# Topological cascade through vortex reconnection

RENZO L. RICCA

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*Novosibirsk (Zoom) – 17 June, 2020*

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## *Motivations*

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- *Quantify energy/helicity transfers in dynamical systems.*

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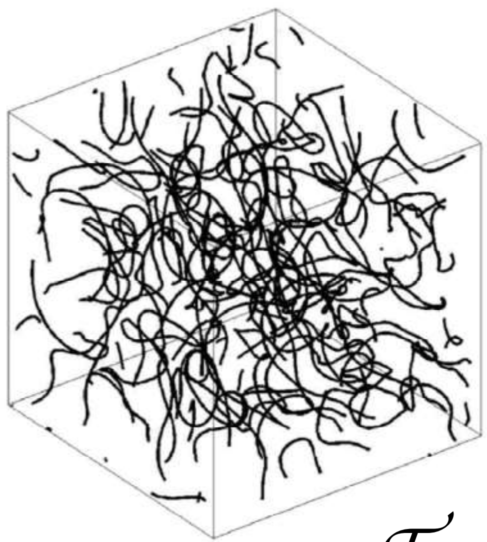
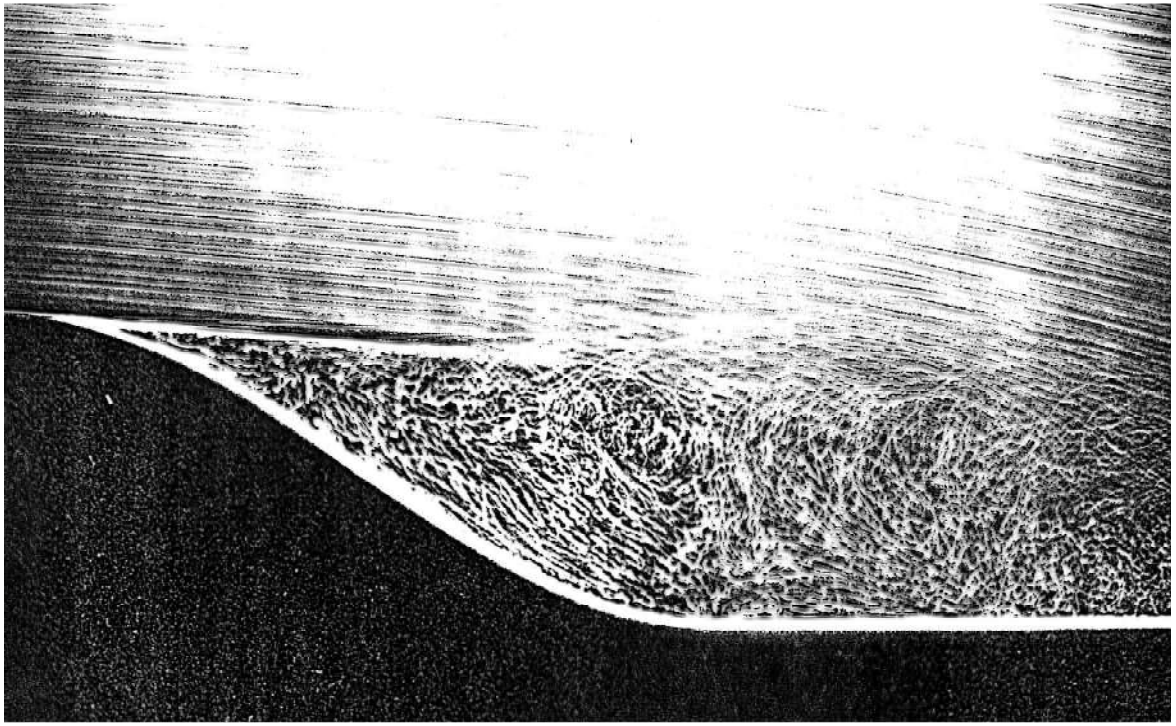


$\mathcal{T}$

# *Fluid structures in classical turbulence*

*Werle, ONERA 1974*
*(Van Dyke 1982)*



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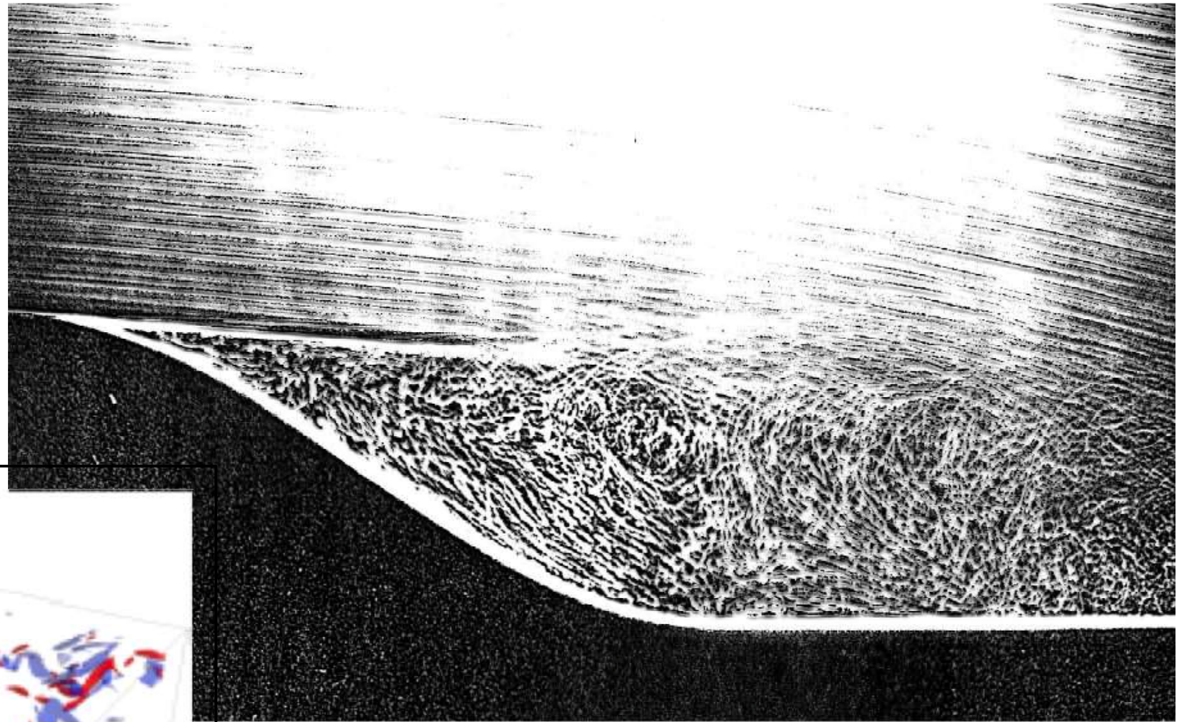
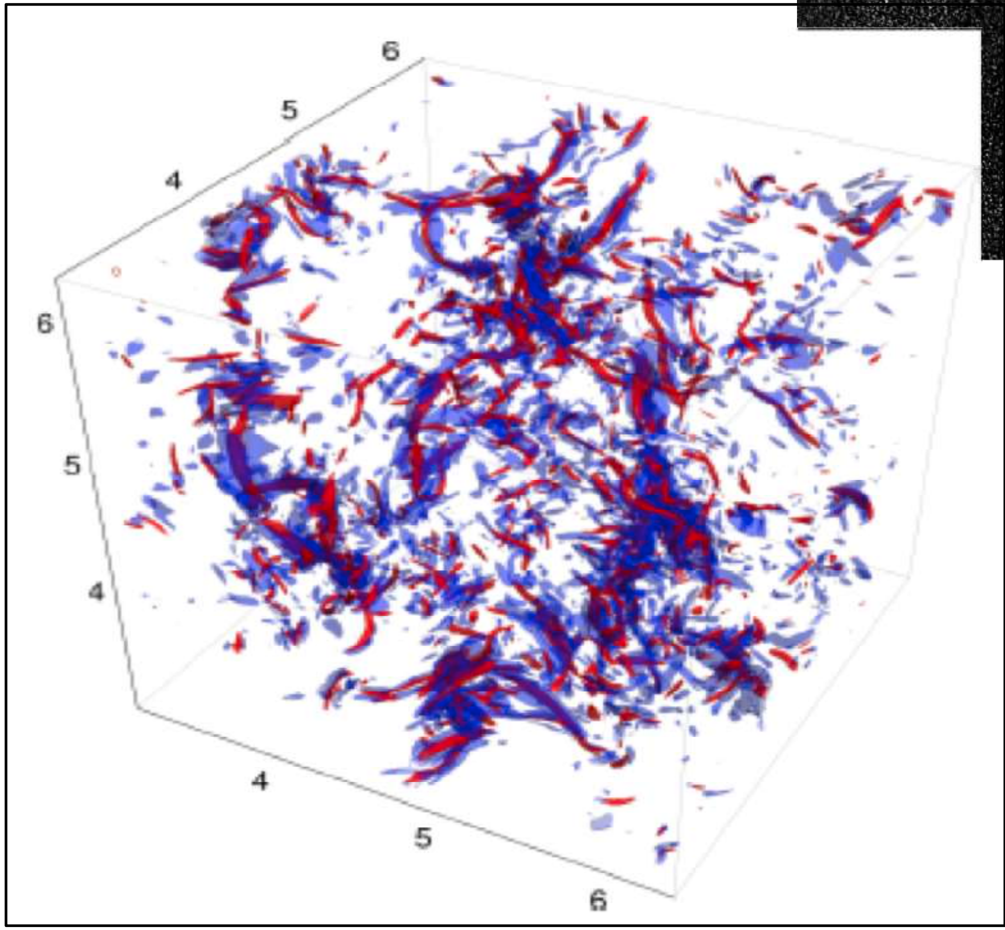


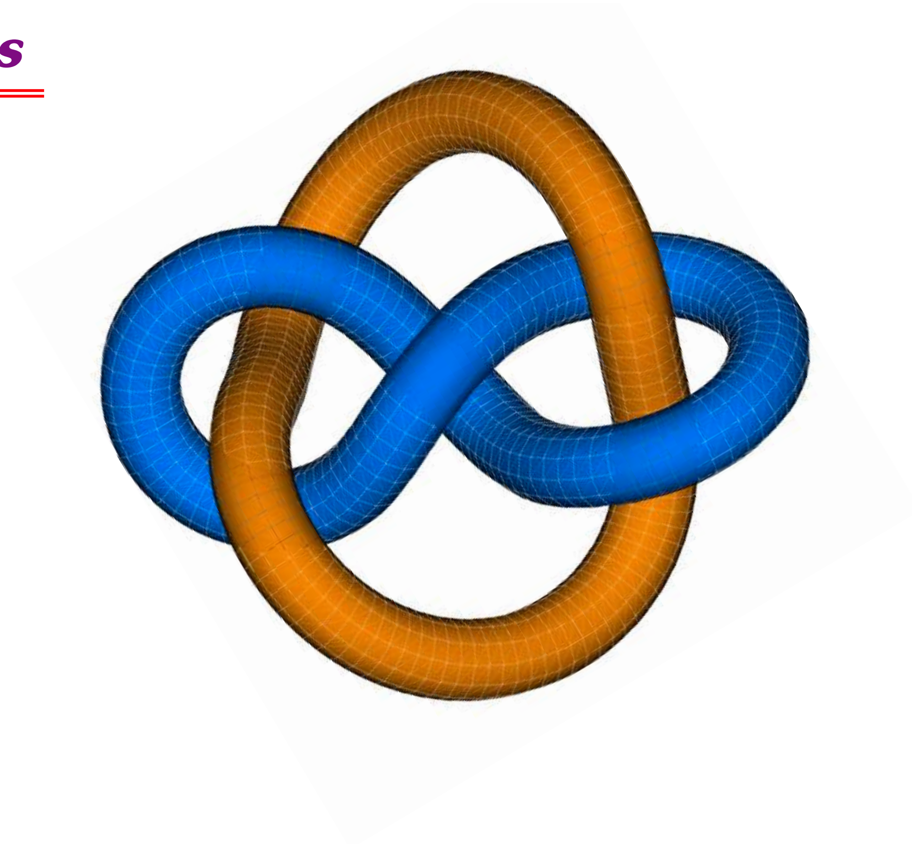



*Picardo et al. (PRF 2019)*

# *Vortex knots as tubular embeddings*

***Let*** $T = C \otimes S$ ***and*** $V = V(T)$***:***

$$T \hookrightarrow K \quad \textit{in} \quad \mathbb{R}^3$$



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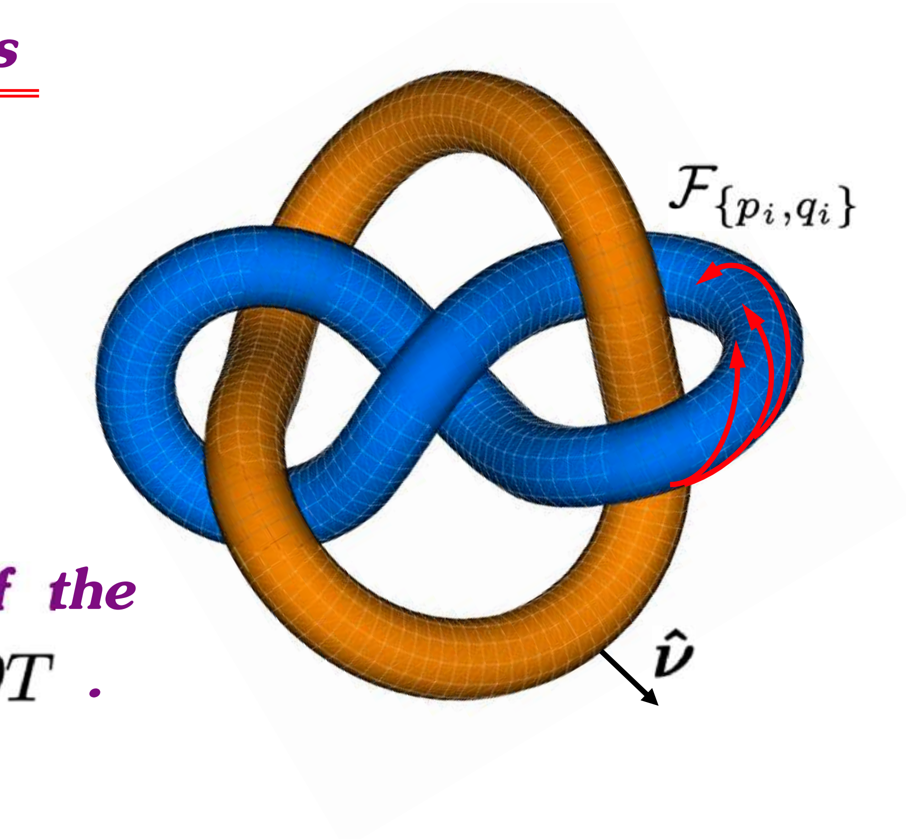
$$K \equiv \mathrm{supp}(\boldsymbol{\omega})$$

*by a standard foliation* $\mathcal{F}_{\{p_i,q_i\}}$ *of the* $\boldsymbol{\omega}$*-lines, such that* $\boldsymbol{\omega} \cdot \hat{\boldsymbol{\nu}} = 0$ *on* $\partial T$ *.*



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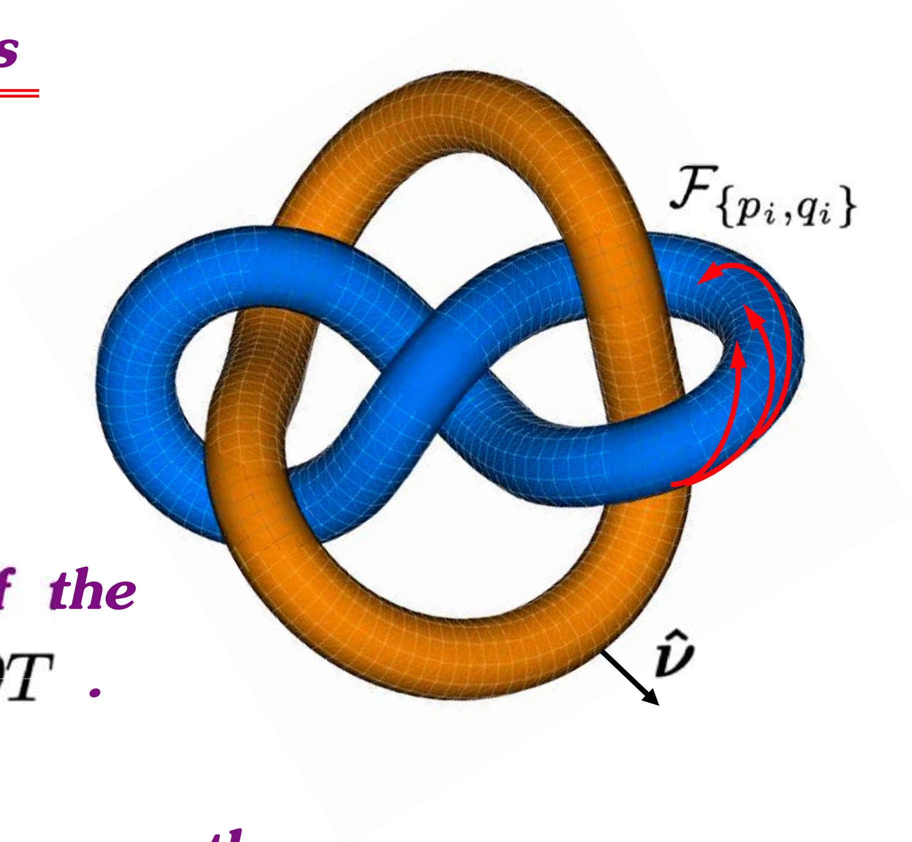


- ***Definition:*** *A vortex tangle* $\mathcal{T}$ *is a smooth immersion in* $\mathbb{R}^3$ *of finitely many disjoint standard tubular knots* $K_i$ *, such that*

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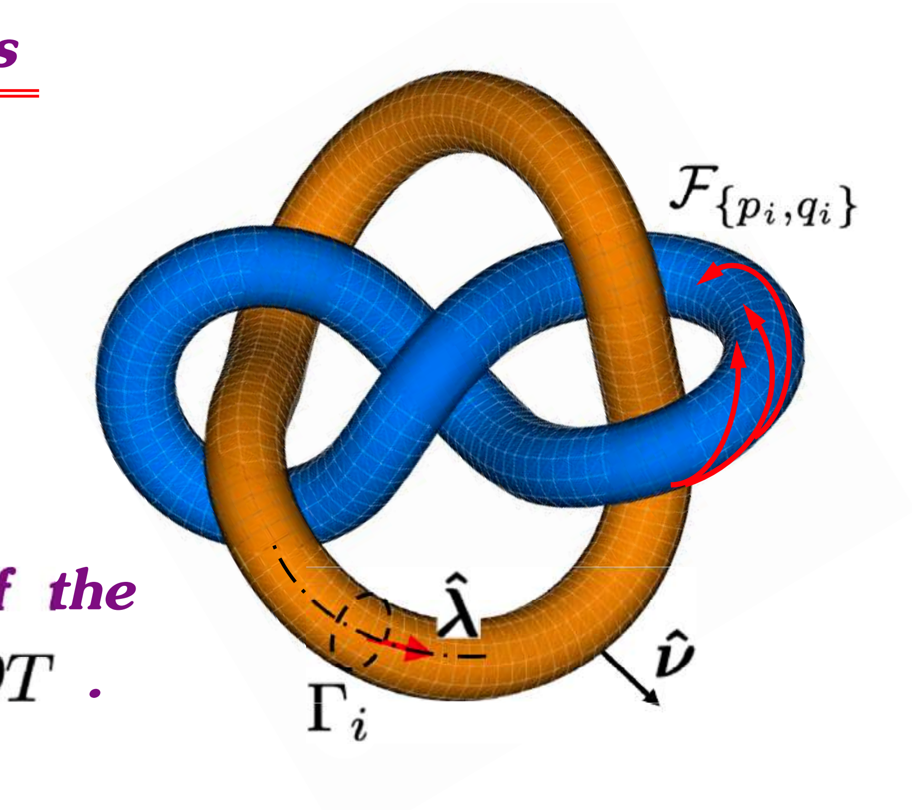


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- *Ideal evolution: circulation and topology preserved*

$$\Gamma_i = \int_{S_i} \boldsymbol{\omega} \cdot \hat{\boldsymbol{\lambda}} \, dS = cst. \; ; \text{ knot type } K_i \text{ conserved.}$$

## *Kinetic helicity and linking numbers*

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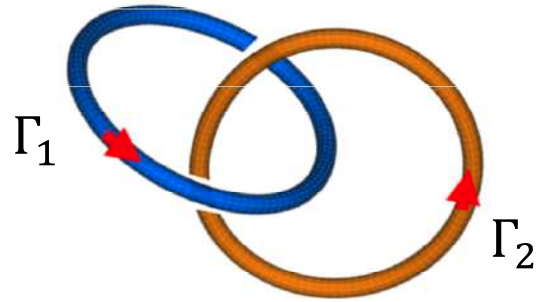

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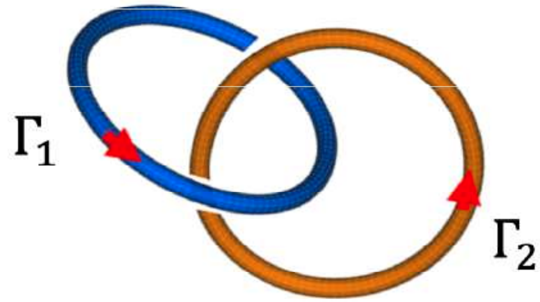

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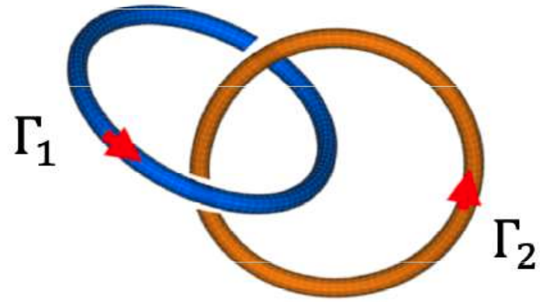


***Self-linking number (Călugăreanu-White):***

$$SL(K) = Wr(C) + Tw(C, C^*) \,,$$

***writhing number*** ***total twist number***

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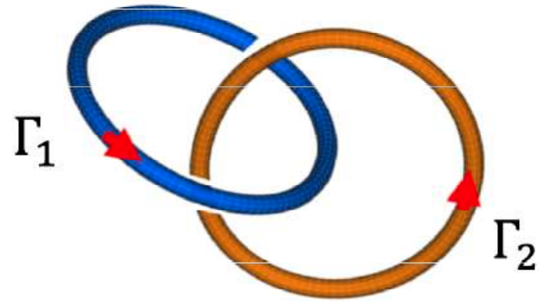


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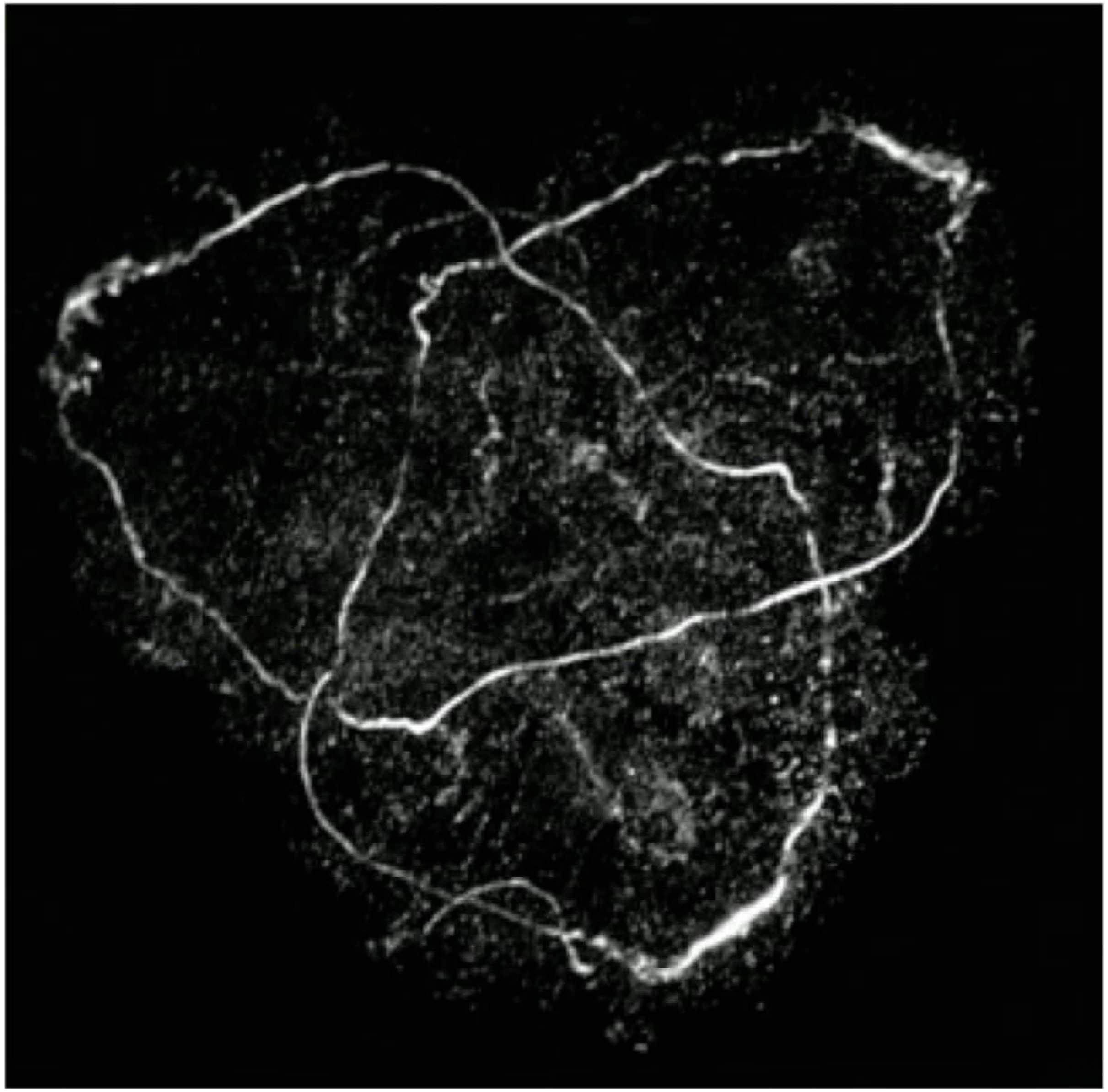
# *Topological cascade from head-on collision of vortex rings*

*(Lim & Nickels, Nature 357 1992)*

# *Production and evolution of a trefoil vortex knot in water*



*(Klecker & Irvine, Nature Physics 9 2013)*

# *Anti-parallel reconnection of vortex tubes in water*

*(Alekseenko et al., JETP Letters 7 2016)*

# *Mechanics of vortex reconnection*

- ***Biot-Savart induction law:***

$$\mathbf{u}_{\mathrm{R}}(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega}\frac{\boldsymbol{\omega}(\mathbf{x}^*)\times(\mathbf{x}-\mathbf{x}^*)}{|\mathbf{x}-\mathbf{x}^*|^3}\,\mathrm{d}V^*$$

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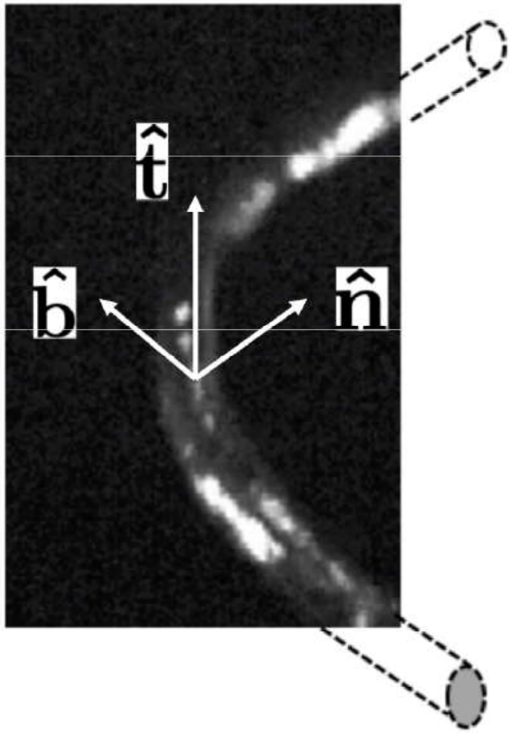

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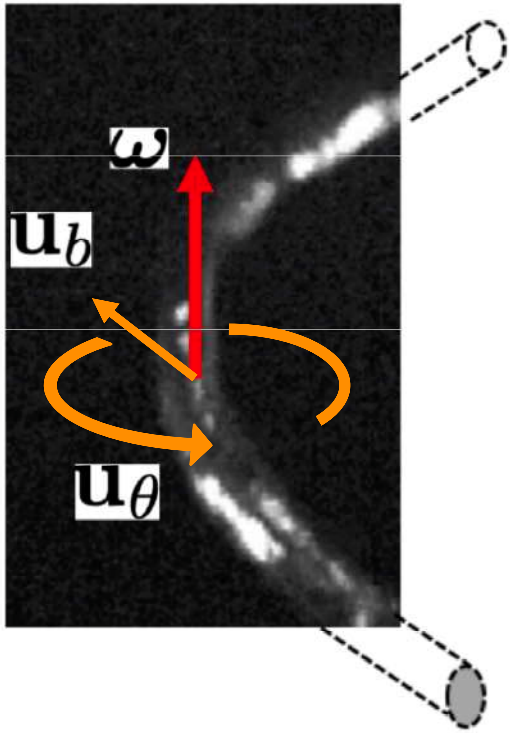

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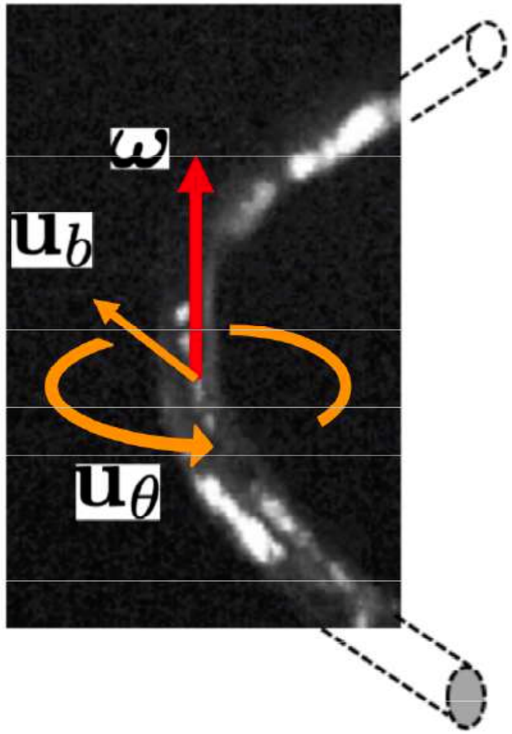
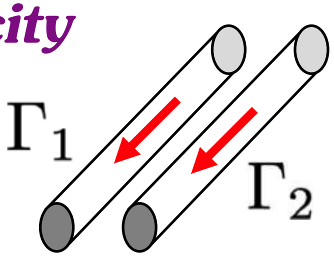
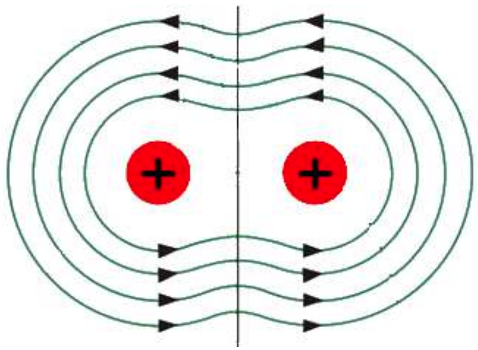



- ***Azimuthal contribution:*** $\mathbf{u}_\theta$

  ***(i) parallel vorticity***



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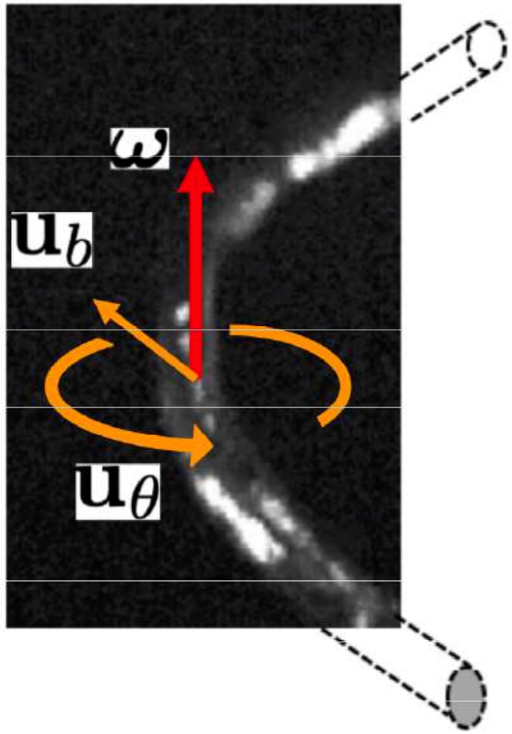


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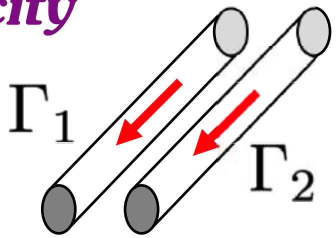


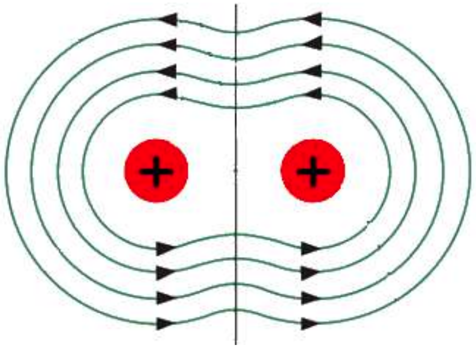

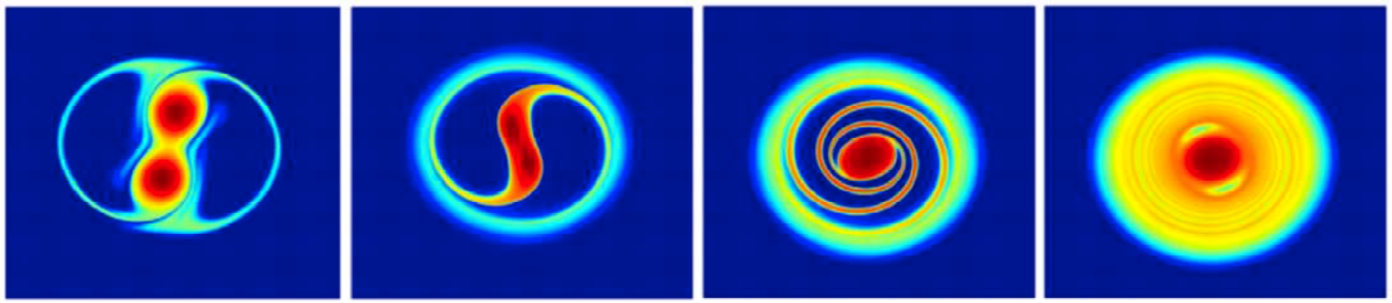

***(Josserand & Rossi 2007)***

# Mechanics of vortex reconnection: pre-reconnection stage

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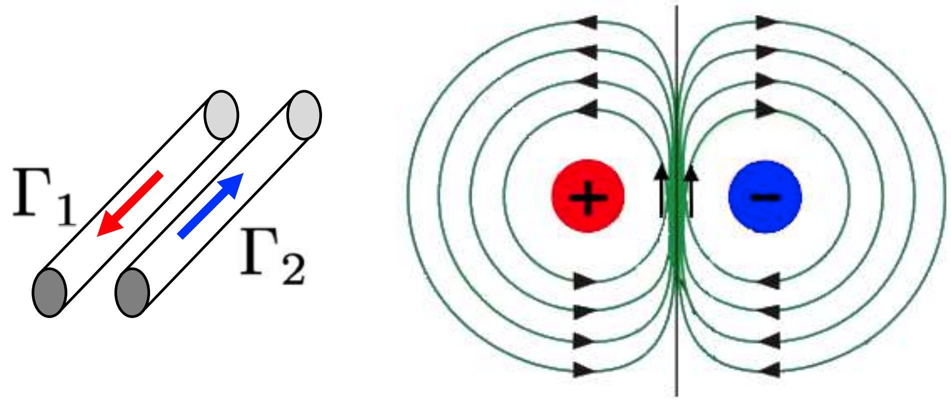

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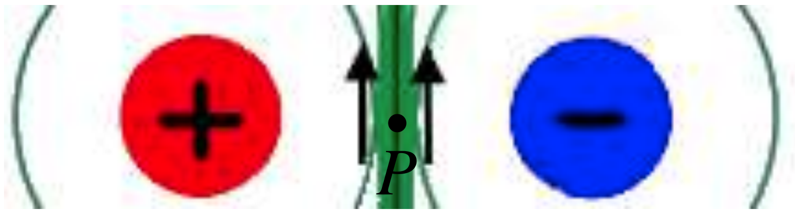


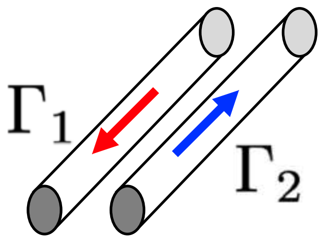


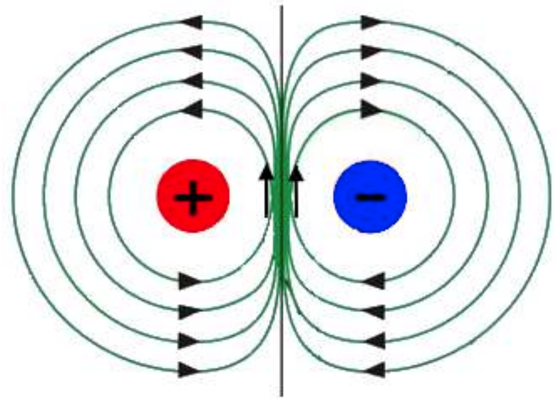

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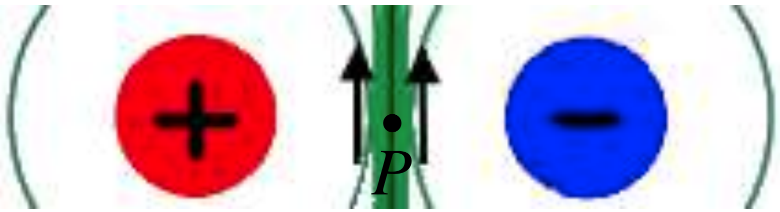


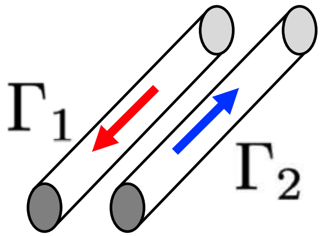


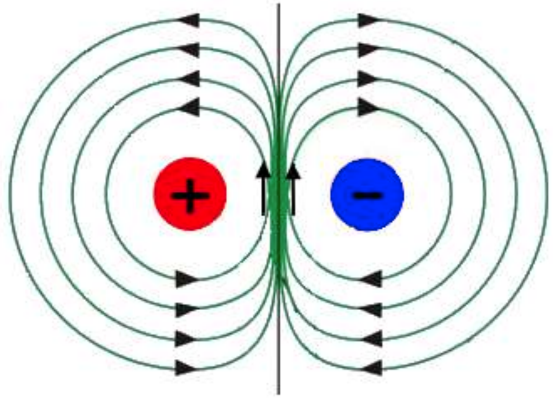

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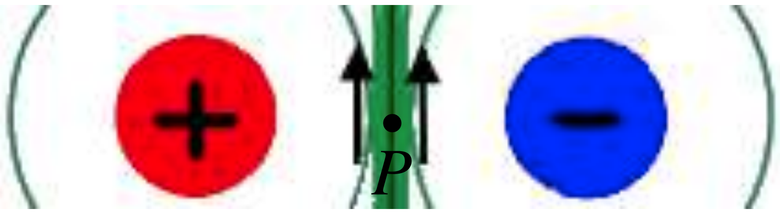


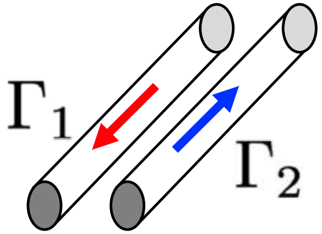


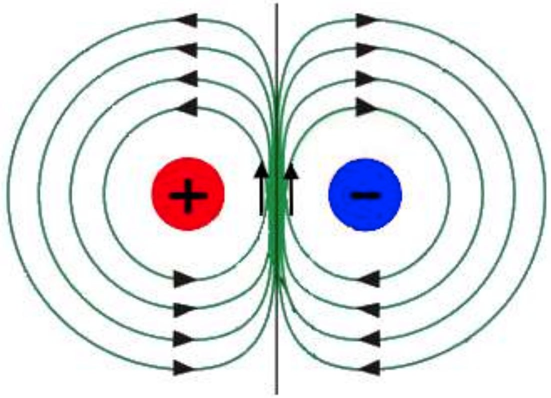

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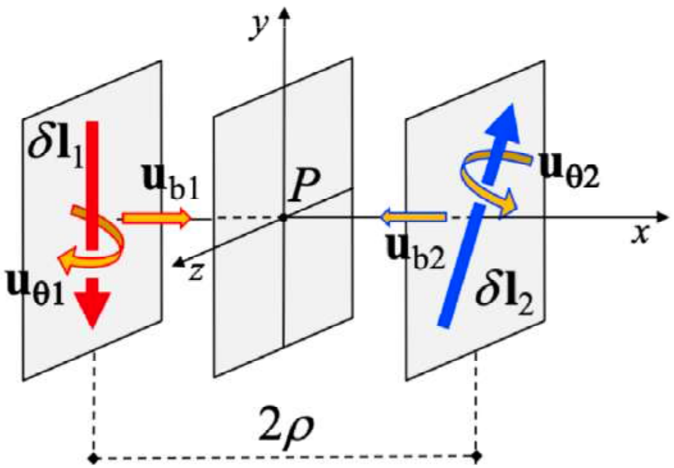
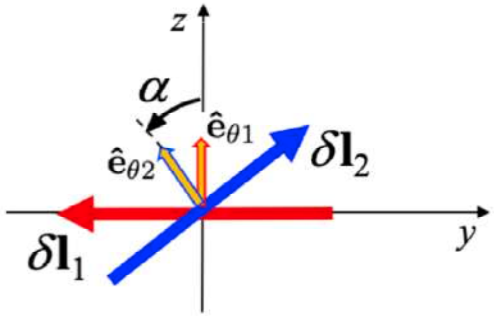
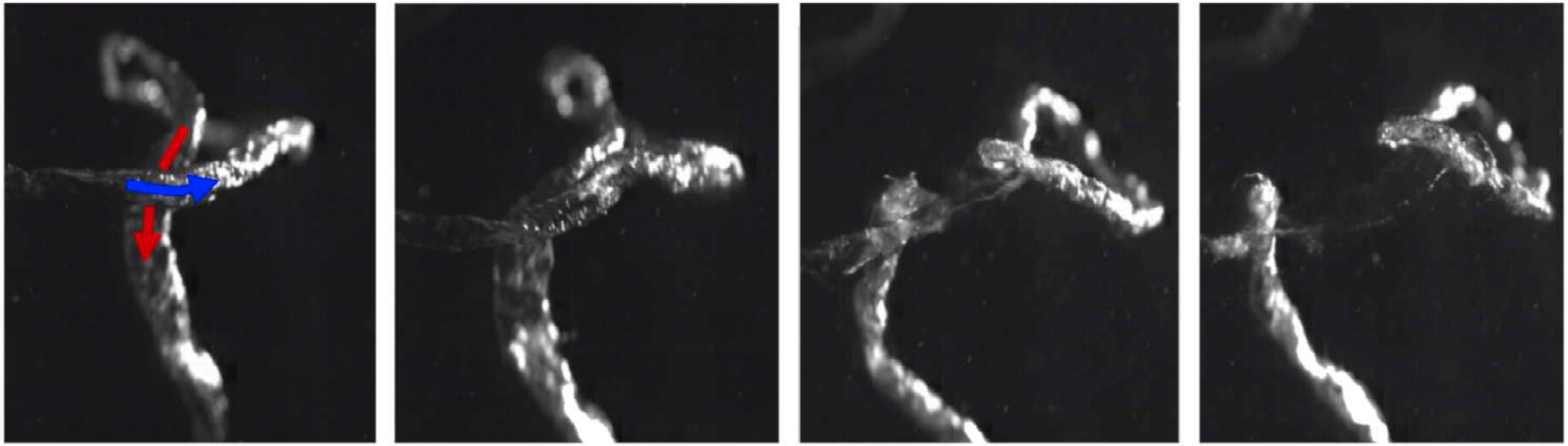



$$|\mathbf{u}_P|^2 = |\mathbf{u}_{\theta 1} + \mathbf{u}_{\theta 2}|^2 = 2\left(\frac{\Gamma}{2\pi\rho}\right)^2 (1+\cos\alpha) : \quad |\mathbf{u}_P|^2_{\max} \iff \alpha = 0 \,.$$

■

# *Observation of anti-parallel reconnection*



*(Alekseenko et al., JETP Letters 7 2016)*

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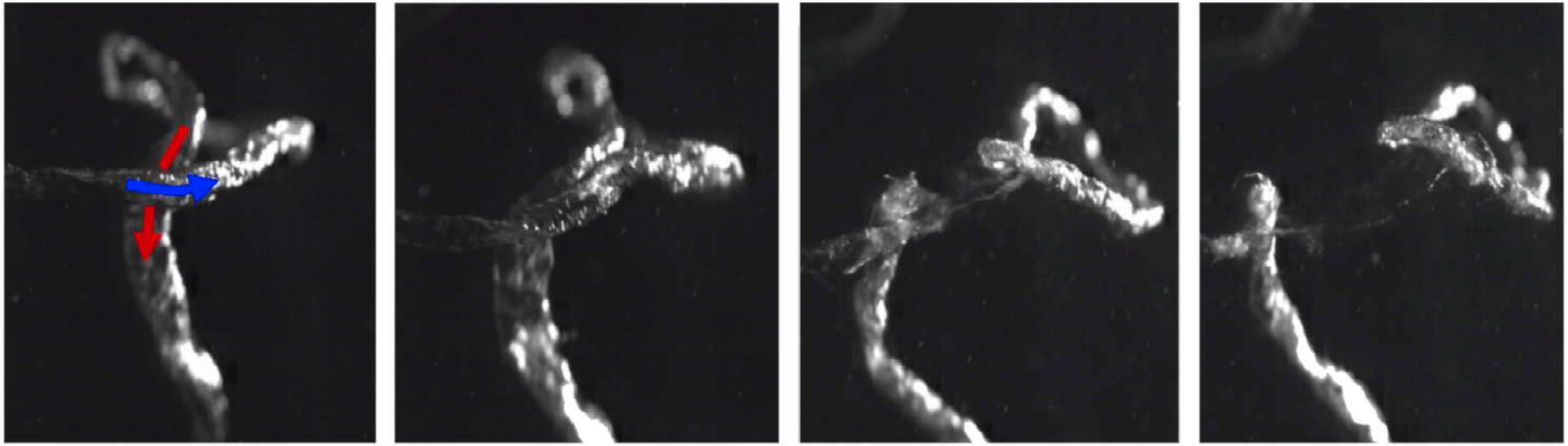

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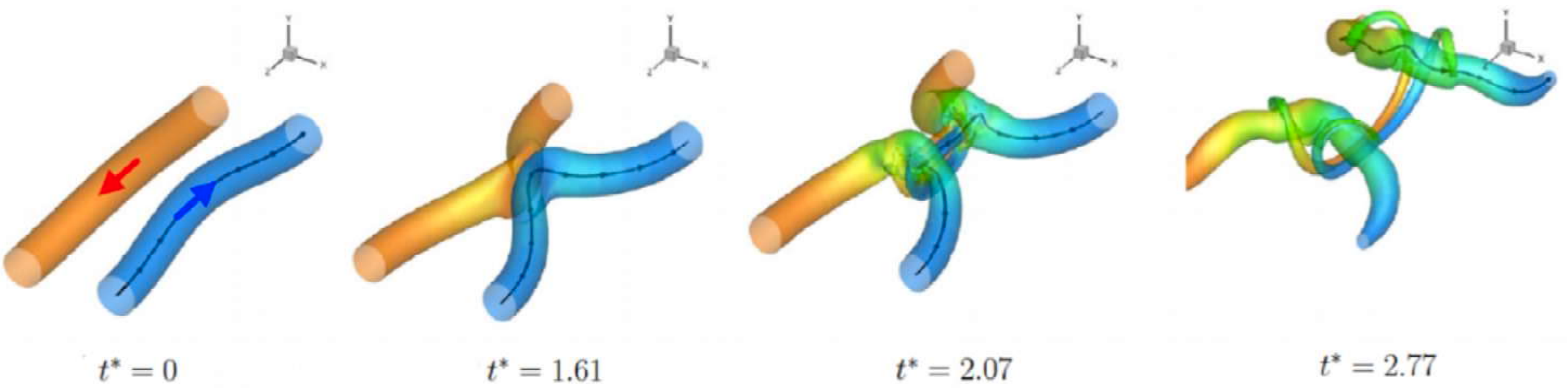
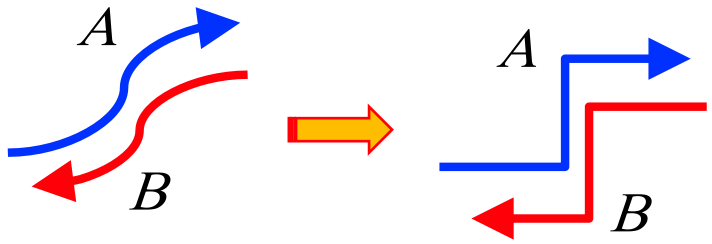



DNS of vorticity iso-surfaces at 40% of maximum initial vorticity
(Hussain & Duraisamy, Phys Fluids 23 2011)

# Writhe conservation and helicity change under reconnection

## Consider anti-parallel reconnection by polygonal curves:



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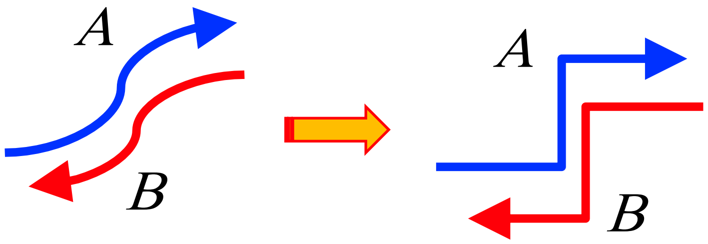


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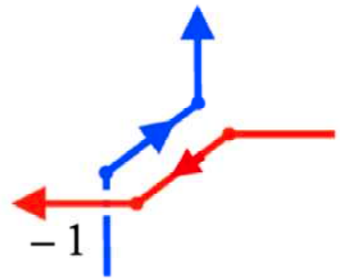

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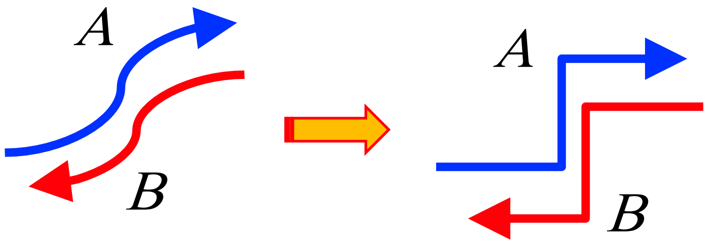


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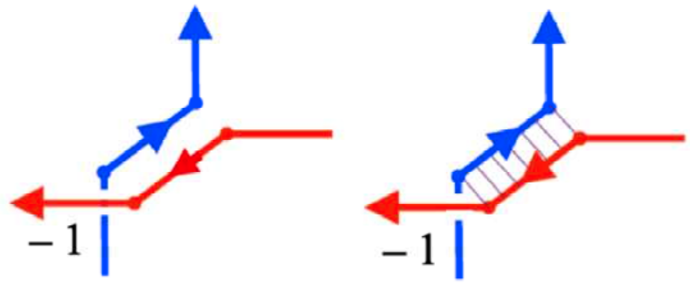

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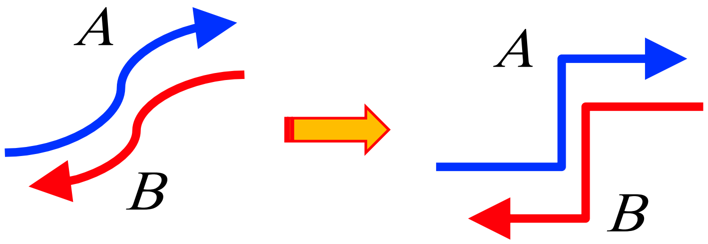


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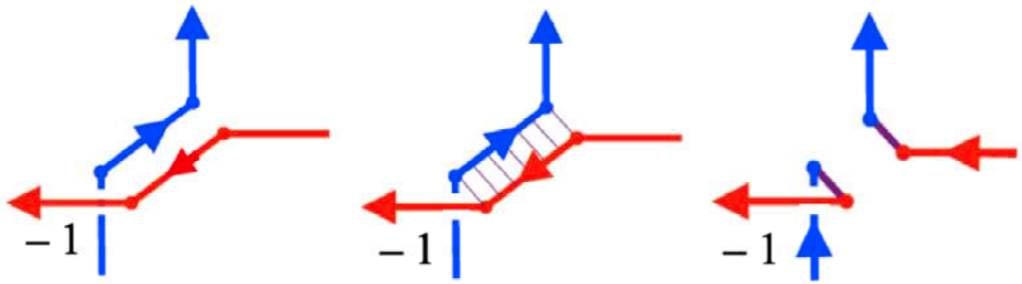

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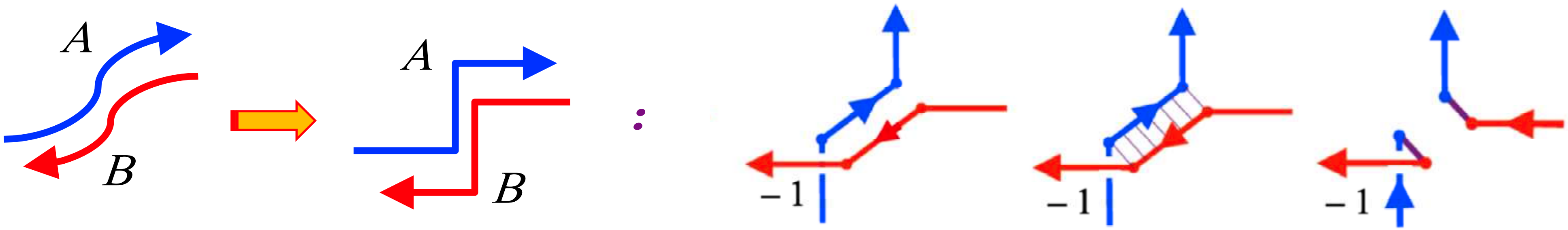


- **Theorem** (Laing et al. 2015). The writhe $Wr$ of 2 disjoint, oriented, polygonal curves $A$ and $B$ is conserved under anti-parallel reconnection of $A$ and $B$: $Wr(A \cup B) = Wr(A \# B)$.

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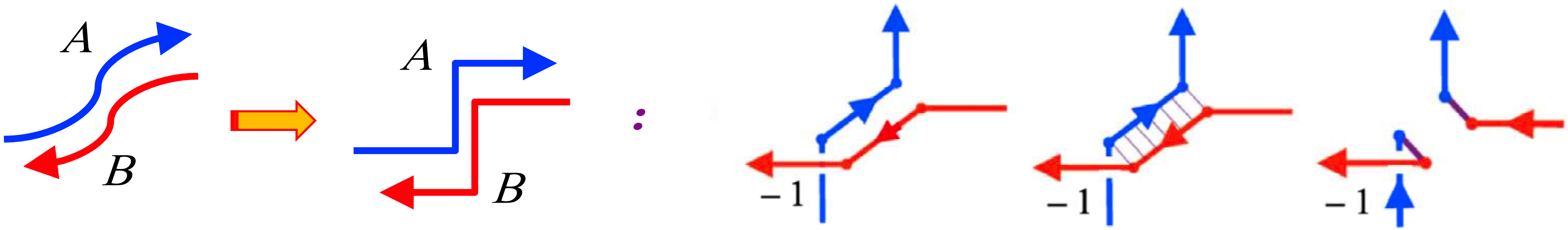


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***Corollary 1. If 2 vortex tubes of axes*** $A$ ***and*** $B$ ***have same circulation, then the writhe helicity is conserved under anti-parallel reconnection.***

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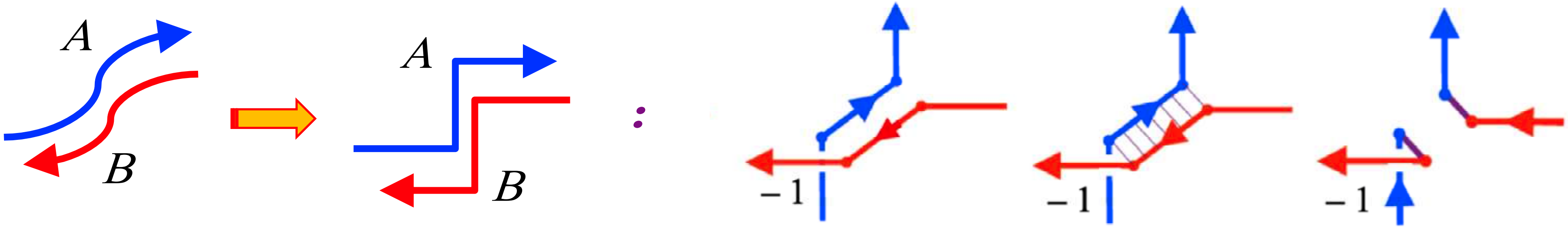


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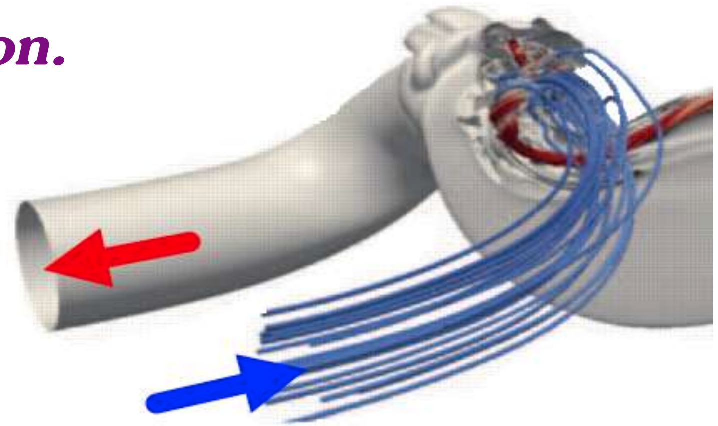
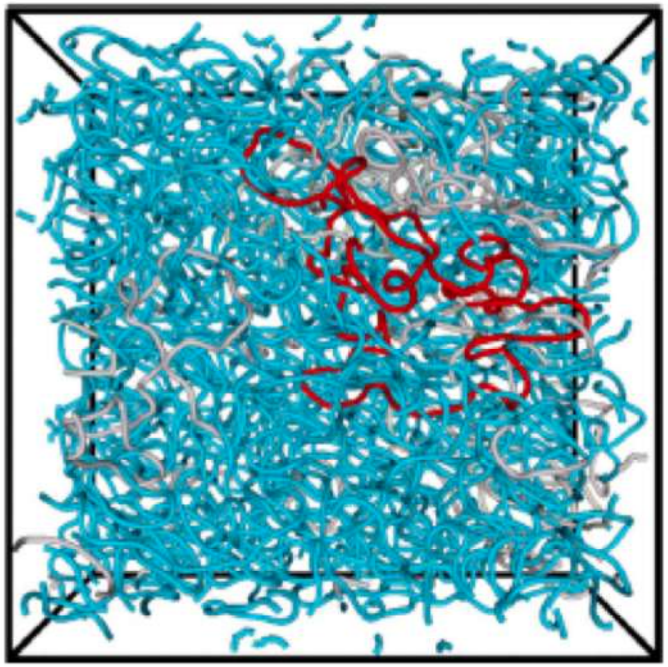
***Corollary 2. Since under anti-parallel reconnection of $A$ and $B$ total torsion is conserved, any change in helicity is solely due to a change in intrinsic twist.***



***(Van Rees et al. 2012)***

# *Vortex defects in condensates and superfluids*



*Taylor & Dennis (Nature Comm 2016)*

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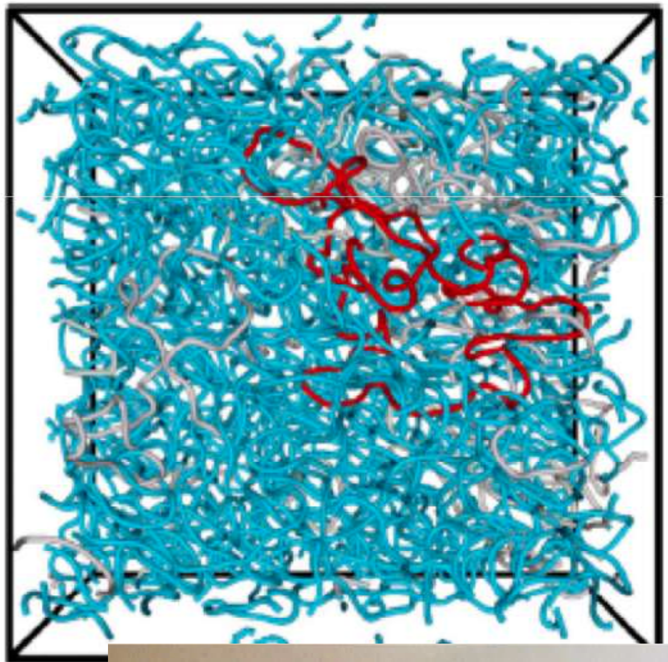

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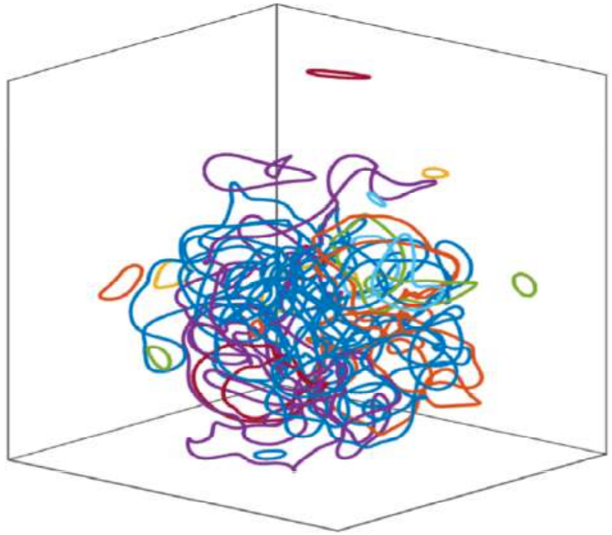

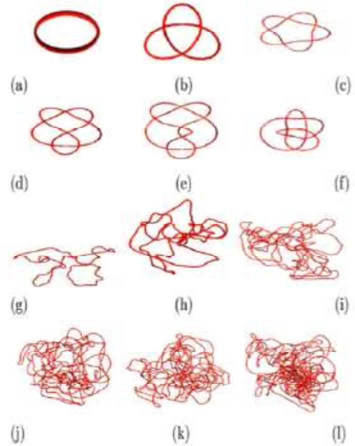


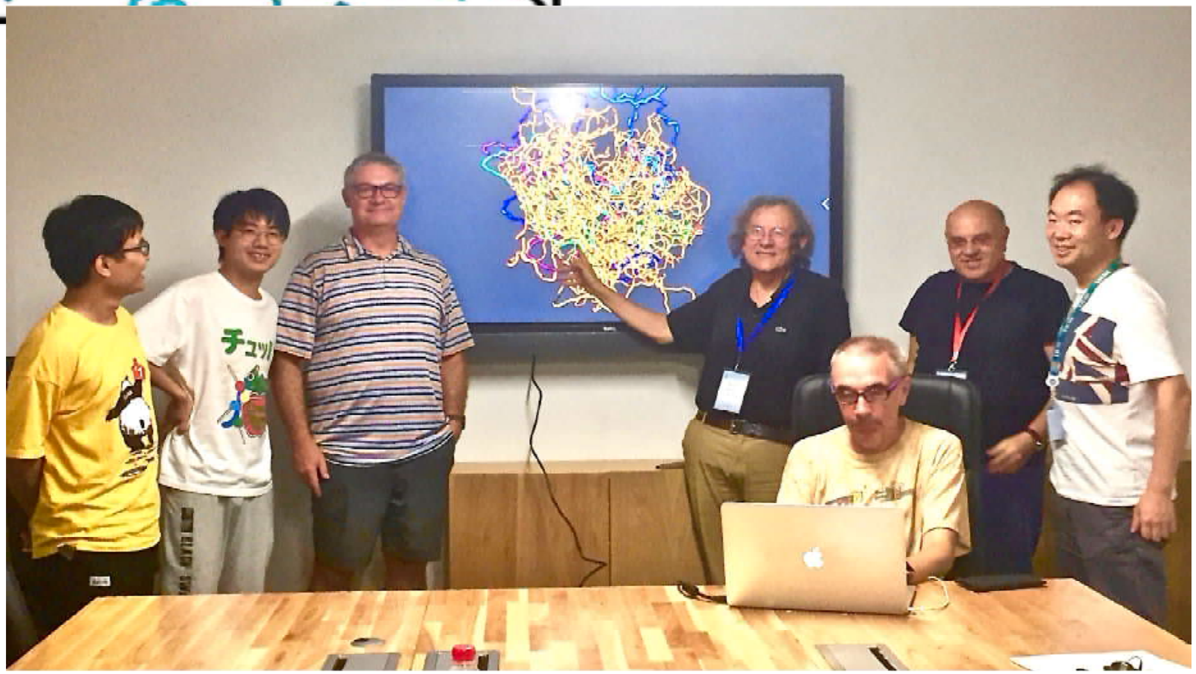

*Cooper et al. (Nature Sci Rep 2019)*

## Tackling structural complexity by knot polynomials

- ***Helicity and linking number limitations:***

  ***(i)*** $H(K) = f(SL_i, Lk_{ij}; \Gamma_i)$ ;

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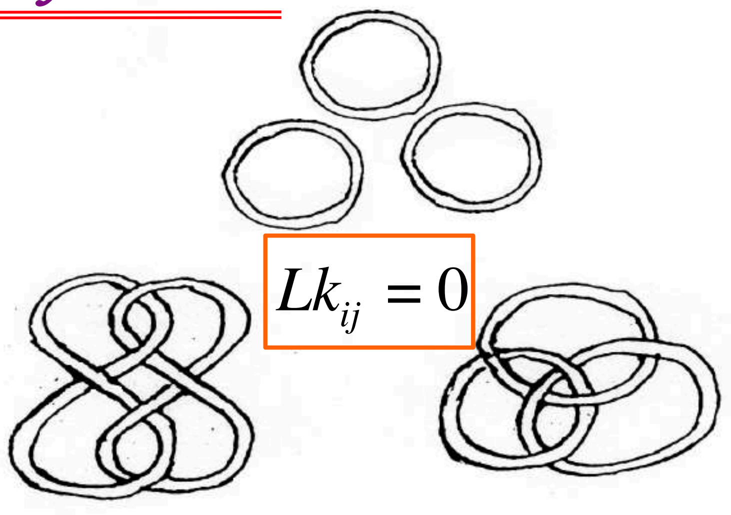


***(Maxwell 1867)***

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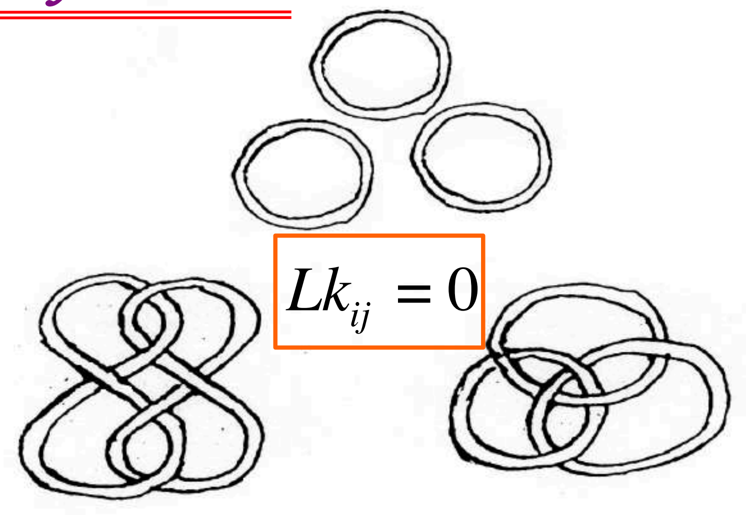


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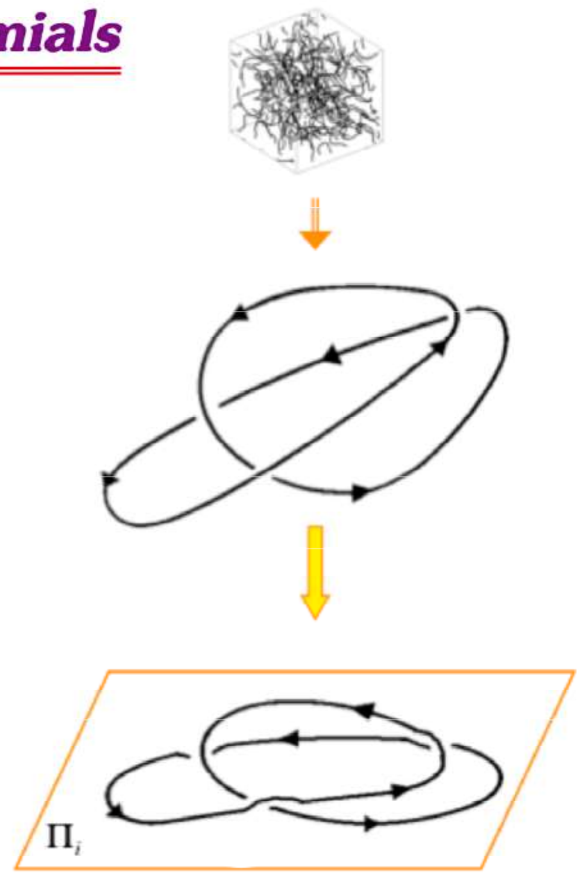

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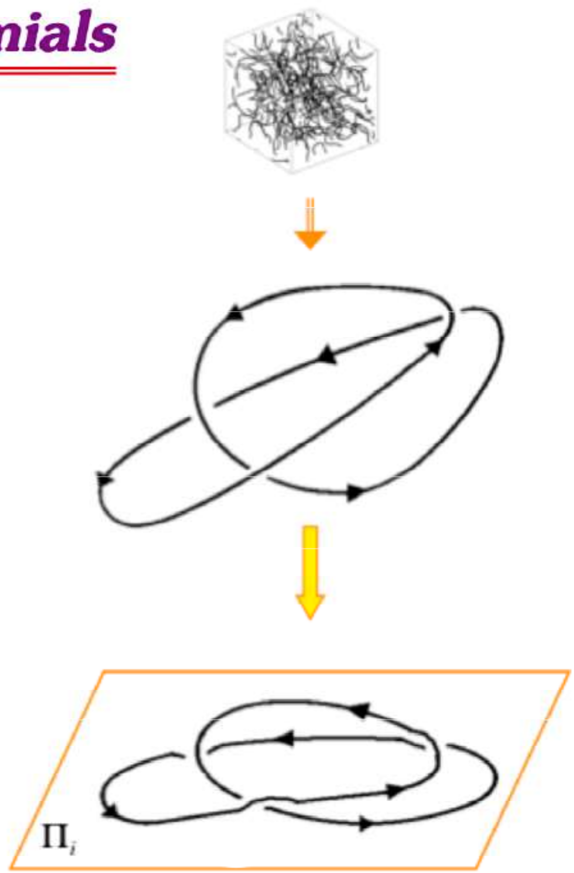


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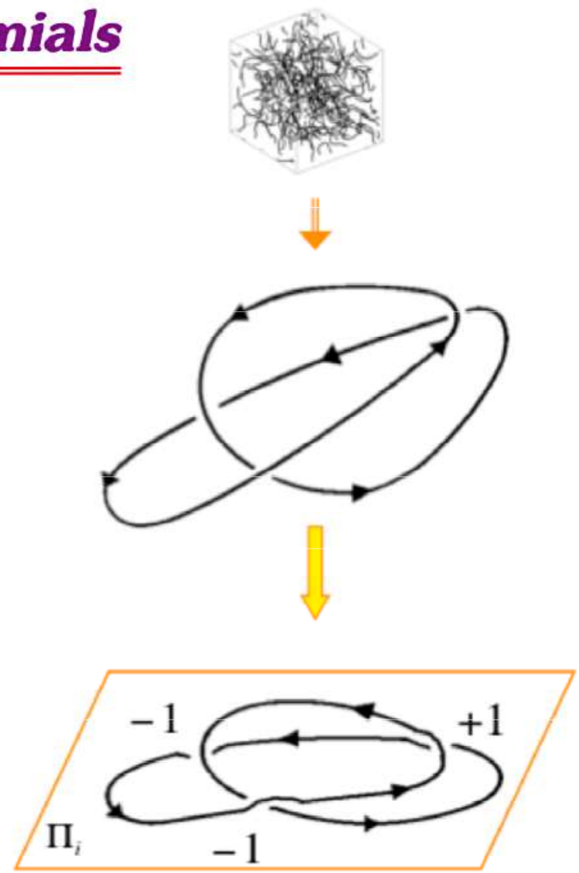

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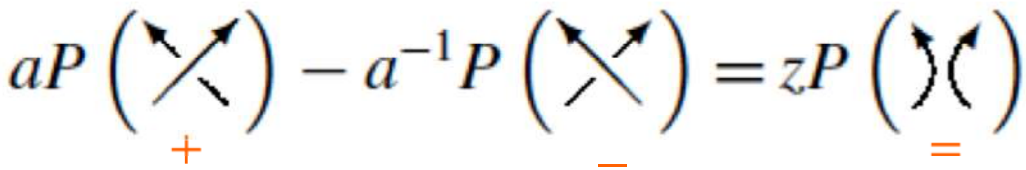


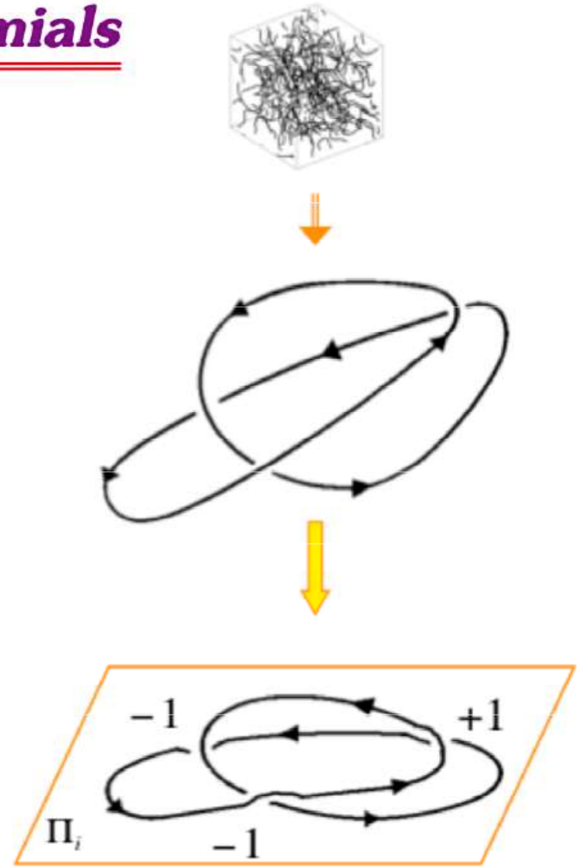


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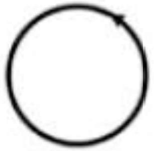
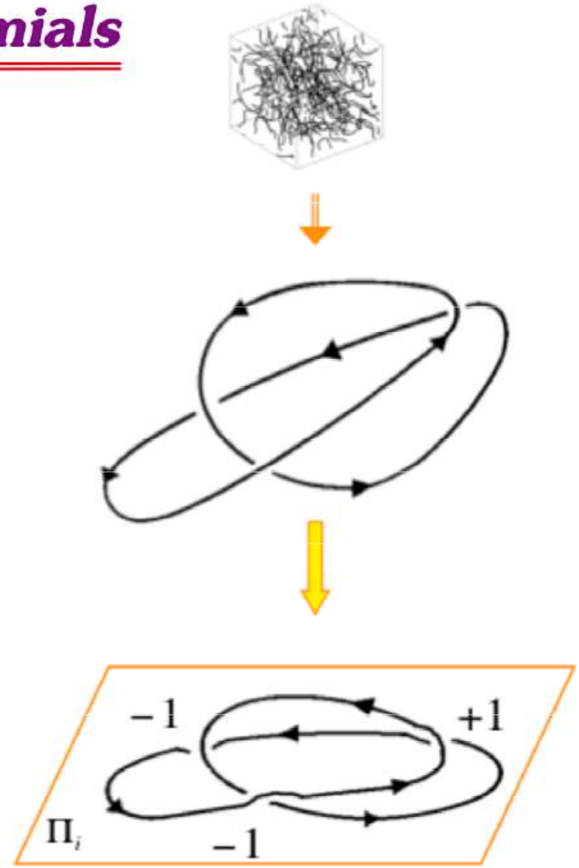


$U_1$

## *Tackling structural complexity by knot polynomials*

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***P1:***

$U_1$ $\sim$ $\gamma_+$

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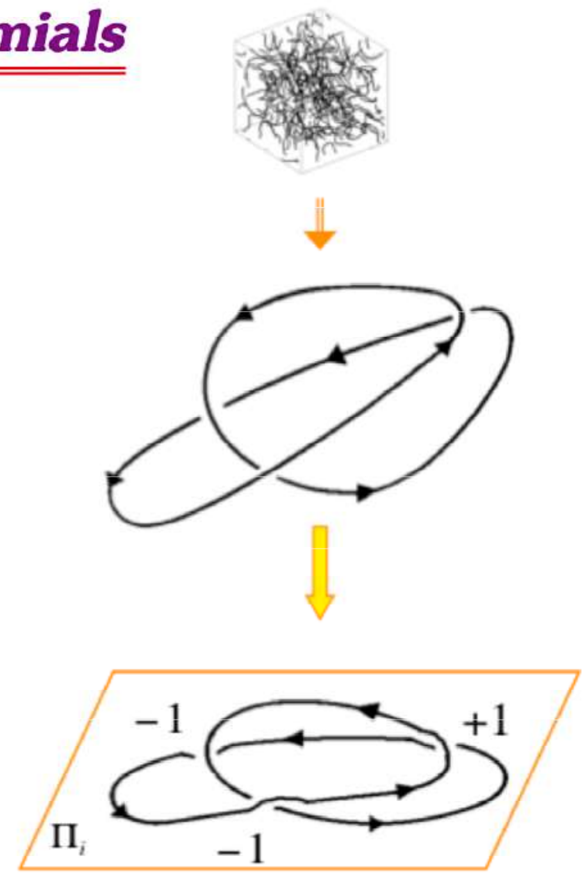


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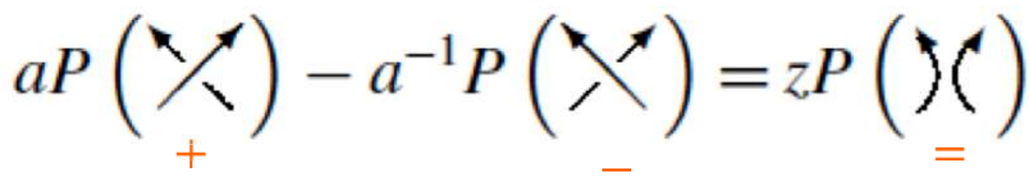


***P1:***

$U_1$ $\gamma_+$ $\gamma_-$

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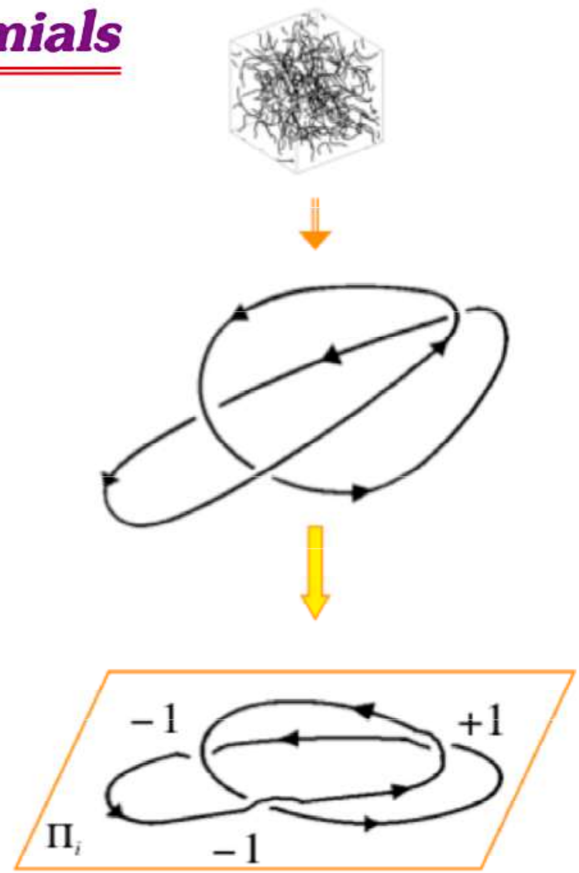


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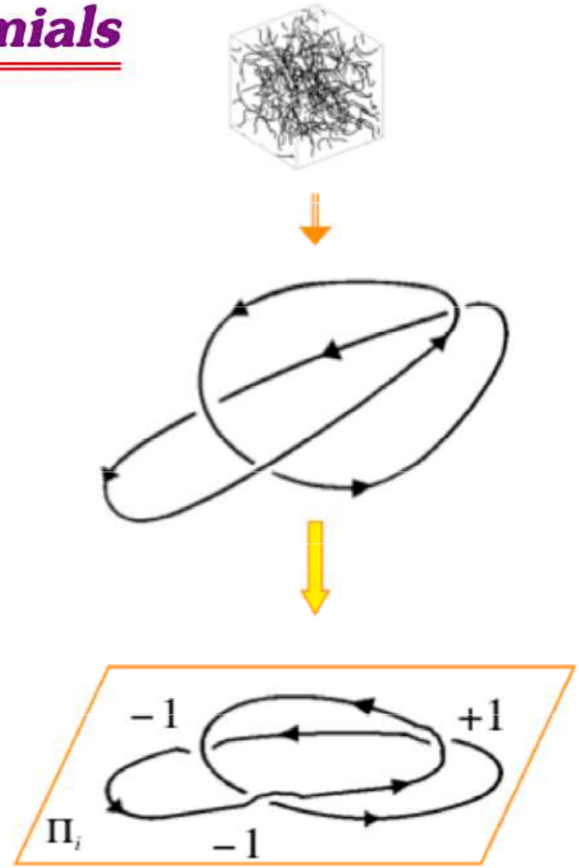


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+ − =

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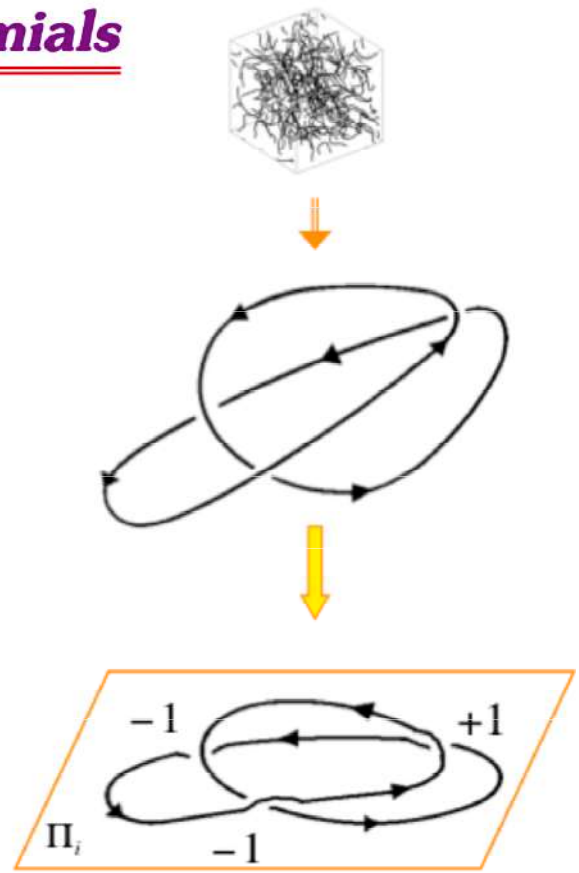


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$+$ $\quad$ $-$ $\quad$ $=$

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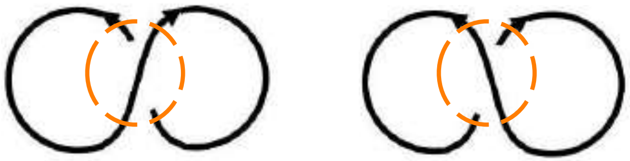
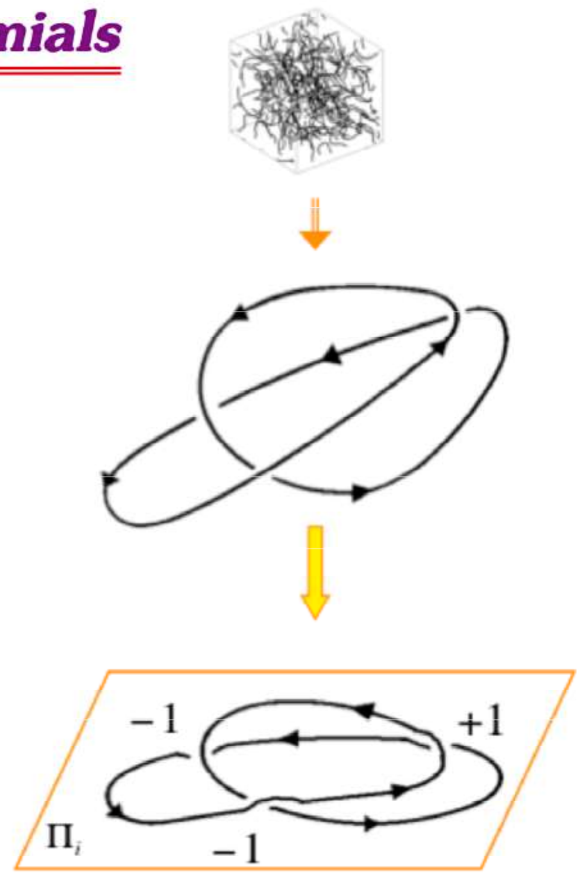
**P2:**



$\gamma_+$ $\quad$ $\gamma_-$

# *Tackling structural complexity by knot polynomials*

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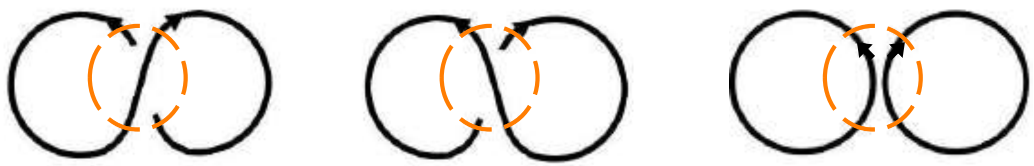

$\gamma_+$ $\gamma_-$ $U_2$

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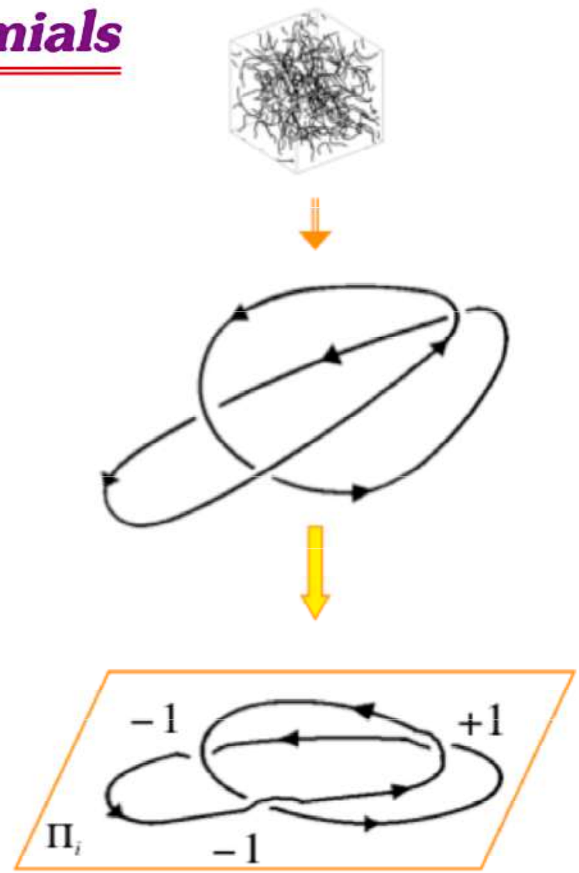


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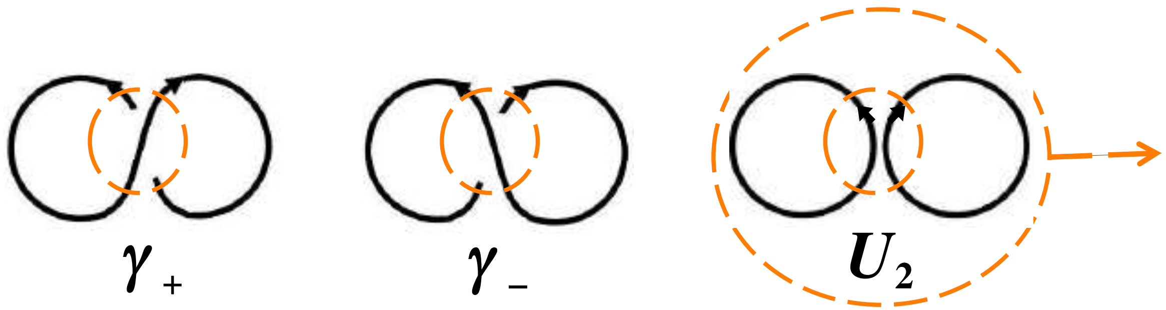
***P2:***



$\gamma_+$ $\quad$ $\gamma_-$ $\quad$ $U_2$

$$P(U_2) = \frac{a - a^{-1}}{z}$$

## *HOMFLYPT polynomial from self-linking*

- ***Theorem* *(Liu & Ricca, JFM 2015).* *If* $K$ *denotes a vortex knot of axis* $C$ *and helicity* $H = H(K)$ *, then***

$$e^{H(K)} = e^{\oint_C \mathbf{u} \cdot d\mathbf{l}}$$

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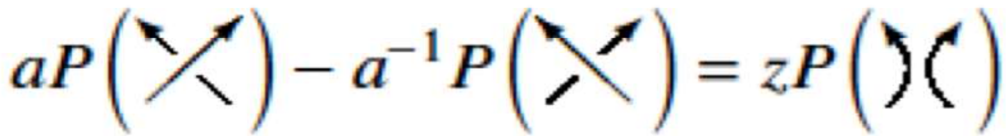

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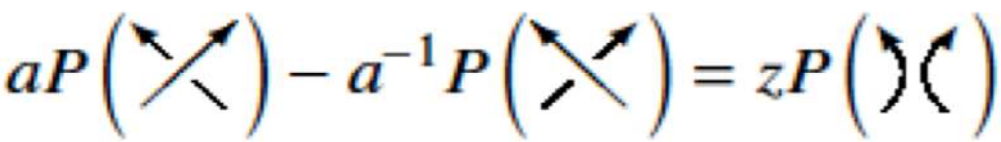



$$\Longleftrightarrow \quad \left[ f(Tw) \right] = g(Wr)\,.$$

- ***Reduction of HOMFLYPT $P_K(a,z)$ to Jones $V_K(t)$:***

## *HOMFLYPT polynomial from self-linking*

- ***Theorem** (Liu & Ricca, JFM 2015). If $K$ denotes a vortex knot of axis $C$ and helicity $H = H(K)$, then*

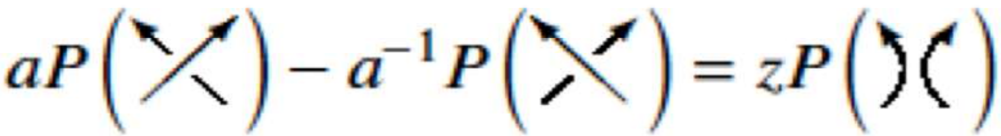
$$e^{H(K)} = e^{\oint_C \mathbf{u}\cdot d\mathbf{l}}$$

*appropriately rescaled satisfies (under a plausible statistical hypothesis) the skein relations of the HOMFLYPT polynomial $P(K) = P_K(a,z)$.*

- ***HOMFLYPT variables in terms of writhe and twist:***

$$a = f(Tw), \qquad z = g(Wr).$$

***Hence, from the HOMFLYPT P2 skein relation, we have:***



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- ***Reduction of HOMFLYPT $P_K(a,z)$ to Jones $V_K(t)$:***

$t = h(a,z)$ : $\quad Wr \propto Tw \quad \Longleftrightarrow$ ***knots and links are "framed".***

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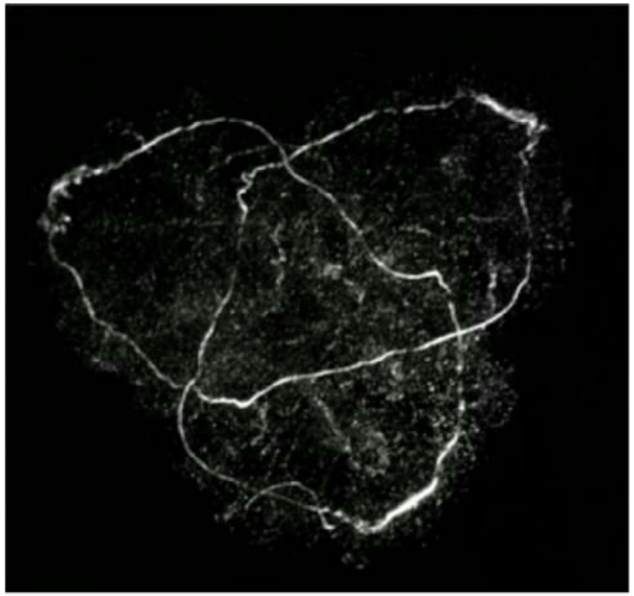
| $T^L$ | $2a^2 + a^2z^2 - a^4$ | 2.36 |
| $T^R$ | $2a^{-2} + a^{-2}z^2 - a^{-4}$ | 1.51 |
| $F^8$ | $a^{-2} - 1 - z^2 + a^2$ | 0.17 |
| $W$ | $-a^{-1}z^{-1} - a^{-1}z + az^{-1} + 2az + az^3 - a^3z$ | 1.59 |
| … | … | … |

# *Vortex trefoil cascade process in water (Kleckner & Irvine 2013)*



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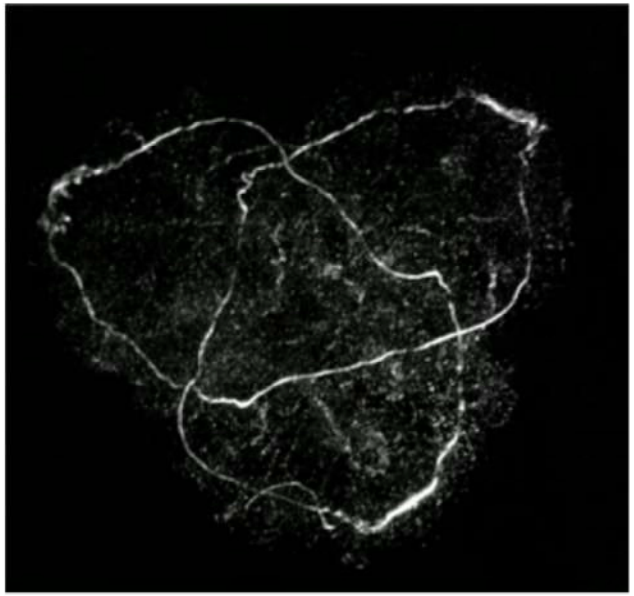

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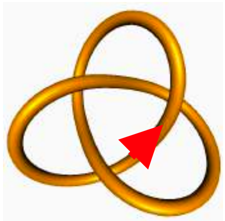

$T\,(2,3)$

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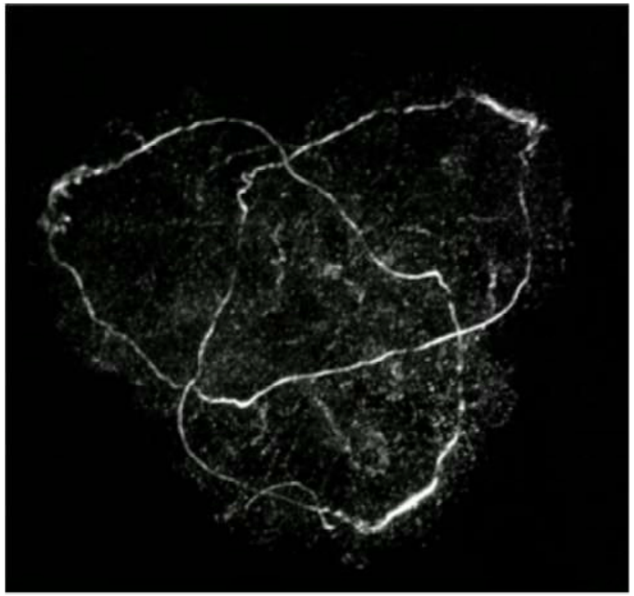

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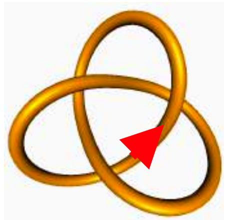

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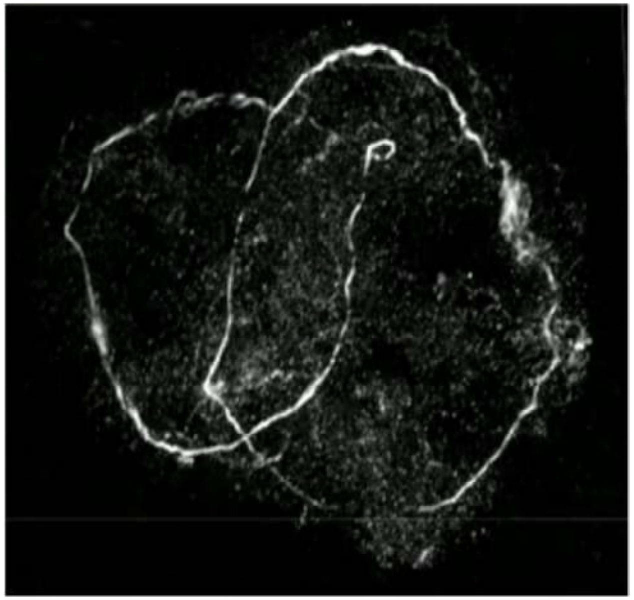

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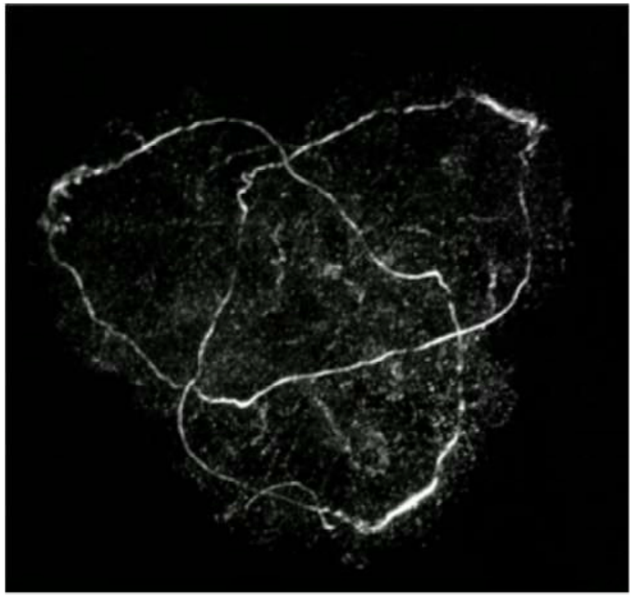

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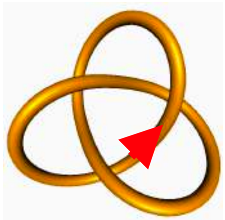

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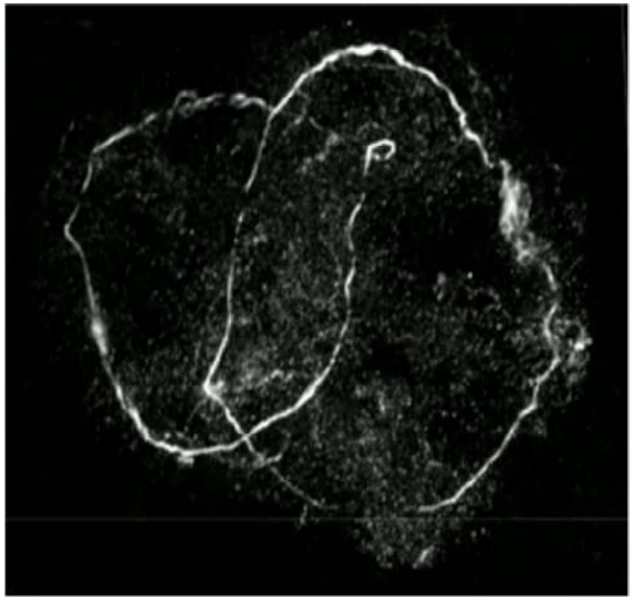

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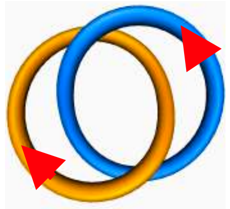

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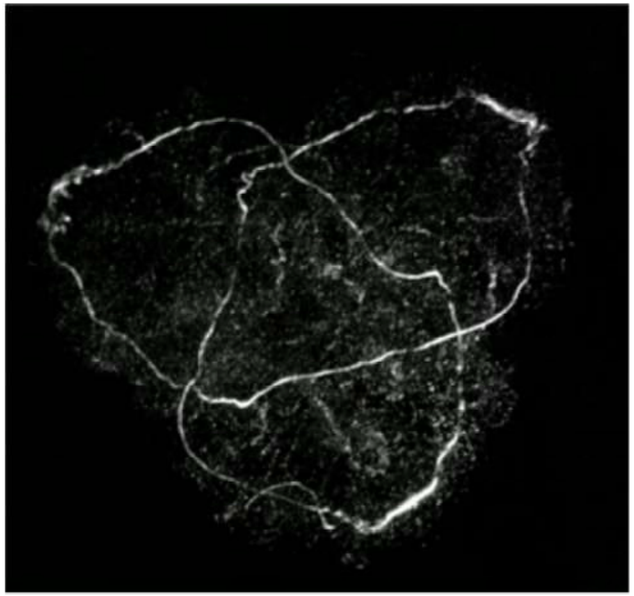

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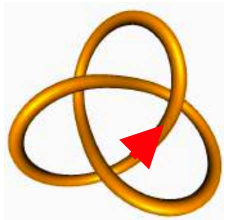

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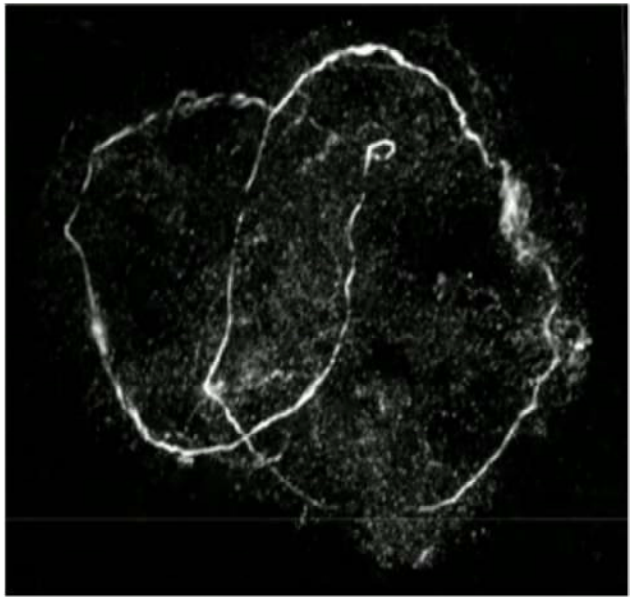

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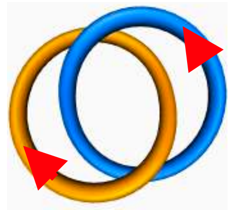

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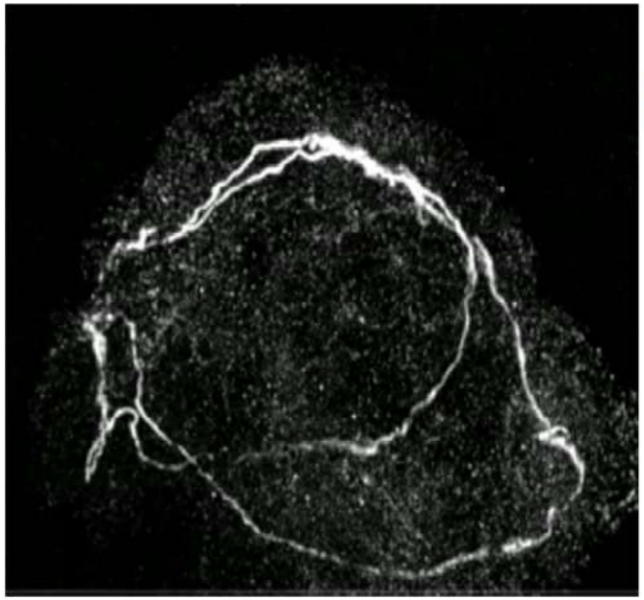

$t = 3$

$T\,(2,1)$

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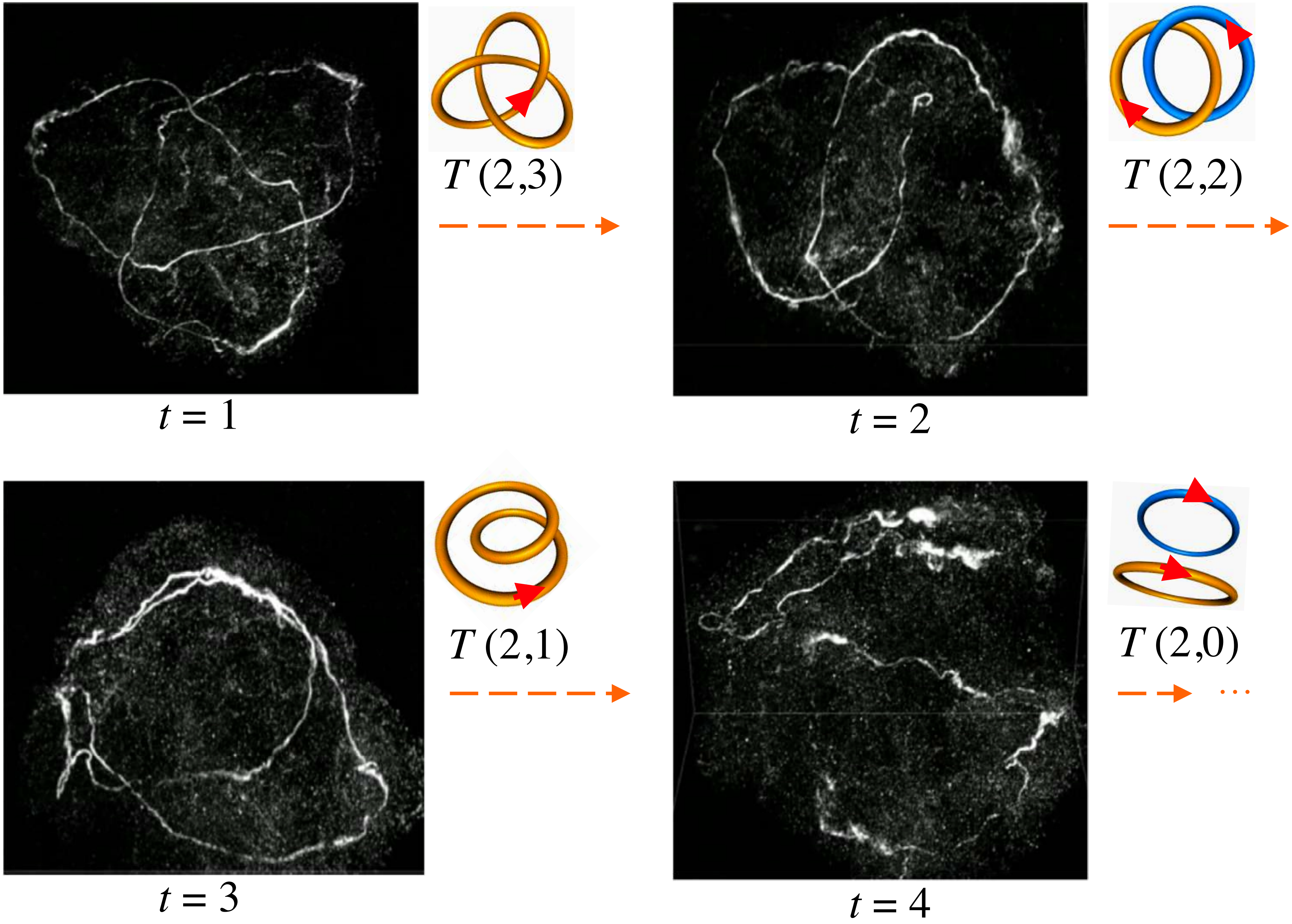

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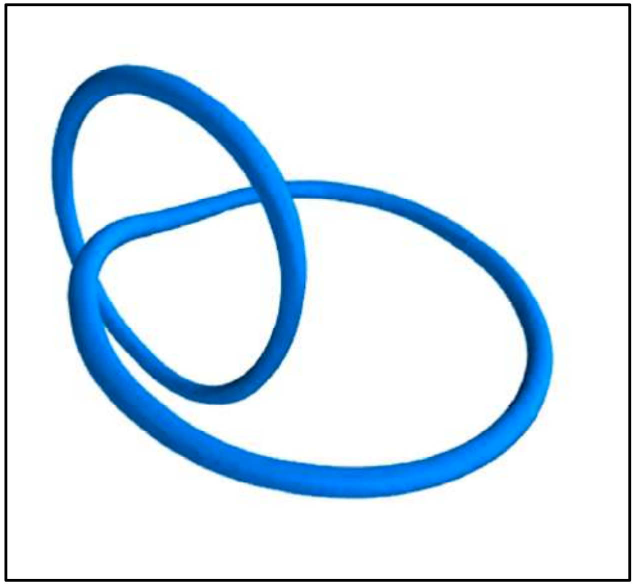

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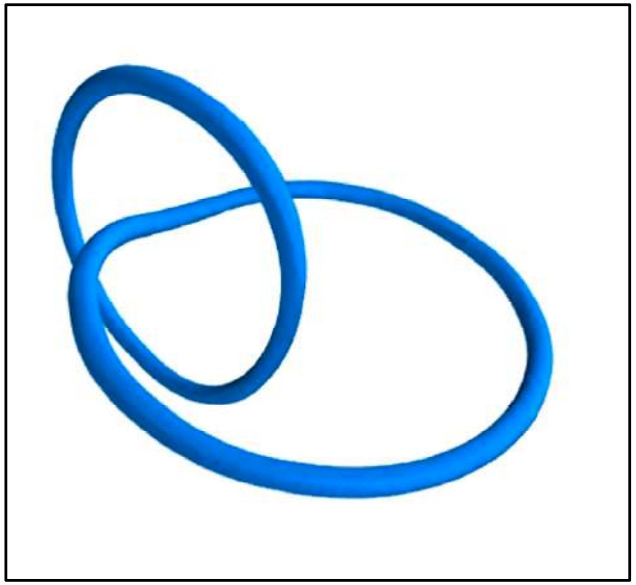

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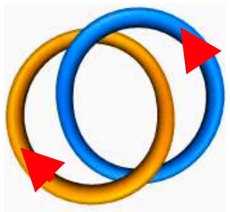

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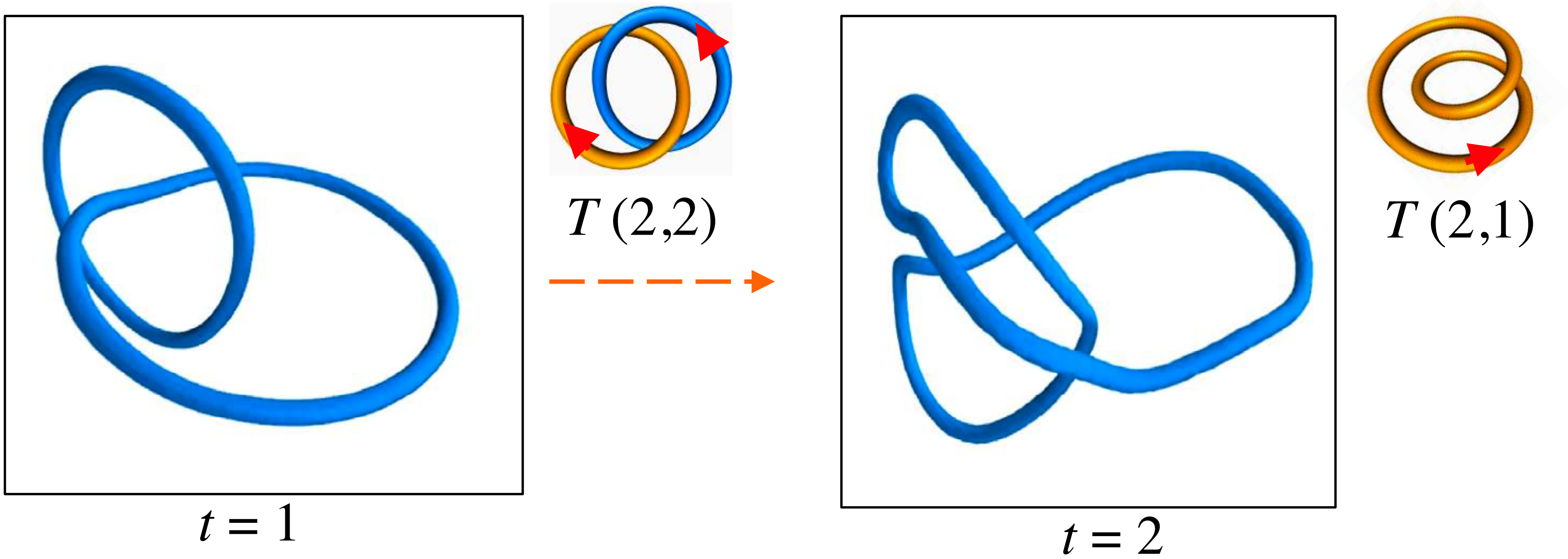

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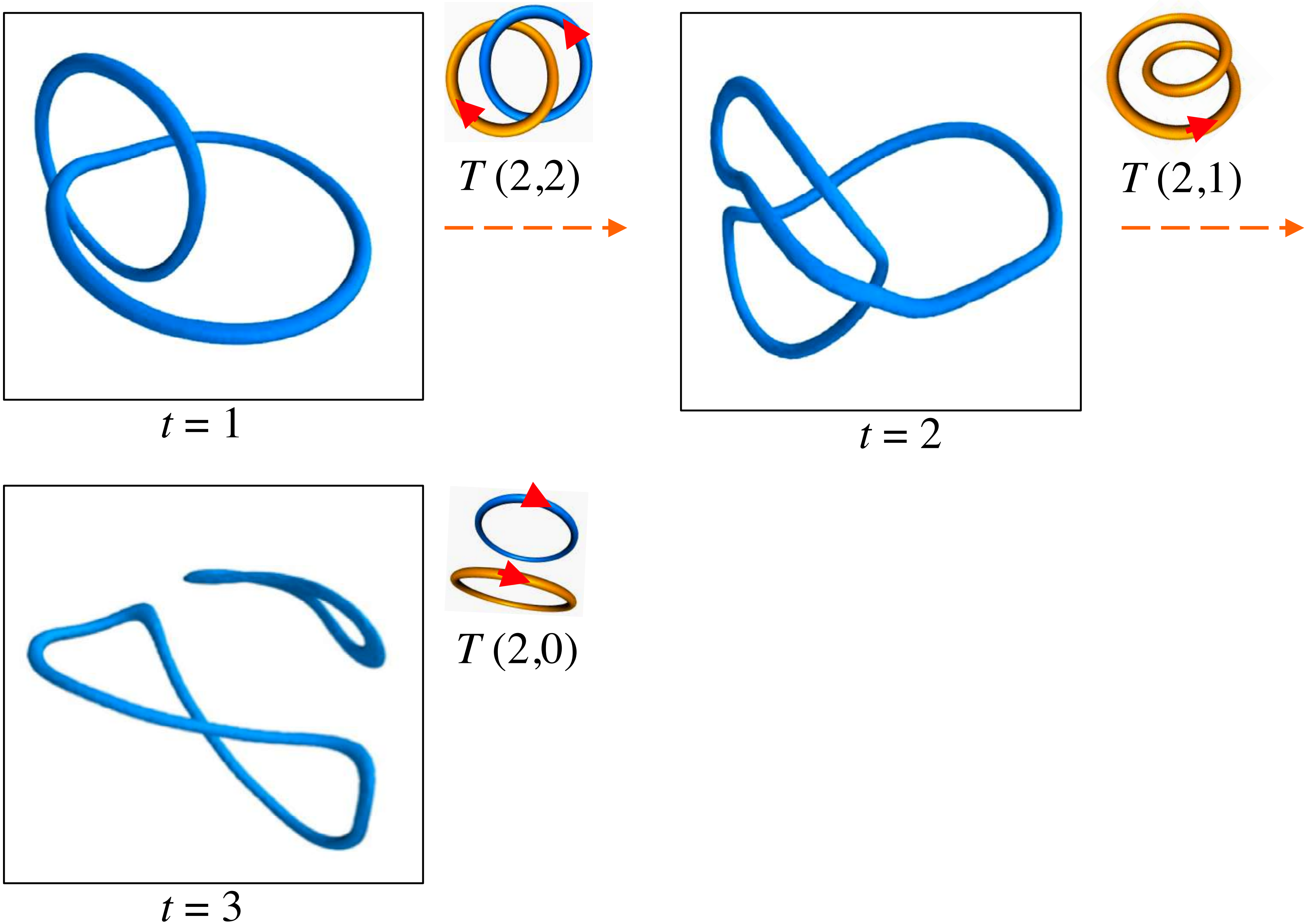

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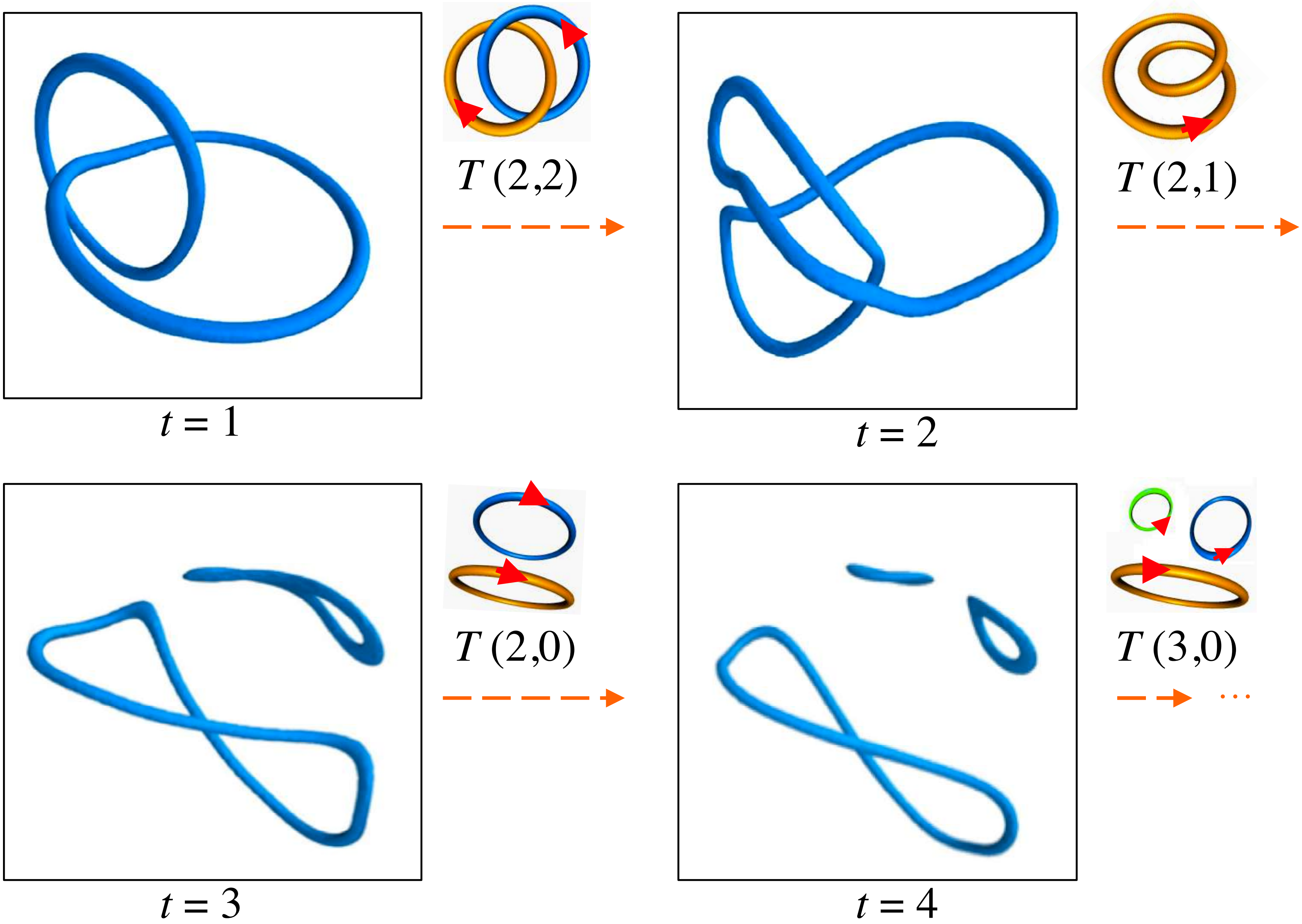

## Ideal cascade of torus knots & links

**Consider the cascade process:**

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# *Ideal cascade of torus knots & links*

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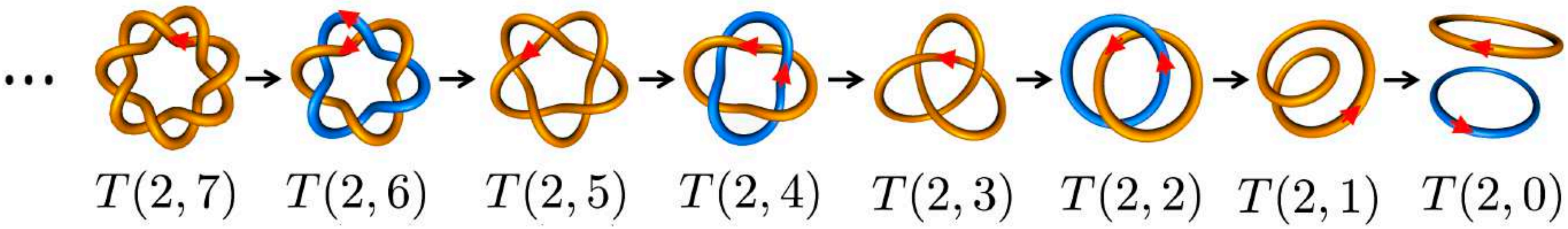

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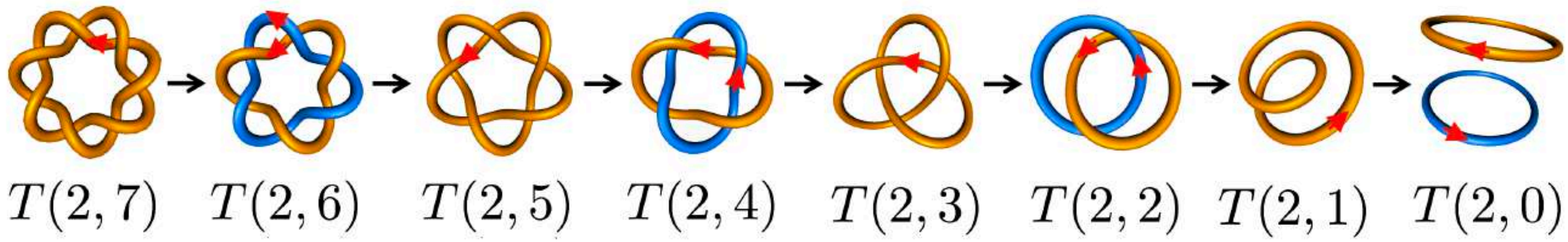


$\{T(2,n)\}$ **:** $\quad \ldots \rightarrow T(2,2n+1) \rightarrow T(2,n) \rightarrow \ldots \rightarrow T(2,0)$ **.**

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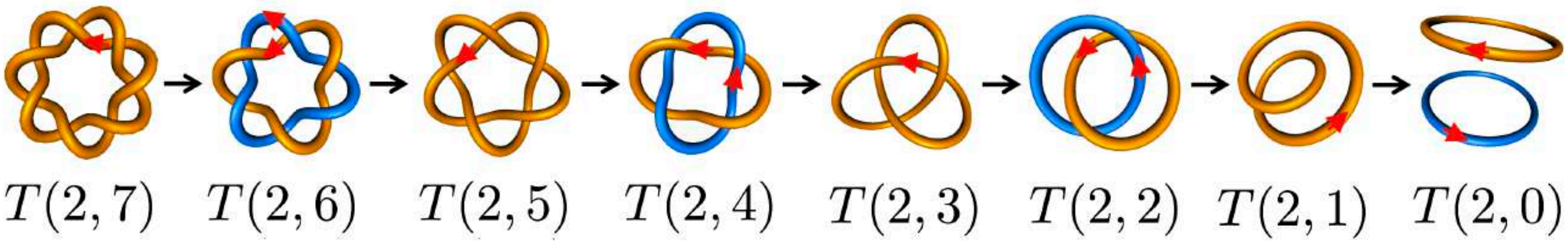


$$\{T(2,n)\}: \quad \ldots \to T(2,2n+1) \to T(2,n) \to \ldots \to T(2,0) .$$

- ***Theorem (Liu & Ricca 2016). HOMFLYPT computation of** $P_{T(2,n)}$ **generates for decreasing** $n$ **a monotonically decreasing sequence of numerical values given by***

$$P_{T(2,3+q)} = A_q(\tau,\omega) P_{T(2,3)} + B_q(\tau,\omega) P_{T(2,2)} \quad (q \in \mathbb{N}) ,$$

***where** $A_q(\tau,\omega)$ **and** $B_q(\tau,\omega)$ **are known functions of** $\tau$ **and** $\omega$**, with initial conditions** $P_{T(2,3)}$ **and** $P_{T(2,2)}$**.***

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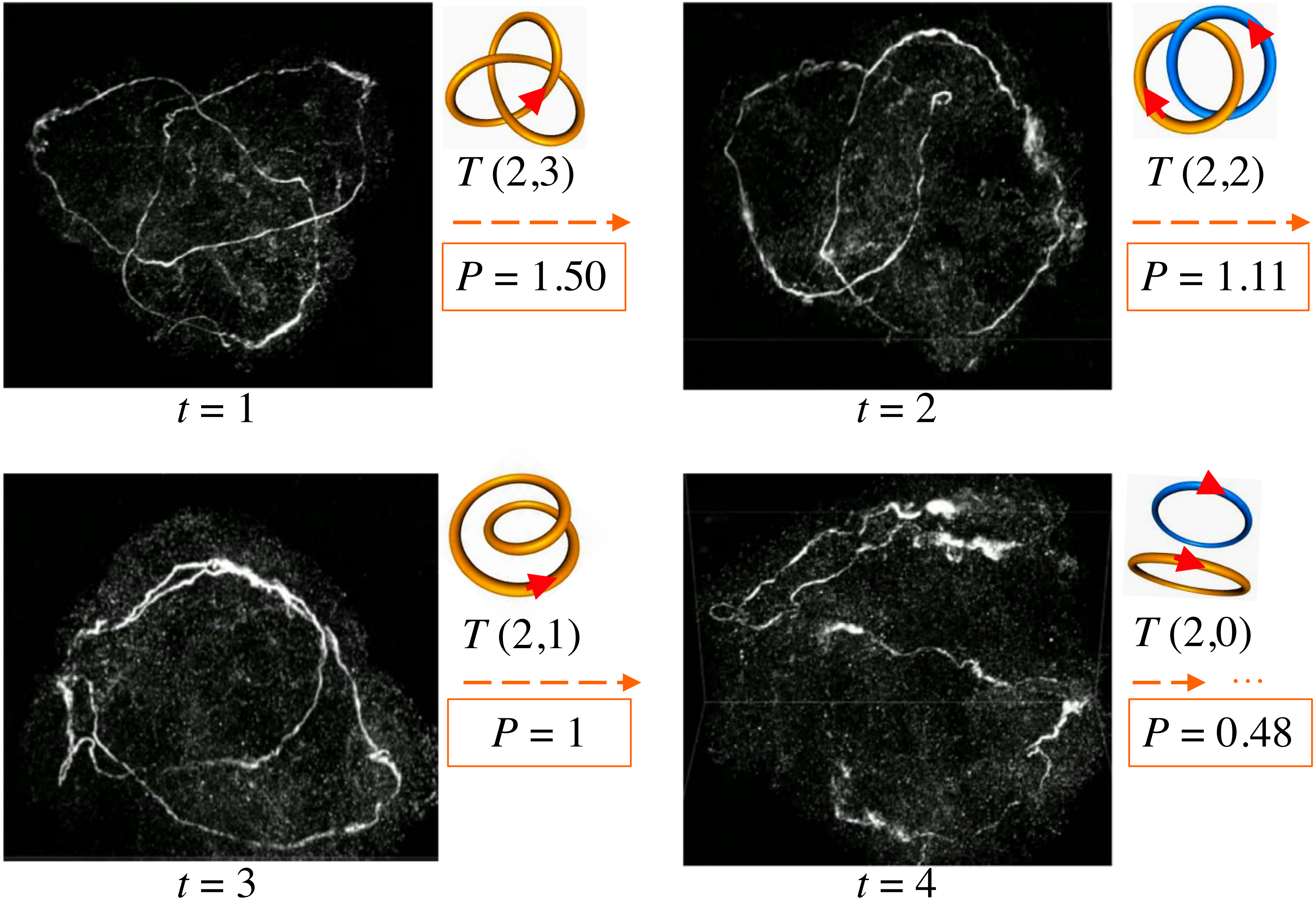

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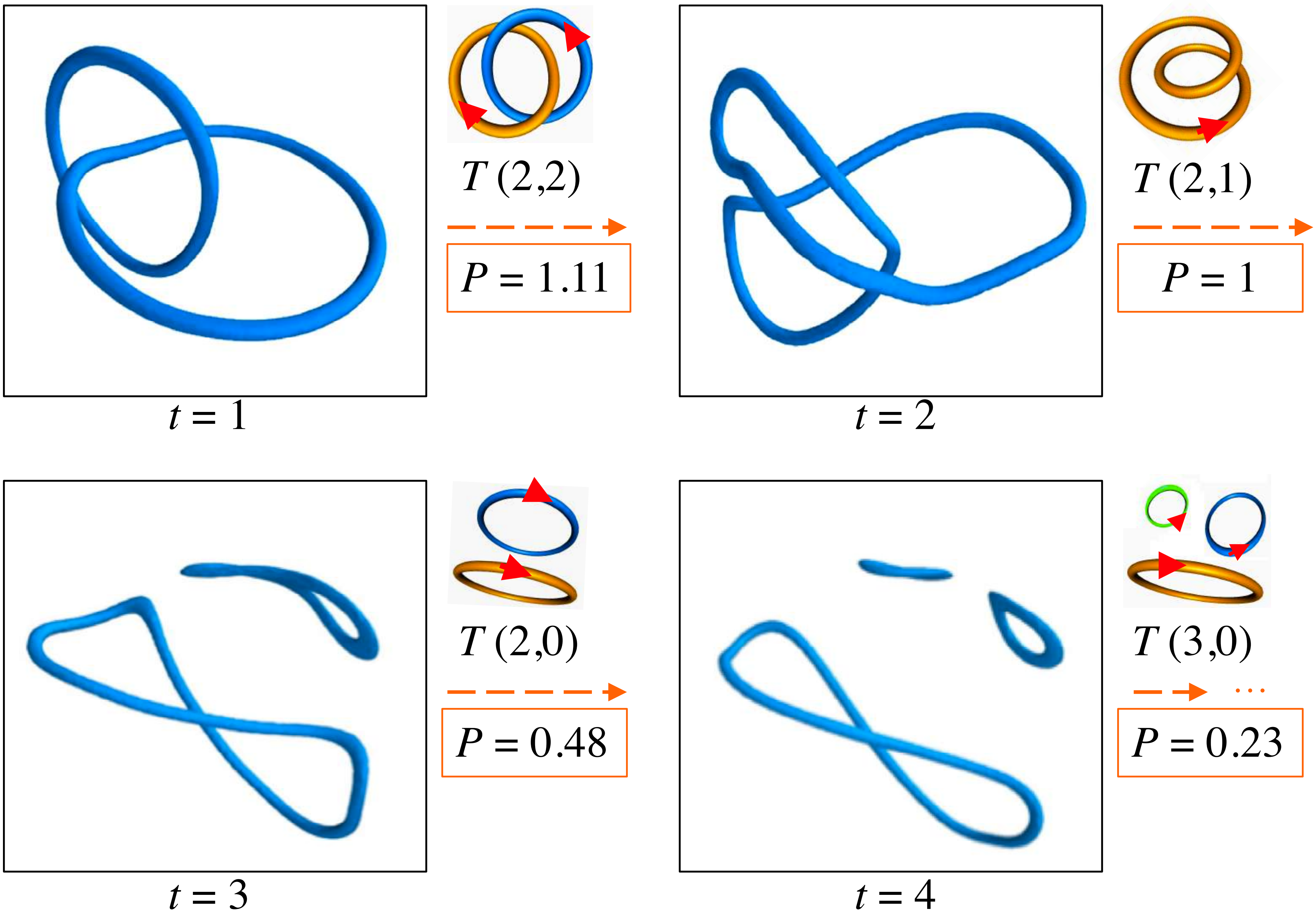
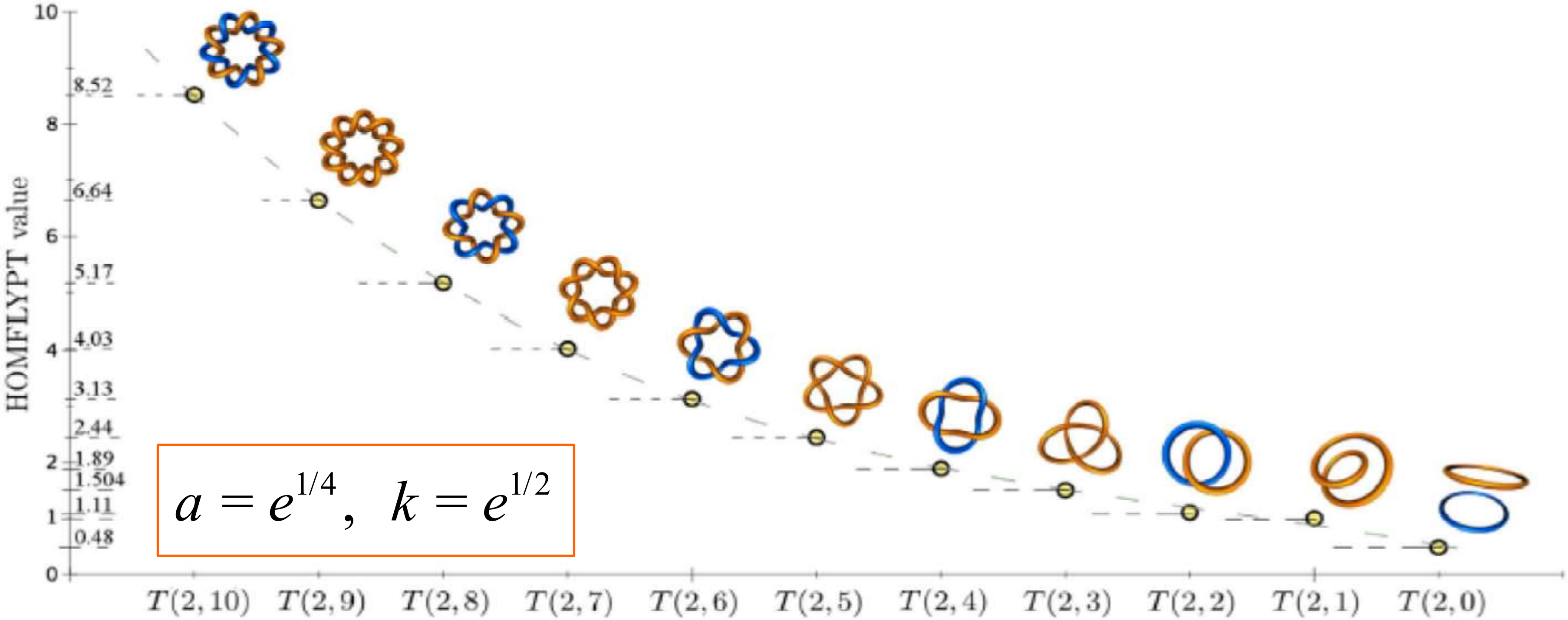


# *HOMFLYPT quantifies topological complexity*



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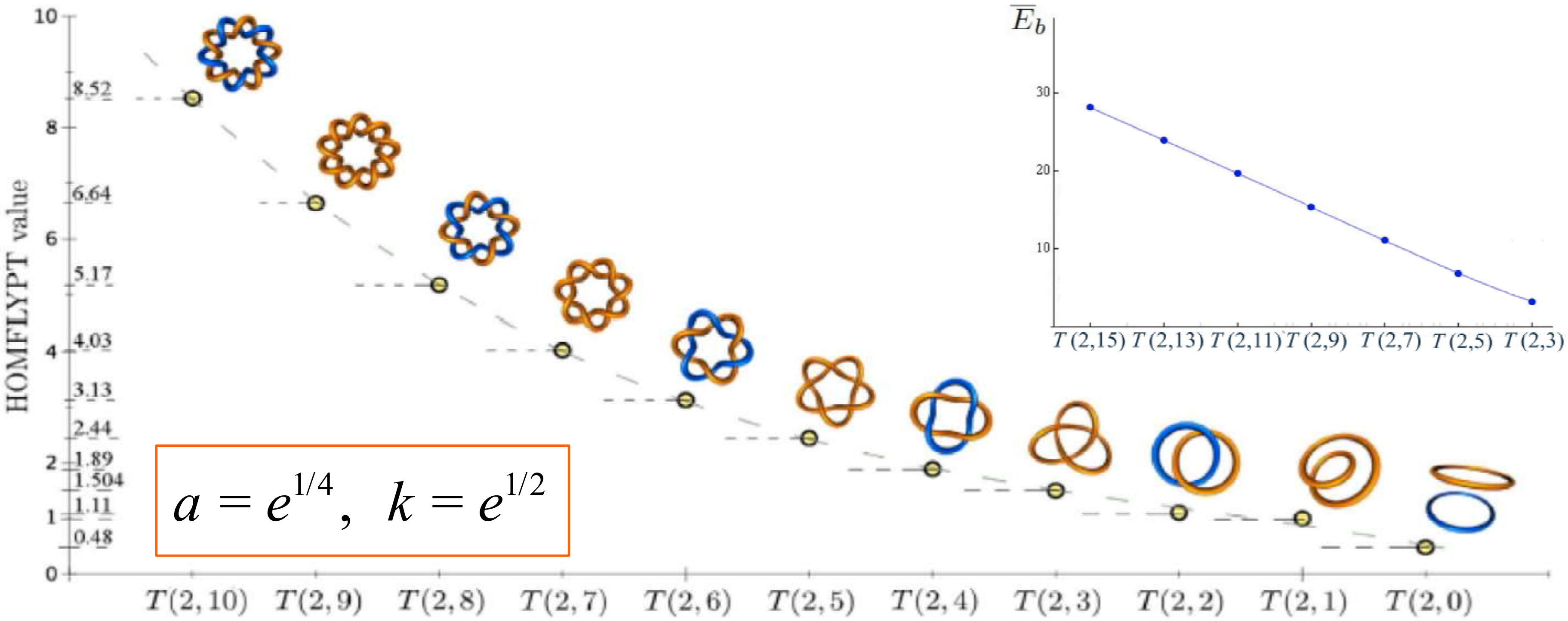

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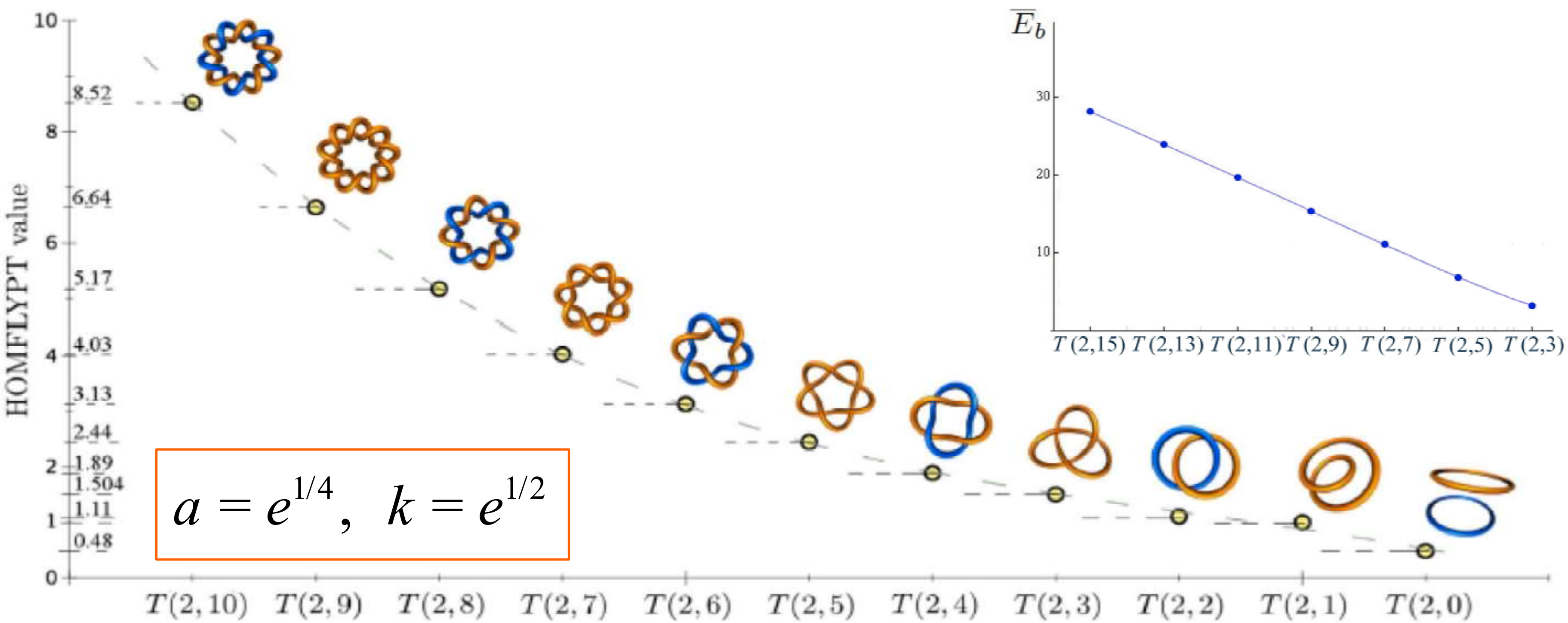


- ***HOMFLYPT best quantifier:***

| Numerical values for torus knots and *co-oriented* torus links ($Wr = Tw = 1/2$) | | | | | | | | | | | |
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| HOMFLYPT: $a = e^{1/4}, k = e^{1/2}$ | 8.52 | 6.64 | 5.17 | 4.03 | 3.13 | 2.44 | 1.89 | 1.50 | 1.11 | 1 | 0.48 |
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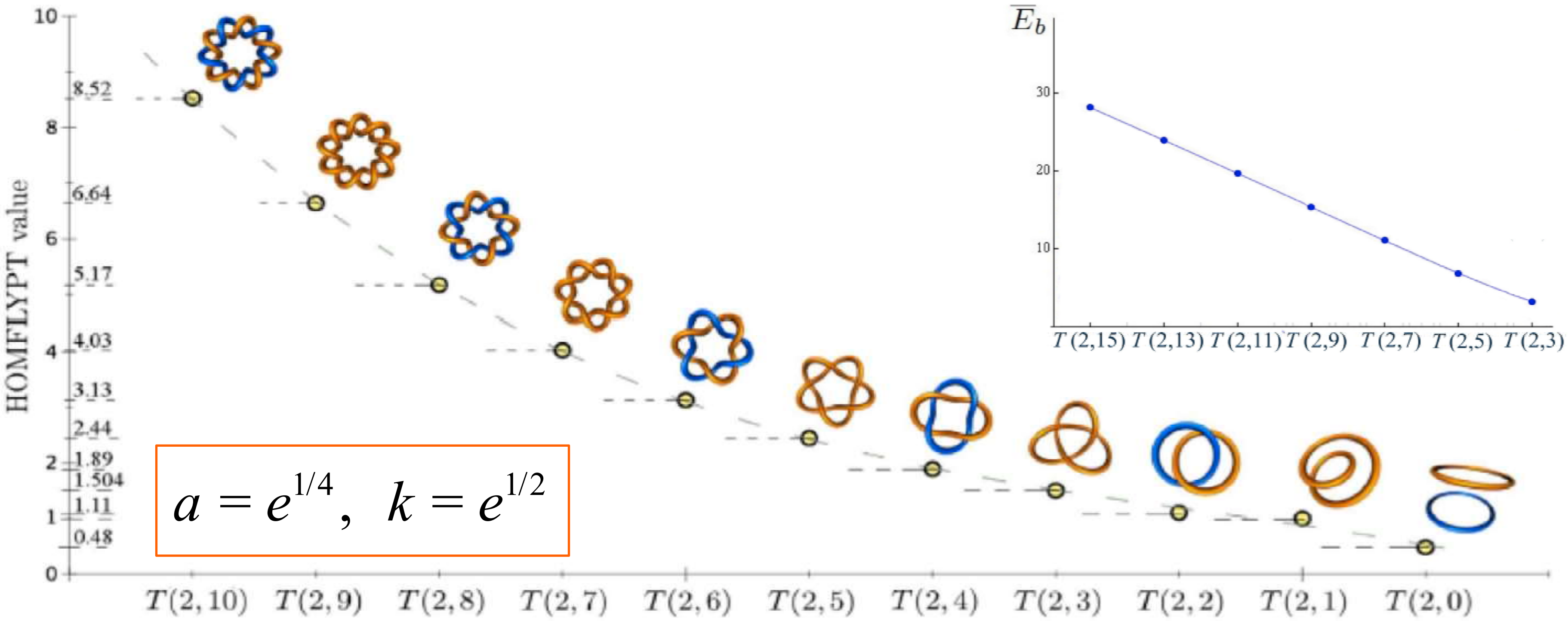


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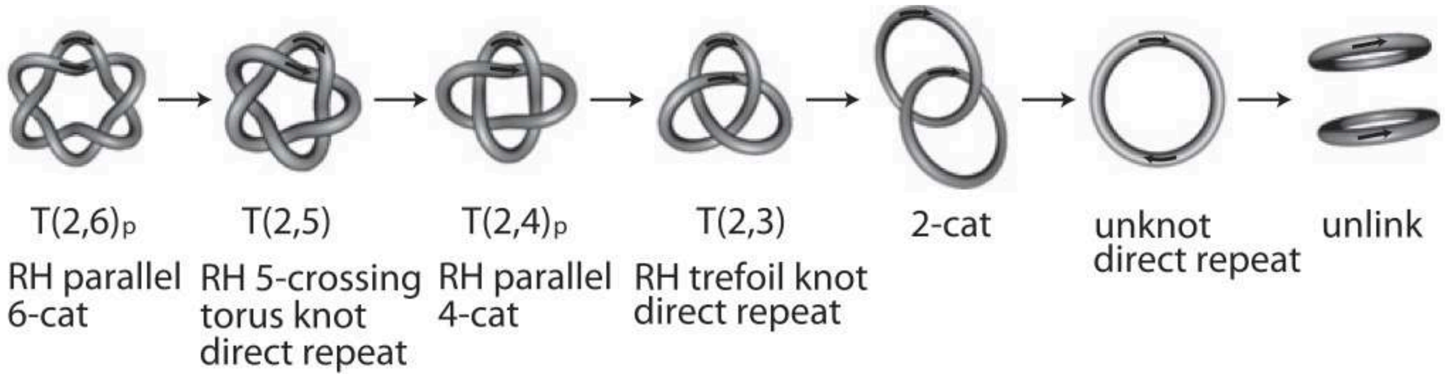

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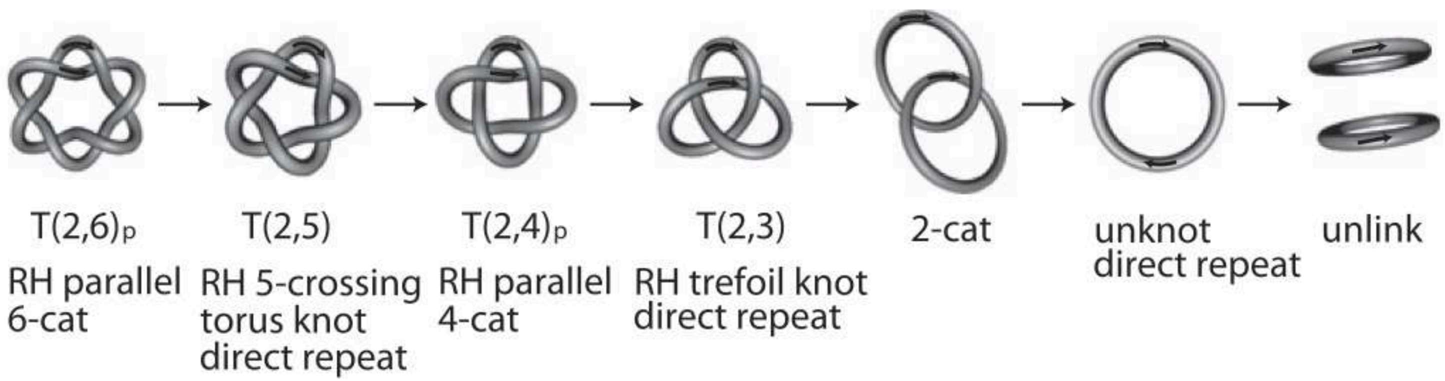


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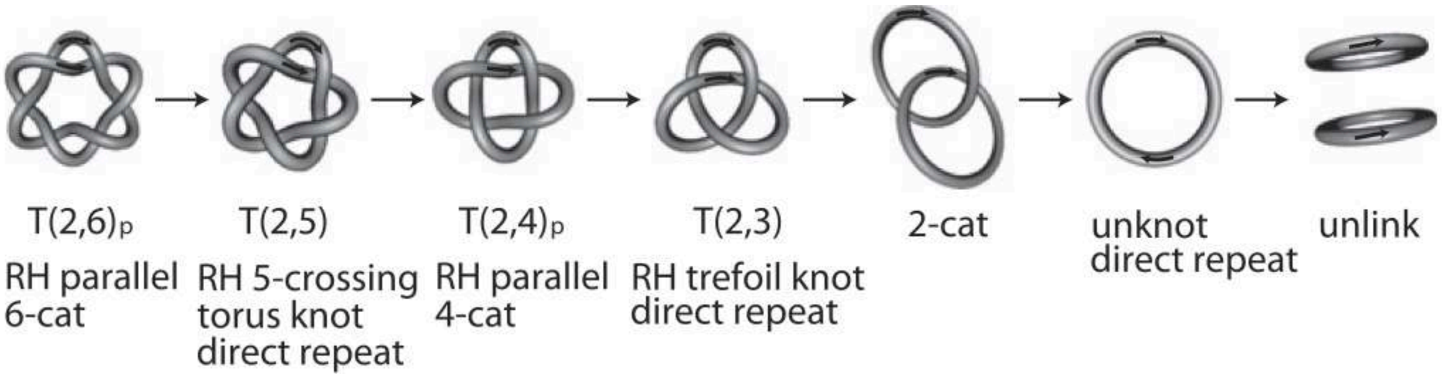


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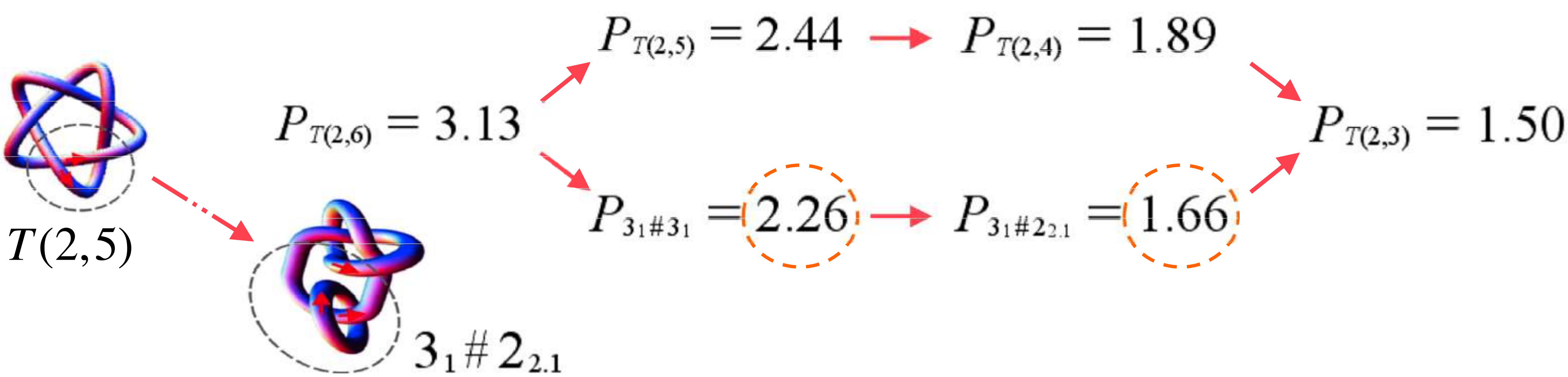

## Selected references


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***Thank you!***