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Virtual quandle for links in lens spaces

(Based on joint work with A. Cattabriga, 2019)

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05.08.2020

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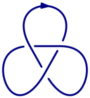

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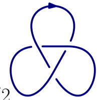

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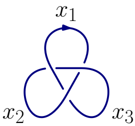

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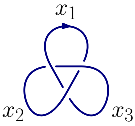

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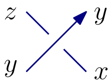

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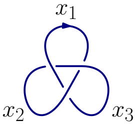

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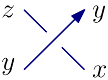

$$G(L) = \langle x_1, x_2, x_3 \mid x_3^{-1}x_1x_3 = x_2, x_1^{-1}x_2x_1 = x_3, x_2^{-1}x_3x_2 = x_1\rangle.$$

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Quandle of a link

Quandle Q is an algebraic system $(Q, *)$ such that

1. $x * x = x$ for all $x \in Q$.
2. The map $S_x : y \mapsto y * x$ is a bijection of Q.
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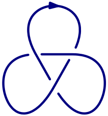

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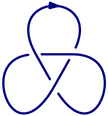
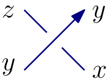
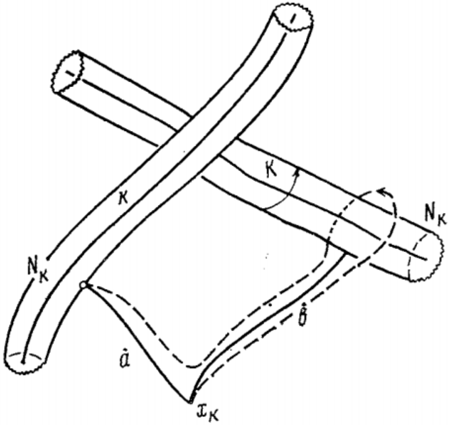
 $\rightarrow Q(L)$

The quandle $Q(L)$ can be found in the following way



Generators: labels on the arcs.
Relations: $x * y = z$ near all crossings.

Geometric definition of a quandle



Quandle is an almost complete invariant

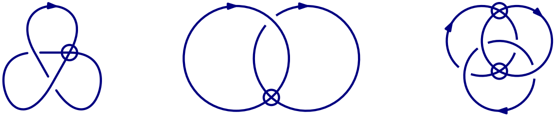
Theorem

Let L_1, L_2 be two links in S^3. Then $Q(L_1)$ and $Q(L_2)$ are isomorphic if and only if L_1, L_2 are weakly equivalent.

D. Joyce, A classifying invariant of knots, the knot quandle
J. Pure Appl. Algebra, V. 23, N. 1, 1982, 37–65.

S. Matveev, Distributive groupoids in knot theory
Mat. Sb. (N.S.), V. 119(161), N. 1(9), 1982, 78–88.

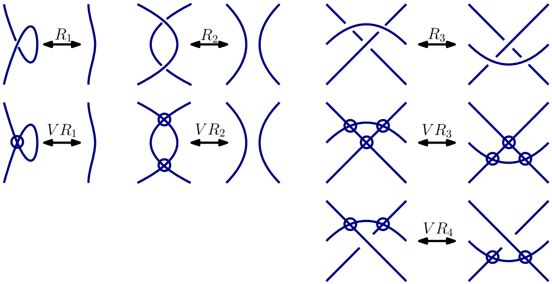
Virtual links



L. Kauffman, Virtual knot theory
European J. Combin., V. 20, N. 7, 1999, 663–690.

Equivalent virtual links

Two virtual knot diagrams are called equivalent if one of these diagrams can be transformed to another one by a finite sequence of moves depicted below.



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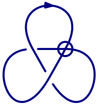

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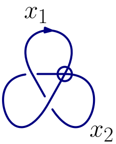

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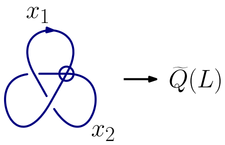

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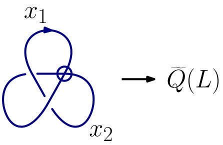

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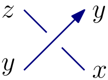

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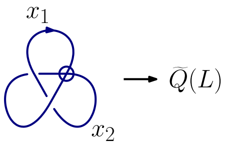

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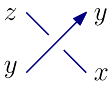

$$\widetilde{Q}(L) = \langle x_1, x_2 \mid x_2 * x_1 = x_1, x_1 * x_1 = x_2 \rangle = \{x_1\}.$$

Virtual quandle of a knot (V. Manturov)

Virtual quandle VQ is an algebraic system $(Q, *, f)$ such that

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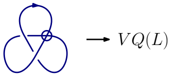

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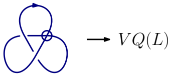

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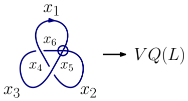

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Generators: labels on the semiarcs.

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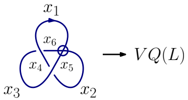

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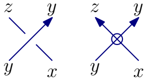

Generators: labels on the semiarcs.
Relations: $y = t$, $x * y = z$ near classical crossings,
$t = f(y)$, $z = f^{-1}(x)$ near virtual crossings.

Lens spaces $L(p, q)$

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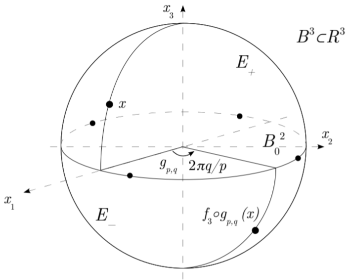

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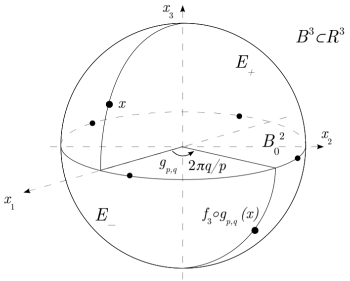

$L(p,q)$ can be obtained by a p/q rational surgery on the unknot in the 3-sphere S^3.

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D. Buck, M. Mauricio, Connect sum of lens spaces surgeries: application to Hin recombination
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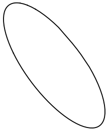
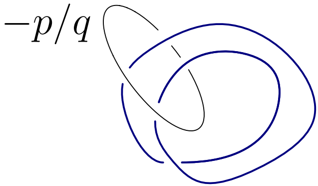
S. Stevan, Torus knots in lens spaces and topological strings
Ann. Henry Poincare, V. 16, 2015, 1937–1967.

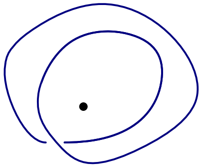
Surgery link



Mixed link diagram



Punctured link diagram



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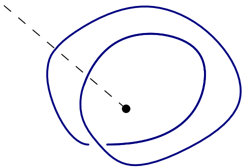
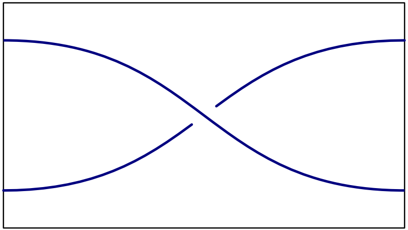


Band diagram



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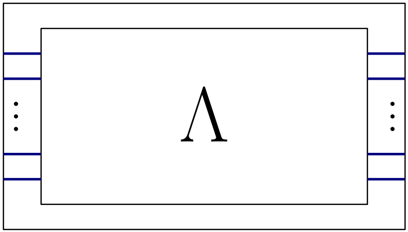

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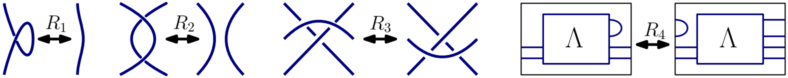

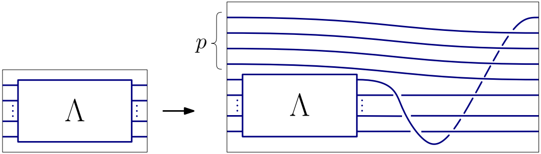

Invariants

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Let $\pi : S^3 \to L(p,q)$ be the universal cover map.
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For a link L in $L(p,q)$ denote by $\widetilde{f}$ the map given by
$\widetilde{f}(L) = f(\pi^{-1}(L))$ is an invariant for links in $L(p,q)$.

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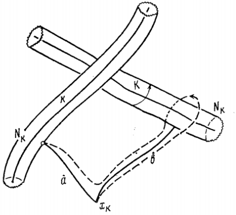

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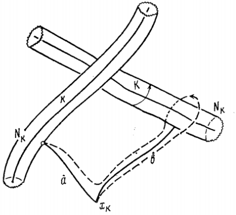

Manfredi, 2018: $Q_1(L) \simeq Q_2(L)$.

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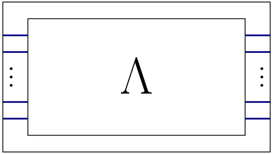

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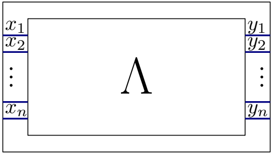
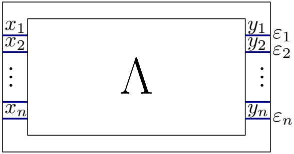


Label the arcs:
Left by $x_1, x_2, \ldots, x_n$
Right by $y_1, y_2, \ldots, y_n$
Remaining by $z_1, z_2, \ldots, z_m$

Virtual quandle for links in lens spaces



Label the arcs:
Left by $x_1, x_2, \ldots, x_n$
Right by $y_1, y_2, \ldots, y_n$
Remaining by $z_1, z_2, \ldots, z_k$

To each right point associate the number ε_i
$\rightarrow | \quad \varepsilon_i = 1$
$\leftarrow | \quad \varepsilon_i = -1$

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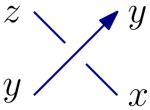

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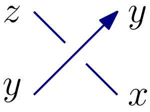

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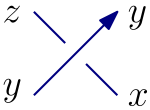

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  $f^p(x) = x$ for all $x \in VQ(K)$.

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Here:
$x *^1 y = x * y$
$x *^{-1} y = S_y^{-1}(x)$, where $S_x : y \mapsto y * x$.

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The virtual quandle $VQ(K)$ is an invariant for links in $L(p, 1)$.

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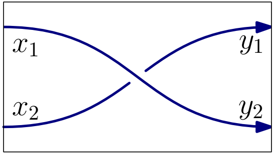
Nice idea.

Example



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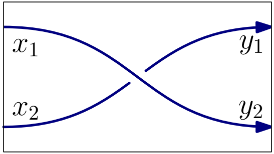

$$\begin{aligned} VQ(K) = \langle x_1, x_2, y_1, y_2 \mid\ & x_1 = y_2, x_2^{x_1} = y_1, \\ & f(x_1) = y_1, f(x_2) = y_2, y_2 y_1 \equiv 1, \\ & \forall x \ \ x f(x) \dots f^{p-1}(x) \equiv 1, \forall x \ \ f^p(x) = x \rangle. \end{aligned}$$

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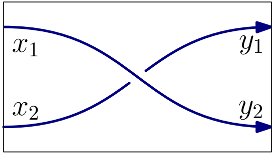

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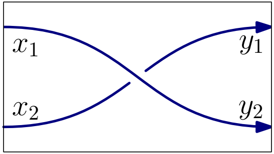

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