What is a braidoid?

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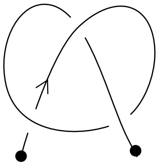
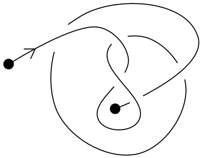
What is a knotoid diagram?





A *knotoid diagram* is an open-ended knot diagram with two endpoints that can lie in different regions of the diagram.

What is a knotoid diagram?

Definition (Turaev)

A *knotoid diagram* K in an oriented surface Σ is an immersion

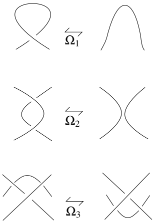
$$K : [0,1] \to \Sigma \text{ such that:}$$

1. each transversal double point is endowed with under/over data, and we call them *crossings* of K,
2. the images of 0 and 1 are two disjoint points regarded as the *endpoints* of K. They are called the *leg* and the *head* of K, respectively.
3. K is oriented from the leg to the head.

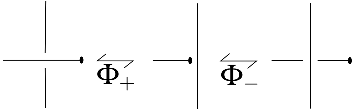
What is a knotoid?

Definition

A *knotoid* is an equivalence class of the knotoid diagrams in Σ considered up to the equivalence relation induced by the knotoid Reidemeister moves and isotopy of Σ.



Knotoid Reidemeister moves



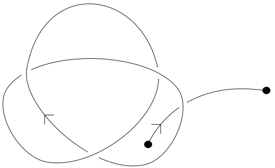
Forbidden knotoid moves

Extending the definition of a knotoid

Definition (Turaev)

A *multi-knotoid diagram* in an oriented surface Σ is a generic immersion of a single oriented unit interval and a number of oriented circles in Σ endowed with under/over-crossing data.

A *multi-knotoid* is an equivalence class of multi-knotoid diagrams determined by the equivalence relation generated by Ω-moves and isotopy of the surface.



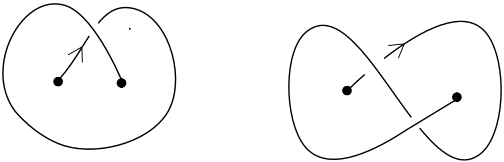
A multi-knotoid diagram

Comparing knotoids in S^2 and $\mathbb{R}^2$

There is a surjective map,

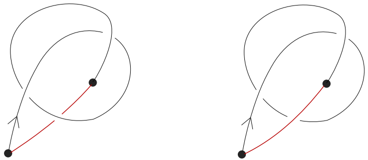
$$\iota : \{ \text{ knotoids in } \mathbb{R}^2 \} \to \{ \text{ knotoids in } S^2 \}$$

which is induced by $\iota : \mathbb{R}^2 \hookrightarrow S^2$. This map is not injective.



Nontrivial planar knotoids which are trivial in S^2

From knotoids to classical knots



There is a surjective map,

$$\omega_{-} : \{\text{ Knotoids }\} \to \{\text{ Classical knots }\}$$

induced by connecting the endpoints of a knotoid diagram with an underpassing arc.

Theorem (Turaev)

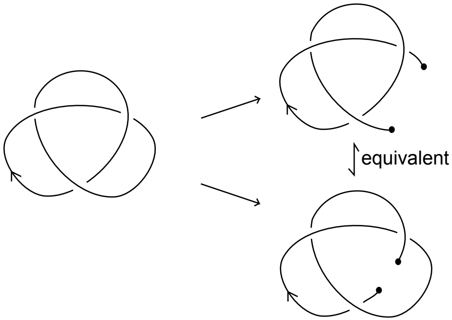
Let κ be a knot and K be a knotoid representative of κ. Then $\pi_1(\kappa) = \pi(K)$.

From classical knots to knotoids

- There is an injective map,

$$\alpha\colon\ \{\text{Classical knots}\} \to \{\text{Knotoids in } S^2\},$$

induced by deleting an open arc which does not contain any crossings from an oriented classical knot diagram.



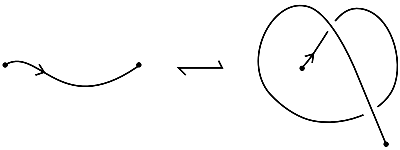
$\Rightarrow$ The theory of knotoids in S^2 is an extension of classical knot theory.

From classical knots to knotoids

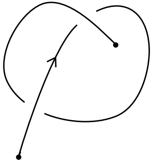
Definition

A knotoid in S^2 that is in the image of α, is called a *knot-type* knotoid. A knotoid that is not in the image of α, is called a *proper* knotoid.

$$\{\text{Knotoids in } S^2\} = \{\text{Knot-type knotoids}\} \cup \{\text{ Proper knotoids}\}$$

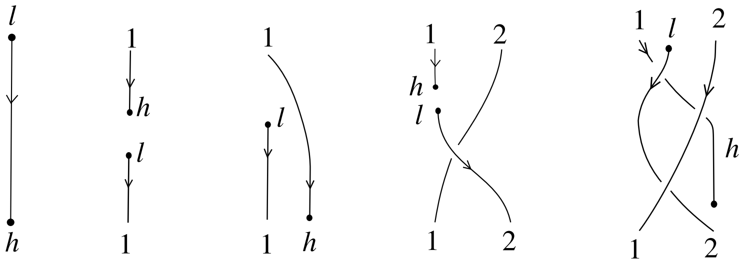


A knot-type knotoid



A proper knotoid

The theory of braidoids



What is a braidoid diagram?

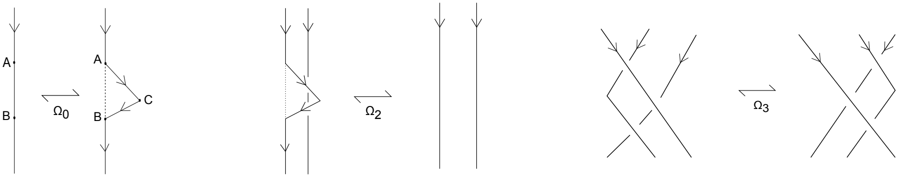
Definition

A *braidoid diagram* B is a system of a finite number of arcs in $[0,1] \times [0,1] \subset \mathbb{R}^2$ that are called the *strands* of B.

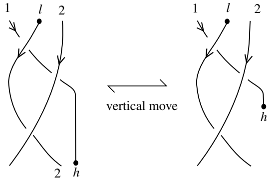
1. There are only finitely many intersection points among the strands, which are transversal double points endowed with over/under data, and are called *crossings*.
2. Each strand is naturally oriented downward, with no local maxima or minima, so that each intersects a horizontal line at most once.
3. A braidoid diagram has two types of strands, the classical strands and the free strands. A *free strand* has one or two ends that are not necessarily at $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. Such ends of free strands are called the *endpoints* of B.

Moves on braidoid diagrams

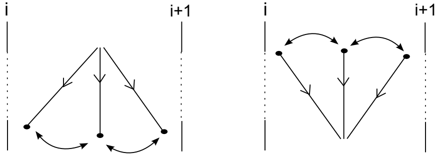
Δ-Moves:



Vertical Moves:



Swing Moves:



Braidoids

Definition

Two braidoid diagrams are said to be *isotopic* if one can be obtained from the other by a finite sequence of Δ-moves, vertical moves and swing moves. An isotopy class of braidoid diagrams is called a *braidoid*.

Definition

A *labeled braidoid diagram* is a braidoid diagram whose braidoid ends are labeled with o or u.
A *labeled braidoid* is an isotopy class of labeled braidoid diagrams up to the isotopy relation generated by the Δ-moves.

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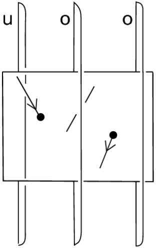
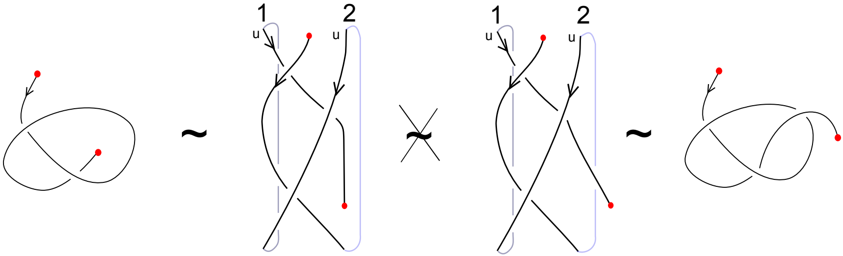
From a braidoid diagram to a knotoid diagram

- We define a closure operation on labeled braidoid diagrams by connecting each pair of corresponding ends accordingly to their labels and within a ‘sufficiently close’ distance:



The closure operation induces a well-defined map from the set of labeled braidoids to the set of planar multi-knotoids.



An unrestricted swing move causing forbidden moves

The analogue of the Alexander Theorem for braidoids

Theorem (The classical Alexander theorem)

Any classical knot/link diagram is isotopic to the closure of a classical braid diagram.

Theorem (G., Lambropoulou)

Any multi-knotoid diagram in $\mathbb{R}^2$ is isotopic to the closure of a labeled braidoid diagram.

From a knotoid diagram to a braidoid diagram

We describe two braidoiding algorithms to prove our theorem.

The idea:

Eliminate the up-arcs of a (multi)-knotoid diagram. We do this by the *braidoiding moves*.

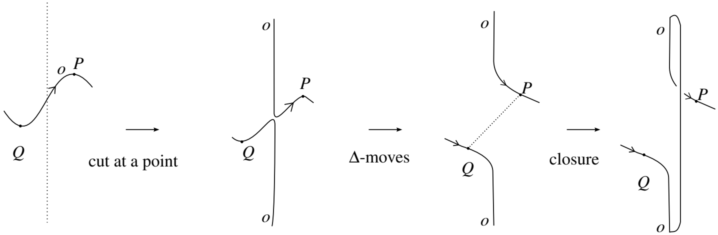
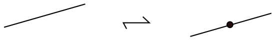


Figure: The germ of the braidoiding move and its closure

- Observe that the closure of each resulting labeled strand is isotopic to the initial up-arc.

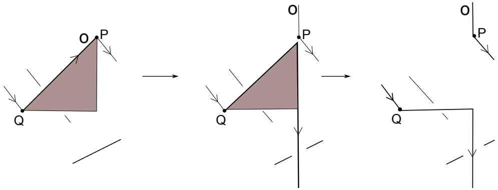
Preparatory notions for the braidoiding algorithms

- Subdivision:

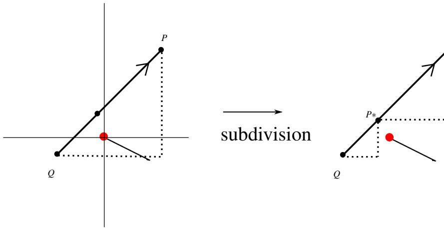


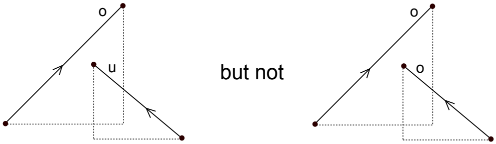
- Up-arcs and free up-arcs:

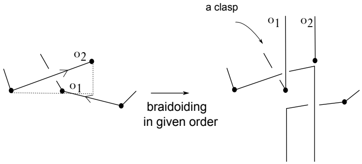
- Sliding triangles and the cut points:



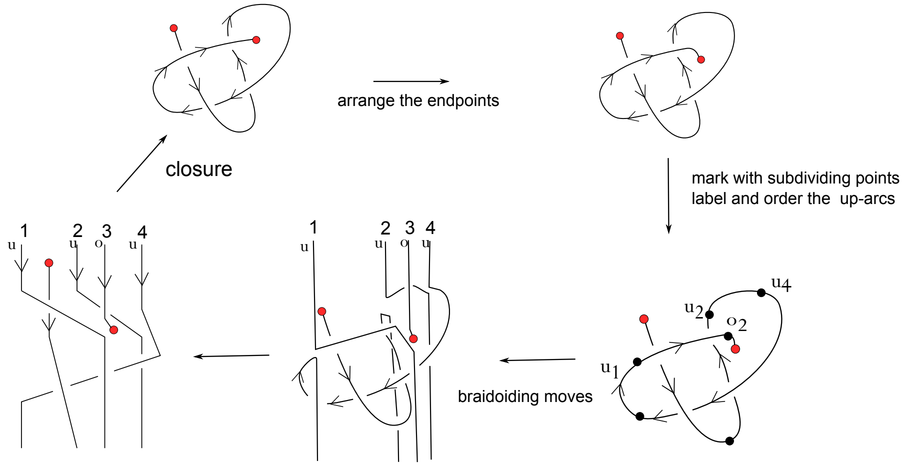
Triangle conditions



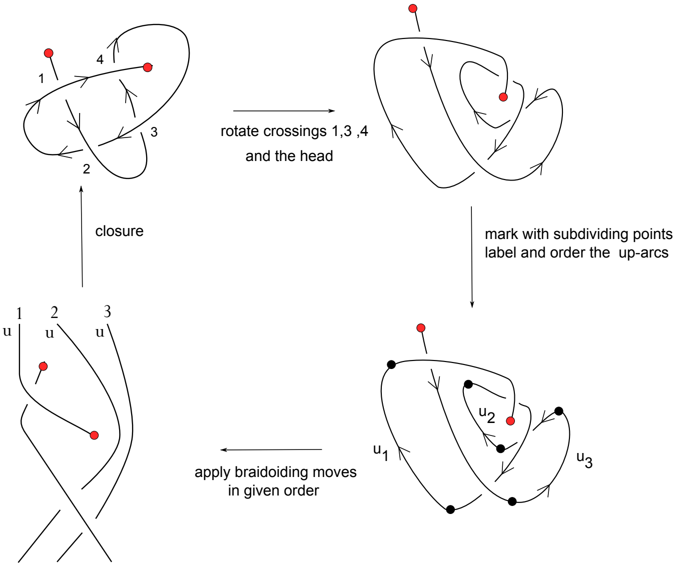




Braidoiding algorithm I:



Braidoiding algorithm II:



A corollary of the braidoiding algoithm II

Definition

A *u-labeled braidoid diagram* is a labeled braidoid diagram whose ends are labeled all with u.

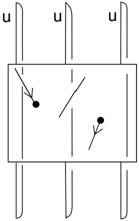
There is a bijection:

$$Label_u : \{\text{Braidoids}\} \rightarrow \{u\text{-labeled braidoids}\}$$

induced by assigning to a braidoid diagram a u-labeled braidoid diagram.

A sharpened version of the theorem

The uniform closure:



Theorem (G.,Lambropoulou)

Any multi-knotoid diagram $\mathbb{R}^2$ is isotopic to the uniform closure of a braidoid diagram.

Markov theorem for classical braids

Theorem (*Markov theorem*)

The closures of two braid diagrams b, b' in $\cup_{n=1}^{\infty} B_n$, represent isotopic links in $\mathbb{R}^3$ if and only if these braids are equivalent by the following operations.

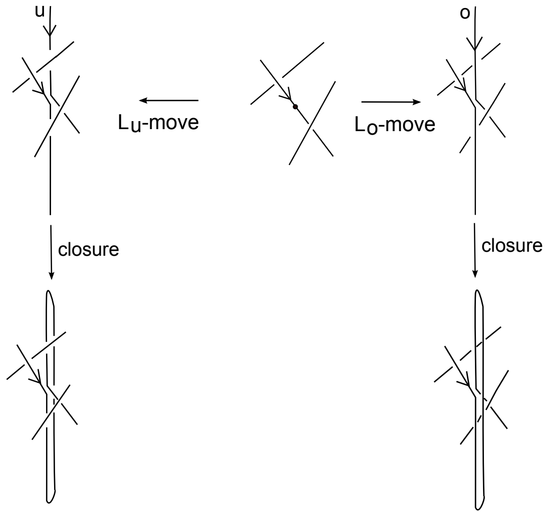
- *Conjugation: For $b, b' \in B_n$, $b' = gbg^{-1}$ for some $g \in B_n$.*
- *Stabilization: For $b \in B_n$, $b' \in B_{n+1}$, $b' = \sigma_n^{\pm} b$.*

Theorem (*One move Markov theorem*, Lambropoulou)

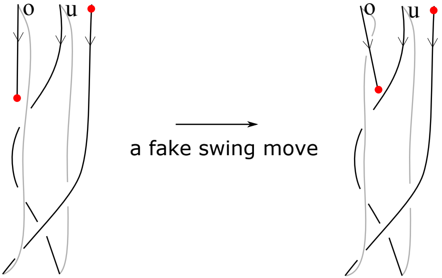
There is a bijection between the set of L-equivalence classes of braids and the set of isotopy classes of (oriented) link diagrams.

From knotoids to braidoids: *L* - equivalence

An *L-move* on a labeled braidoid diagram *B* is the following operation:



Fake swing moves



An analogue of the Markov theorem for braidoids

L-equivalence $=< L$- moves, fake swing moves, isotopy moves $>$.

Theorem (G.,Lambropoulou)

The closures of two labeled braidoid diagrams are isotopic (multi-)knotoids in $\mathbb{R}^2$ if and only if the labeled braidoid diagrams are L-equivalent.

A sketch for the proof

$\Rightarrow$: From our previous observation closure induces a well-defined map:
cl:{L-eqv. classes of labeled braidoids}$\rightarrow${Multi-knotoids}.

$\Leftarrow$: The braidoiding algorithm I induces a well-defined map,

br:{Multi-knotoids in $\mathbb{R}^2$} $\rightarrow${L- eqv. classes of labeled braidoids}.

For this we need to check:

- **Static Part**: Choices done for applying the algorithm such as subdivision, labeling of free up-arcs,
- **Moving Part**: The isotopy moves for knotoid diagrams including the moves displacing the endpoints.

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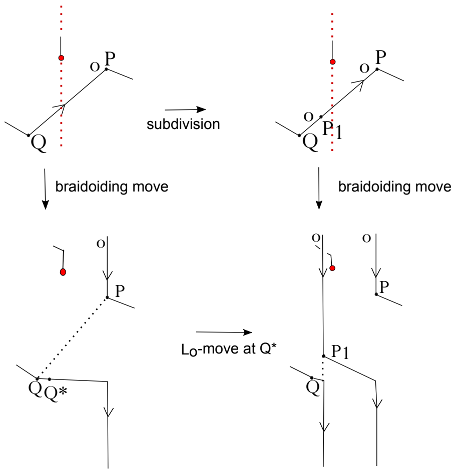
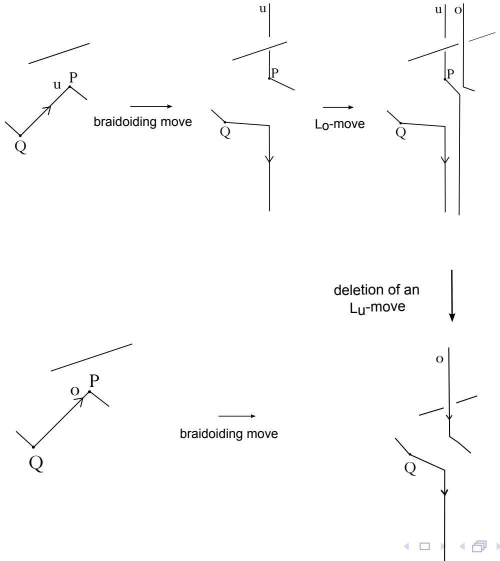
Lemma

Adding subdivision points to an up-arc results in L-equivalent braidoid diagrams.



Lemma

Changing the labeling of a free up-arc results in L-equivalent braidoid diagrams.

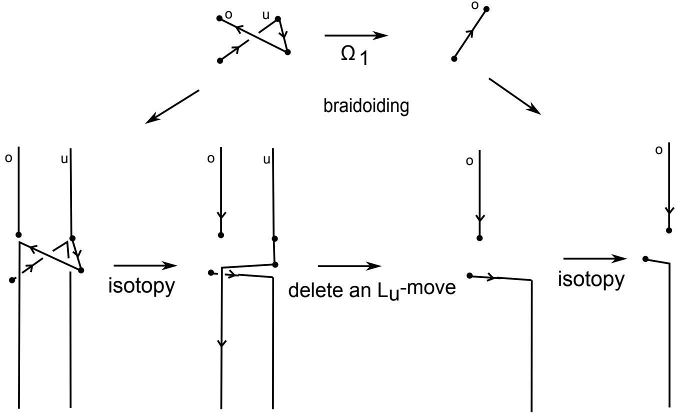


Corollary

Given any two subdivision S_1, S_2 of a knotoid diagram K with any admissible labeling, then the resulting braidoid diagrams are L-equivalent.

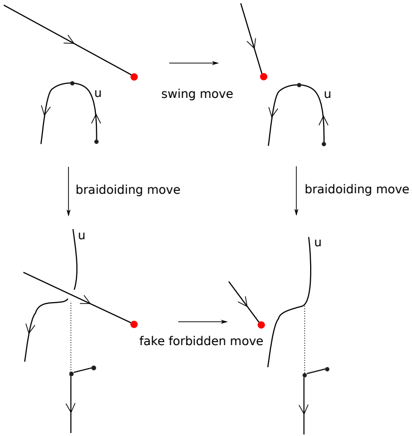
Lemma

Knotoid isotopy moves on a knotoid diagram transforms to L-equivalence moves under braidoiding moves.



Lemma

Planar isotopies displacing the endpoints result in L-equivalent braidoid diagrams.



Lemma: Fake forbidden moves are generated by L-moves and planar isotopies.

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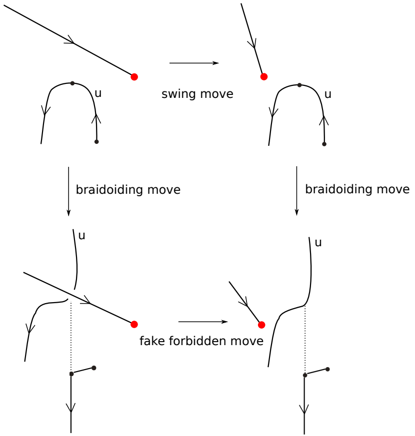
Planar isotopies displacing the endpoints result in L-equivalent braidoid diagrams.



Lemma: Fake forbidden moves are generated by L-moves and planar isotopies.

Corollary

The map br is well-defined.

It is not hard to see that

$$br \circ cl_L(B) = \tilde{B}$$

and

$$cl_L \circ br(K) = \tilde{K},$$

for $B, \tilde{B}$ are isotopic braidoid diagrams and $K, \tilde{K}$ are isotopic multi-knotoid diagrams.

This completes the proof of our theorem.

References

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Thank you for your attention!