* instead of rays, the construction would not work.

The maxima exchange transformation

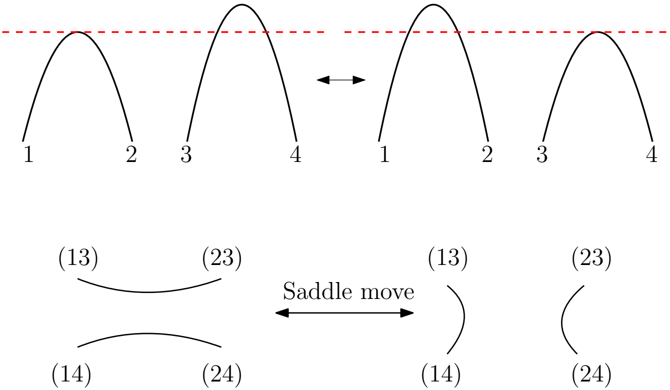

Figure: 4. Maxima exchange move

Link-homotopy

Now we move to the construction of the mapping from links up to link-homotopy to 3-free links. Our general strategy remains the same: we consider an appropriate diagrammatic language for links, and construct the 3-free link by "reading" the diagram along a certain family of straight lines. The link-homotopy case is exceptionally good in the sense that here we can avoid the problem of "exchange of two maxima" move.

We shall present links as closure of braids up to some special moves. This special form of link diagrams was described by Habegger and Lin [3].

Habegger-Lin diagrams

Let us call the *bow operation B* (see Fig. 5) the following map from braids on n strands into tangles with $2n$ ends. Consider a braid diagram β situated vertically on the plane. To perform the bow operation one takes the upper ends of the braid and bends them to the right, until they appear on the same line as the bottom ends of the braid. Hence we get a "rainbow" $B(\beta)$ with $2n$ ends lying on a horizontal line.

Armed with the bow operation, we can define the "concatenation" of a braid β on n strands and a braid γ on $2n$ strands. By definition we set

$$\beta + \gamma = B(\beta)\gamma.$$

The bow operation

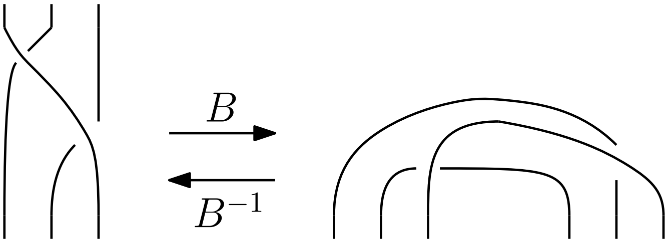

Figure: 5. The bow operation B and its inverse

Note that the "inverse bow operation" B^{-1} does not always produce a braid. Nevertheless, in the cases which are interesting for us, this operation shall be well defined.

Habegger-Lin diagrams (cont.)

We shall be interested in the following three types of $2n-$strand braids:

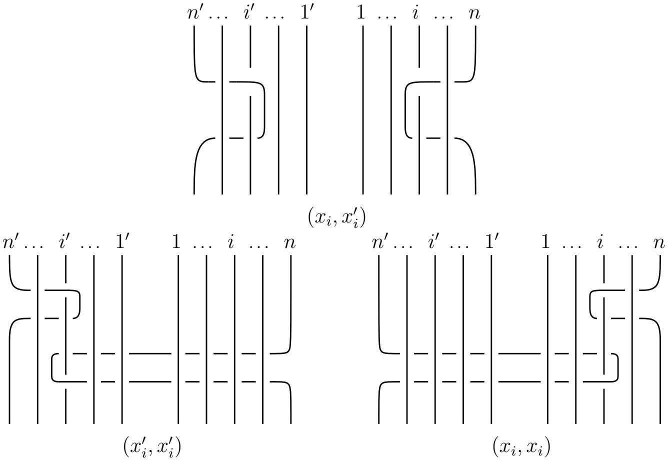

Figure: 6. Braids needed for the Habegger-Lin theorem

Habegger-Lin diagrams (cont.)

Let β be a braid on n strands, and γ be a braid on $2n$ strands of any of the types in Fig. 6. It can be checked that in this case $B^{-1}(\beta+\gamma)$ is a braid.

The Habegger-Lin theorem may be formulated in the following way:

Theorem 4.1

Two braids β_0, β_1 have link-homotopic closures if and only if they may be connected by a sequences of braids $\beta_0 = \xi_0, \xi_1, \dots, \xi_n = \beta_1$ such that for any i the transformation $\xi_i \to \xi_{i+1}$ is either an Artin move or is of the form

$$\xi_i \to B^{-1}(\xi_i + \gamma),$$

where γ is one of the braids in Fig. 6.

Mapping links up to link-homotopy to 3-free links

Now we are ready to construct our mapping. For a link L consider a braid β whose closure $\bar{\beta}$ is L. The closure $\bar{\beta}$ lies on a plane inside an annulus A. Let us put the basepoint P *inside* the annulus. Now we send out a ray beginning at P in general position with respect to the closure $\bar{\beta}$ (that is, it does not go through the crossings of the diagram and is transverse to it). Finally, we start rotating the ray (increasing the angle) and record our 3-free link in the same way as we did in case of braid closures.

Main theorem on links up to link-homotopy

Let us consider equivalence classes of 3-free links modulo the (i, i') *crossing inversion* relation:

$$a_{ii'j_1} \dots a_{ii'j_k} a_{j_1 ii'} \dots a_{j_k ii'} = 1,$$

where $i \in \{1, \dots, n\}$, i' is the "dual" strand in the Habegger-Lin sense, and the set $\{j_1, \dots, j_k\}$ is any subset of the set $\{1, \dots, n, 1', \dots, n'\} \setminus \{i, i'\}$. Here $a_{ii'j}$ denotes a crossing (6-valent vertex of a diagram) on the strands $(ii', ij, i'j)$ in that order. Denote the set of equivalence classes of 3-free links modulo the (i, i') crossing inversion for all i by $\mathcal{F}_3$. The following holds:

Theorem 4.2

The described mapping is a well defined mapping from links up to link-homotopy to $\mathcal{F}_3$.

Example 4.3

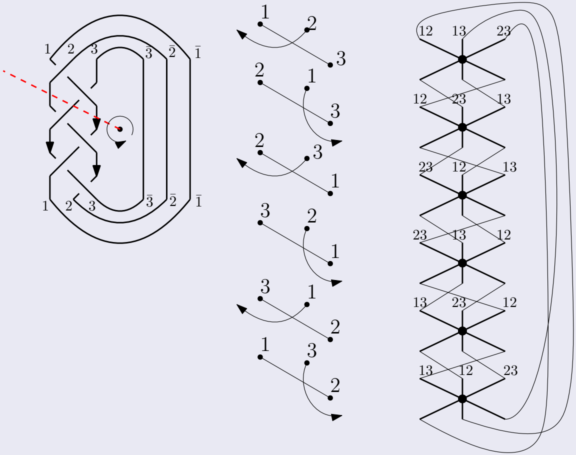

Figure: 7. Borromean link and corresponding 3-free link

On the main theorem of the section

Theorem 4.2 holds for the following reason. We need to check that two diagrams of the same link give us equivalent 3-free diagrams.

First, if two diagrams $\bar{\beta}_1, \bar{\beta}_2$ are closures of braid-equivalent braids β_1, β_2, then the claim is proved in the same way as was done in Theorem 3.1.

Thanks to the Habegger-Lin theorem 4.1, apart from Reidemeister moves we need to study the transformations $\beta \to B^{-1}(\beta + \gamma)$ with γ being one of the basic braids from Fig. 6. It turns out that this transformation does not use the “maxima exchange” move. Note, that the rotating ray will pass through extrema of the diagrams (in the sense of being tangent to the diagram), but the aforementioned extrema do not change positions during the transformation. Therefore, the corresponding 3-free diagrams undergo only the allowed moves.

On the main theorem (cont.)

The last thing we need to understand is what happens with the 3-free diagram when we perform the link-homotopy-exclusive move: change the sign of a selfcrossing of a component of the studied link. The only case when that may appear in the language of the Habegger-Lin diagrams is when we have a crossing X between the strand i and its dual strand i' (which are one and the same after closing the braid).

If we look at this crossing in the presence of a third strand, say, j, we get a 3-free diagram vertex V on the strands $(ii', ij, i'j)$. Changing the sign of the crossing X we *invert the order of the strands in the vertex* V. So for the mapping to be well defined, we need to factorise the 3-free links by such a transformation. Its most general form (for the presence of any number of additional strands around the crossing X) is exactly the (i, i') crossing inversion realtion.

A mapping to 2-free links

Now let us consider the following “strand deletion” mapping from 3-free links to 2-free links (that is, the usual free links). Consider a 3-free links with the components labelled $1, \dots, N$. Fix a component i and delete it. On the level of diagrams, that transforms all 6-vertices the i component goes through, into (framed) 4-vertices. Now, for each 6-vertex not incident to the i component, remove them altogether. Thus we get a map g form 3-free links to 2-free links.

Theorem 4.4

This mapping g is a well defined mapping from 3-free links to free links.

A mapping to 2-free links (cont.)

This theorem may be verified by a direct check of the 3-free moves.

Now, consider a classical link up to link-homotopy in the Habegger-Lin form. Map it to a 3-free link as described above (this mapping is not well defined). Map the result into 2-free links. Is the resulting mapping from classical links to free links well defined? Yes it is. The reason being that in a sense the projection mapping erases the difference between the order of the strands in a 6-valent vertex, and the link-homotopy move ceases to spoil the original mapping to 3-free links.

Theorem 4.5

The mapping $f \circ g$ is a well defined mapping from classical links up to link-homotopy into free links.

An invariant of 3-free links : Basic idea

In knot theory, the basic, but one of important, problem is to know whether a given knot (or link) is trivial. Note that, if a given knot (or link) is trivial, then every crossing c can be removed by the first Reidemeister move or has another crossing c' such that c and c' can be canceled by the second Reidemeister move.

Especially, for braids, if a given braid is trivial, then it must be possible to make pairs of crossings such that each pair of two crossings are canceled by second and third Reidemeister moves. In other words, if there is an “essential” crossing of a braid, which cannot be canceled by the second Reidemeister moves, then we can say that the given braid is non-trivial. One can find some results in sections 8 and 9 in [1].

In this section we will apply the above idea for 3-free links. In the end of this section we will show that the 3-free link obtained from Borromean ring is non-trivial and, therefore, Borromean ring is non-trivial by using G_n^3-like structure.

Definition 5.1

A 3-free link with $2n$ end points is called *a 3-free braid on n-strands*, if it can be placed on $[0,1] \times [0,1]$ satisfying following conditions:

1. n end points are placed on $[0,1] \times \{0\}$ and remained n points are on $[0,1] \times \{1\}$.
2. 6-vertices can be placed for edges to go down strictly from $[0,1] \times \{1\}$ to $[0,1] \times \{0\}$.

We can easily see that the 3-free link described in Example 4.3 is a 3-free braid on 3 strands.

$\hat{G}_n^3$: G_n^3-like group

Definition 5.2

Let $\mathbf{A}_n = \{a_{(i,j,k)} | i,j,k \in \bar{n}\}$. The group $\hat{G}_n^3$ is defined by

$$\hat{G}_n^3 = \langle \mathbf{A}_n | R1, R2, R3 \rangle,$$

where

- R1: $a_{ijk} a_{kji} = 1$
- R2: $a_{ijk} a_{stu} = a_{stu} a_{ijk}$ for $|\{i,j,k\} \cap \{s,t,u\}| < 2$,
- R3: $a_{ijk} a_{ijl} a_{ikl} a_{jkl} = a_{jkl} a_{ikl} a_{ijl} a_{ijk}$ for any $i < j < k < l$.

Strategy

It is not difficult to see that there is one-to-one correspondence between 3-free braids on $\binom{n}{2}$ strands and $\hat{G}_n^3$, see Fig 8.

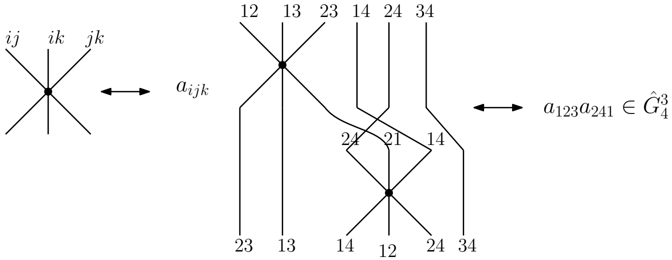

Figure: 9. a_{ijk} and triple points of a 3-free link

From now on, instead of 6-valent vertices we consider generators corresponding to 6-valent vertices in words from $\hat{G}_n^3$ and verify whether a generator in a given word can be canceled by the relation $a_{ijk}a_{kji} = 1$ or not by using so-called "indices" of generators.

Index for a_{ijk} valued in $\mathbb{Z}$

Let $\beta \in \hat{G}_n^3$, where $\beta = Fa_{ijk}B$ for some $F, B \in \hat{G}_n^3$. For the generator a_{ijk} in β and for $s \in \bar{n}$, we define $i_\beta^s(a_{ijk})$ by

1. $i_\beta^s(a_{ijk}) = \sharp_F a_{ijs} + \sharp_F a_{jis} - \sharp_F a_{sij} - \sharp_F a_{sji} + \sharp_F a_{isk} - \sharp_F a_{ksi} + \sharp_F a_{sjk} + \sharp_F a_{skj} - \sharp_F a_{jks} - \sharp_F a_{kjs}$, if $s \neq i, j, k$
2. $i_\beta^s(a_{ijk}) = \sharp_F a_{jik} - \sharp_F a_{kij}$, if $s = k$
3. $i_\beta^s(a_{ijk}) = \sharp_F a_{ikj} - \sharp_F a_{jki}$, if $s = i$
4. $i_\beta^s(a_{ijk}) = 0$ if $s = j$,

where $\sharp_F a_{stu}$ is the number of a_{stu} in F. For simplicity we denote the set of generators,

$$A_{ijk}^s = \{a_{ijs}, a_{jis}, a_{sij}, a_{sji}, a_{isk}, a_{ksi}, a_{sjk}, a_{skj}, a_{jks}, a_{kjs}\}$$

which are appeared in the definition of i_*^s. We denote $\sharp_F A_{ijk}^s = \sharp_F a_{ijs} + \sharp_F a_{jis} - \sharp_F a_{sij} - \sharp_F a_{sji} + \sharp_F a_{isk} - \sharp_F a_{ksi} + \sharp_F a_{sjk} + \sharp_F a_{skj} - \sharp_F a_{jks} - \sharp_F a_{kjs}$.

Lemma 5.3

$i^s_\beta(a_{ijk})$ is not changed by applying relations from $\hat{G}^3_n$ to β.

Proof. Let $\beta = Fa_{ijk}B$. We may assume that a_{ijk} is not disappeared by applying relations to β.
If relations $a_{stu}a_{opq} = a_{opq}a_{stu}$ and $a_{opq}a_{opr}a_{oqr}a_{pqr} = a_{pqr}a_{oqr}a_{opr}a_{opq}$ in F or in B, then it is easy to see that $i^s_\beta(a_{ijk})$ is not changed, because $\sharp_F A^s_{ijk}$ is not changed.
If a relation $a_{opq}a_{qpo} = 1$ is applied in F and $a_{opq}, a_{qpo} \notin A^s_{ijk}$, then it is easy to see that $i^s_\beta(a_{ijk})$ is not changed.
If a relation $a_{opq}a_{qpo} = 1$ is applied in F and $a_{opq}, a_{qpo} \in A^s_{ijk}$, then $i^s_\beta(a_{ijk})$ is not changed, since a_{opq} and a_{qpo} have different sign in $\sharp_F A^s_{ijk}$.
If a relation $a_{opq}a_{qpo} = 1$ is applied in B, then it is trivial that $i^s_\beta(a_{ijk})$ is not changed.

Let us show that $i^s_\beta(a_{ijk})$ is not changed, when we apply the relation $a_{ijk}a_{ijl}a_{ikl}a_{jkl} = a_{jkl}a_{ikl}a_{ijl}a_{ijk}$, which a_{stu} is contained in. Say $\beta = Fa_{ijk}a_{ijl}a_{ikl}a_{jkl}B$ and $\beta' = Fa_{jkl}a_{ikl}a_{ijl}a_{ijk}B$

1. For a_{ijk} if $s \neq l$, then it is easy to see that $i^s_\beta(a_{ijk}) = i^s_{\beta'}(a_{ijk})$. When $s = l$, assume that $i^l_\beta = N$. Notice that in β' before the a_{ijk} there are a_{jkl} and a_{ijl}. By definition of i^l_*,
$$i^l_{\beta'} = N + \sharp a_{ijl} - \sharp a_{jkl} = N + 1 - 1 = N.$$
2. For a_{ijl} we just need to consider $i^k_\beta(a_{ijl})$ and $i^k_{\beta'}(a_{ijl})$. Notice that $i^k_\beta(a_{ijl}) = \sharp_F A^k_{ijl} + 1$ (here we add 1 because of a_{ijk} in $a_{ijk}a_{ijl}a_{ikl}a_{jkl}$) and $i^k_\beta(a_{ijl}) = \sharp_F A^k_{ijl} + 1$ (here we add 1 because of a_{ikl} in $a_{jkl}a_{ikl}a_{ijl}a_{ijk}$. That is, $i^k_\beta(a_{ijl}) = i^k_{\beta'}(a_{ijl})$.
3. For a_{ikl} and a_{jkl} analogously we can show that $i^k_\beta(a_{ikl}) = i^k_{\beta'}(a_{ikl})$ and $i^k_\beta(a_{jkl}) = i^k_{\beta'}(a_{jkl})$.

That completes the proof.

Remark 5.4

Let $\beta = Fa_{ijk}a_{kji}B$. By definition of $i^s_\beta(a_{ijk})$, it is easy to see that $i^s_\beta(a_{ijk}) = -i^s_\beta(a_{kji})$. In other words, if two generators a_{ijk} and a_{kji} in β can be canceled, then $i^s_\beta(a_{ijk}) = -i^s_\beta(a_{kji})$ for all $s \in \bar{n}$.

Example 5.5

Let $\beta = a_{123}a_{213}a_{231}a_{321}a_{312}a_{132} \in \hat{G}^3_3$. If a_{123} and a_{321} can be canceled by applying relations for $\hat{G}^3_3$, then $i^s_\beta(a_{123}) = -i^s_\beta(a_{321})$ for all $s \in \bar{3}$ as asserted in the previous remark. It is easy to see that $i^1_\beta(a_{321}) = 0$, but $i^1_\beta(a_{123}) = \sharp_F a_{231} - \sharp_F a_{132} = 1$, that is, $i^3_\beta(a_{321}) = 0 \neq -1 = -i^3_\beta(a_{231})$, therefore a_{123} and a_{321} cannot be canceled. Therefore, $\beta = a_{123}a_{213}a_{231}a_{321}a_{312}a_{132}$ is not trivial in $\hat{G}^3_3$.

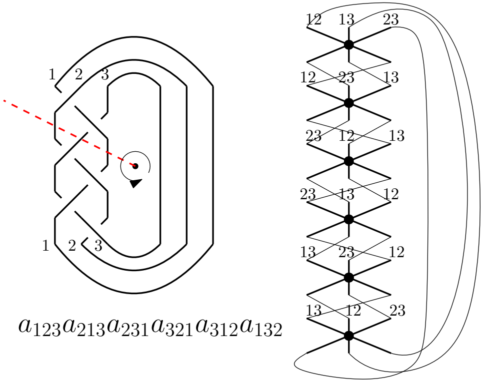

Figure: 10. Borromean link and its corresponding 3-free link. Since $a_{321}a_{312}a_{132}a_{123}a_{213}a_{231}$ is not trivial in $\hat{G}_3^3$, Borromean link is not trivial.

Unsolved Problems

1. How to construct invariants of links (knots) up to *isotopy* using the above approach? How to handle the problem of two maxima?
2. How to get concordance invariants of links?
3. How to get invariants of higher dimensional links up to link homotopy using the above approach?

- V. O. Manturov, D. Fedoseev, S. Kim, I. Nikonov, "Invariants and pictures. Low-dimensional topology and combinatorial group theory", World Scientific, Singapore, 2020
- J. S. Birman, "On the stable equivalence of plat representations of knots and links", *Can. Math. J.*, XXVIII, No. 2 (1976), pp. 264-290
- N. Habegger, X.-S. Lin, "The classification of links up to homotopy", Journal of the AMS, 2, American Mathematical Society, 3 (2): pp. 389-419 , 1990, doi:10.2307/1990959
- V. O. Manturov, I. M. Nikonov, "Homotopical Khovanov homology", J. Knot Theory and its Ramifications, 24 (13), 2015