Virtually symmetric representations and marked Gauss diagrams

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Two links L_1 and L_2 are said to be *ambient isotopic* if there is exist an ambient isotopy $H : \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ such that $H(L_1, 0) = L_1$ and $H(L_1, 1) = L_2$.

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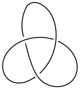

Figure: Trefoil knot diagram.

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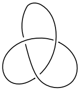

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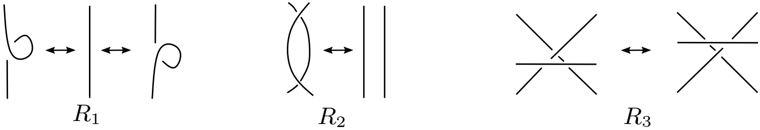
Two link diagrams D_1 and D_2 are said to be *equivalent* if they are related by a finite sequence of moves shown below, upto planar isotopy:



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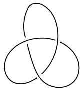

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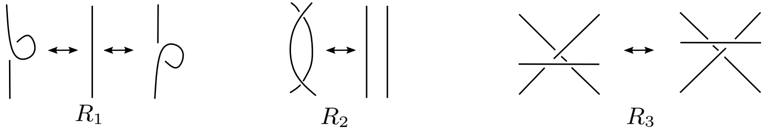

Figure: Reidemeister moves.

Theorem (K. Reidemeister)

Two links are ambient isotopic iff any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves.

Classical link group

Classical link group of link L: Fundamental group of link complement $\pi_1(\mathbb{S}^3 - L)$ and it is a link invariant.

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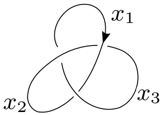

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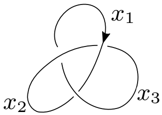

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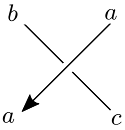

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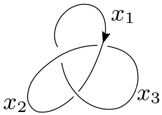

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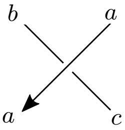

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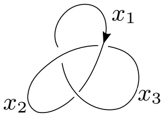

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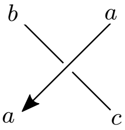

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$$\pi_1(\mathbb{S}^3 - T) = \langle x_1, x_2, x_3 \mid\mid x_2 = x_1^{x_3}, x_3 = x_2^{x_1}, x_1 = x_3^{x_2} \rangle.$$

L. H. Kauffman, *Virtual knot theory*, European J. Combin. **20** (1999), no. 7, 663–690.

A *virtual link diagram* is a generic immersion of finite disjoint oriented circles into a plane where double points are either classical crossings or decorated with a circle around it, called a *virtual crossing*.

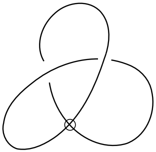

Figure: A virtual knot diagram.

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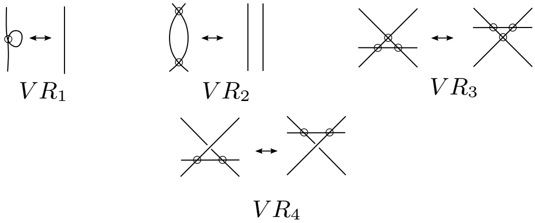

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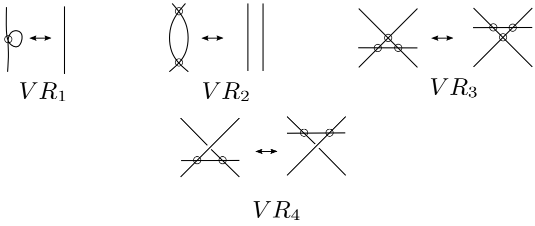

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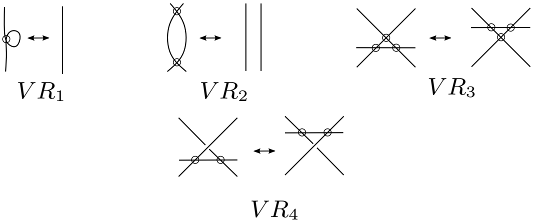

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Theorem (L. Kauffman)

Virtual links are proper generalization of classical links.

A *Gauss diagram* consists of finite number of disjoint circles oriented anticlockwise with finite number of signed arrows whose head and tail lies on circles.

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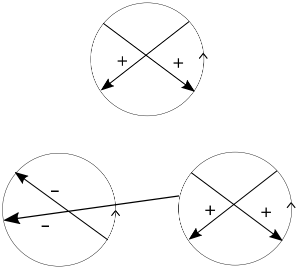

Gauss diagrams

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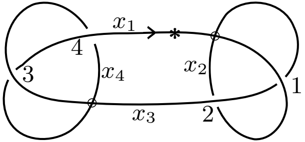

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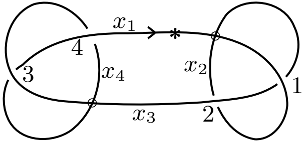

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Oriented Gauss code for K: $1\mathrm{O} - 2\mathrm{U} - 1\mathrm{U} - 2\mathrm{O} - 3\mathrm{O} - 4\mathrm{U} - 3\mathrm{U} - 4\mathrm{O} -$

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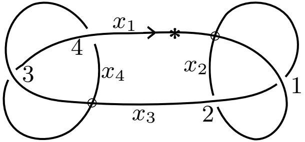

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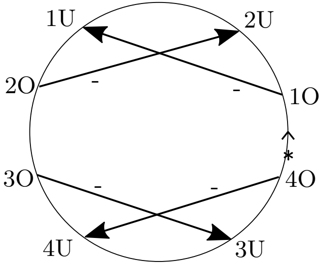

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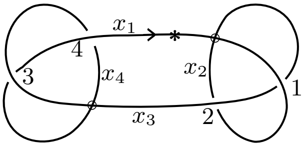

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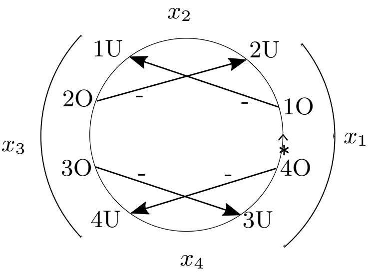

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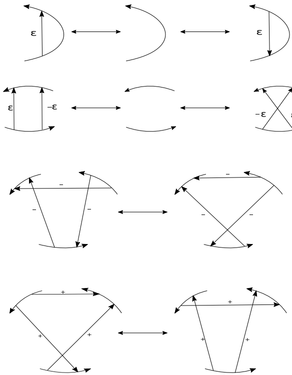

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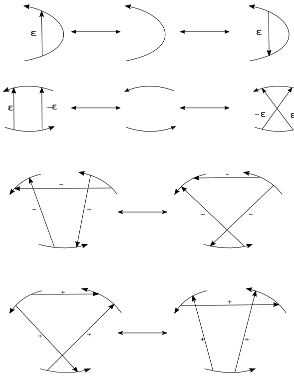

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There is one-to-one correspondence between virtual links and equivalence classes of Gauss diagrams.

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Virtual link group (L. Kauffman)

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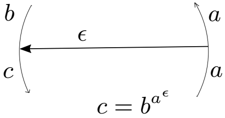
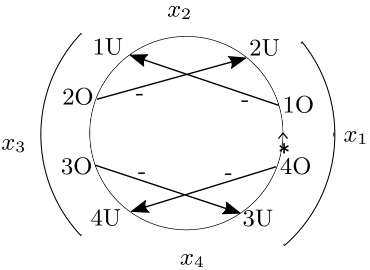
- for each arrow add a relation as shown below.



$$G_K(D) = \langle x_1, x_2, \ldots, x_n \ \ || \ \ \text{one relation for each arrow} \rangle.$$

Example



$$G_K(D) := \langle x_1, x_2, x_3, x_4 \;\; || \;\; x_2 = x_1^{x_3^{-1}}, x_3 = x_2^{x_1^{-1}}, x_4 = x_3^{x_1^{-1}}, x_1 = x_4^{x_3^{-1}} \rangle.$$

Peripheral structure for virtual links using group $G_K(D)$

Let D be a Gauss diagram and $G_K(D)$ be the group associated to it.

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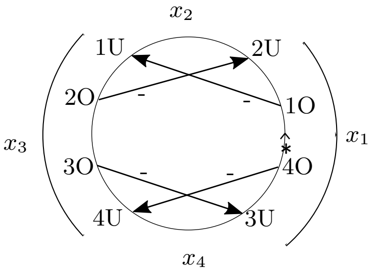
- Peripheral structure: Conjugacy class of peripheral pair.

Example



- Meridian $m = x_1$.
- Longitude $l = x_3^{-1} x_1^{-1} x_1^{-1} x_3^{-1} x_1^4$.

Vik. S. Kulikov, *Alexander polynomials of plane algebraic curves*, Russian Acad. Sci. Izv. Math. **42** (1994), no. 1, 67–89.

Vik. S. Kulikov, *A geometric realization of C-groups*, Russian Acad. Sci. Izv. Math. **45** (1995), no. 1, 197–206.

A group G is called a C-*group* if it admits a presentation $\langle X \quad || \quad \mathcal{R}\rangle$, where $X = \{x_1, x_2, \dots, x_n\}$ and relations $\mathcal{R}$ are of the type $w_{i,j}^{-1} x_i w_{i,j} = x_j$, for some $x_i, x_j \in X$ and some words $w_{i,j}$ in $X^{\pm 1}$.

For example, the fundamental group of a link complement in $\mathbb{S}^3$ is a C-group.

A C-group G is said to be irreducible if its abelianization is $\mathbb{Z}$.

For example, the fundamental group of a knot complement in $\mathbb{S}^3$ is an irreducible C-group.

S. Kim results

S. G. Kim, *Virtual knot groups and their peripheral structure*, J. Knot Theory Ramifications **9** (2000), no. 6, 797–812.

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Neuwirth Problem for virtual knots

Let G be a group and $\mu \in G$. Suppose G is finitely generated by the conjugates of μ. An element $\lambda \in G$ is said to be realizable if there exists a virtual knot K and an onto homomorphism $\rho : G_K(K) \to G$ such that $\rho(m) = \mu$ and $\rho(l) = \lambda$. Let Λ_G denotes the set of realizable elements in G. Is Λ_G a non-empty set?

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Theorem

The set Λ_G is a non-empty subgroup of G.

Theorem

$\Lambda_G = G' \cap Z(\mu)$, where G' is the commutator subgroup of G and $Z(\mu)$ is the centralizer of μ in G.

Let B_n denotes the braid group on n strands.

- Generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$,
- Relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for} \quad i \in \{1, 2, \ldots, n-2\};$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{where} \quad |i-j| \geq 2 \quad \text{for} \quad i, j \in \{1, 2, \ldots, n-1\};$$

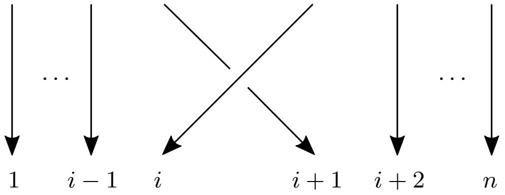

Figure: Generator σ_i.

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Artin representation: $\psi : B_n \to Aut(F_n)$ defined as

$$\psi(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i. \end{cases}$$

Artin representation is a faithful representation.

The *virtual braid group* VB_n is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \rho_1, \rho_2, \ldots, \rho_{n-1}$ with following relations:

$$
\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for} \quad i \in \{1,2,\ldots,n-2\};\\
\sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{where} \quad |i-j| \geq 2 \quad \text{for} \quad i,j \in \{1,2,\ldots,n-1\};\\
\rho_i^2 &= 1 \quad \text{for} \quad i \in \{1,2,\ldots,n-1\};\\
\rho_i\rho_j &= \rho_j\rho_i \quad \text{where} \quad |i-j| \geq 2 \quad \text{for} \quad i,j \in \{1,2,\ldots,n-1\};\\
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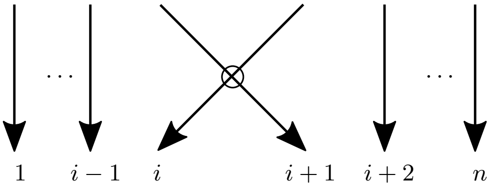

Figure: Generator ρ_i.

Notations

- $F_n := \langle x_1, x_2, \ldots, x_n \rangle$ is the free group of rank n.
- $F_{n,1} := F_n * \mathbb{Z}$, where $\mathbb{Z} = \langle v \rangle$ is the free abelian group of rank 1.
- $F_{n,n} := F_n * \mathbb{Z}^n$, where $\mathbb{Z}^n = \langle v_1, v_2, \ldots, v_n \rangle$ is the free abelian group of rank n.
- $F_{n,n+1} := F_n * \mathbb{Z}^{n+1}$, where $\mathbb{Z}^{n+1} = \langle v_1, v_2, \ldots, v_n, u \rangle$ is the free group of rank $n+1$.
- $F_{n,2} = F_n * \mathbb{Z}^2$, where $\mathbb{Z}^2 = \langle u, v \rangle$ is the free abelian group of rank 2.
- $F_{n,2n+1} := F_n * \mathbb{Z}^{2n+1}$, where $\mathbb{Z}^{2n+1} = \langle u_1, u_2, \ldots, u_n, v_0, v_1, v_2, \ldots, v_n \rangle$ is the free abelian group of rank $2n+1$.

Two representations $\psi : VB_n \to Aut(H)$ and $\tilde{\psi} : VB_n \to Aut(H)$ are said to be equivalent if there exist an automorphism $\phi : H \to H$ such that $\tilde{\psi}(\beta) = \phi^{-1} \circ \psi(\beta) \circ \psi$ for all $\beta \in VB_n$.

Examples

Generalized Artin representation: $\psi_A : VB_n \to Aut(F_{n,1})$ is defined as

$$\psi_A(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \psi_A(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v^{-1}}, \\ x_{i+1} \mapsto x_i^{v}. \end{cases}$$

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Silver-Williams representation: $\psi_{SW} : VB_n \to \mathrm{Aut}(F_{n,n+1})$ is defined as

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$$\psi_{SW}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \psi_{SW}(\rho_i) : \begin{cases} u_i \mapsto u_{i+1}, \\ u_{i+1} \mapsto u_i. \end{cases}$$

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Boden-Dies representation: $\psi : VB_n \to \mathrm{Aut}(F_{n,2})$ is defined as

$$\psi_{BD}(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-u}, \\ x_{i+1} \mapsto x_i^{u}, \end{cases} \qquad \psi_{BD}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v^{-1}}, \\ x_{i+1} \mapsto x_i^{v}. \end{cases}$$

Bardakov-Mikhalchishina-Neshchadim

The representation $\psi_M : VB_n \longrightarrow \mathrm{Aut}(F_{n,2n+1})$ is defined as:

$$\psi_M(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1}^{u_i} x_i^{-v_0 u_{i+1}}, \\ x_{i+1} \longmapsto x_i^{v_0}, \end{cases} \qquad \psi_M(\sigma_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\psi_M(\sigma_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i, \end{cases}$$

$$\psi_M(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longmapsto x_i^{v_{i+1}}, \end{cases} \qquad \psi_M(\rho_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\psi_M(\rho_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i. \end{cases}$$

This representation generalizes the previous representations.

Definition

A representation $\varphi : VB_n \to \mathrm{Aut}(H)$ of the virtual braid group VB_n into the automorphism group of some group (or module) $H = \langle h_1, h_2, \ldots, h_m \;\; || \;\; \mathcal{R} \rangle$ is called *virtually symmetric* if for any generator ρ_i, $i = 1, 2, \ldots, n-1$, its image $\varphi(\rho_i)$ is a permutation of the generators $h_1, h_2, \ldots, h_m$.

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Examples of representations equivalent to virtually symmetric representations

- Generalized Artin representation,
- Silver-William representation,
- Boden-Dies representation,
- Bardakov-Mikhalchishina-Neshchadim representation.

Equivalent to Bardakov-Mikhalchishina-Neshchadim

$$\tilde{\psi}_M : VB_n \longrightarrow \mathrm{Aut}(F_{n,2n+1})$$

$$\tilde{\psi}_M(\sigma_i) : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \end{cases} \qquad \tilde{\psi}_M(\sigma_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

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$$\tilde{\psi}_M(\rho_i) : \begin{cases} x_i \longmapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longmapsto x_i^{v_{i+1}}, \end{cases} \qquad \tilde{\psi}_M(\rho_i) : \begin{cases} v_i \longmapsto v_{i+1}, \\ v_{i+1} \longmapsto v_i, \end{cases}$$

$$\tilde{\psi}_M(\rho_i) : \begin{cases} u_i \longmapsto u_{i+1}, \\ u_{i+1} \longmapsto u_i. \end{cases}$$

The representation $\phi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ is defined by the action on the generators:

$$\phi_M(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \phi_M(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\phi_M(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto x_i^{v_{i+1}}, \end{cases} \qquad \phi_M(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

Virtually symmetric representation equivalent to the previous representation

A virtually symmetric representation

The representation $\varphi_M : VB_n \to Aut(F_{n,n})$ is defined as below is equivalent to the representation $\phi_M : VB_n \to Aut(F_{n,n})$.

$$\varphi_M(\sigma_i) : \begin{cases} x_i \mapsto x_i \;\; x_{i+1}^{v_i} \;\; x_i^{-1}, \\ x_{i+1} \mapsto x_i^{v_{i+1}^{-1}}, \end{cases} \qquad \varphi_M(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\varphi_M(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \varphi_M(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

We will use this representation to define *virtual link groups* and will show the advantage of it over the previous representation which is not a virtually symmetric representation.

Virtual link group

The braid approach

Let $\beta \in VB_n$. Define groups

$$
\begin{aligned}
G_M(\beta) &:= \langle x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_n \;\mid\; [v_i, v_j] = 1, \\
&\qquad x_i = \varphi_M(\beta)(x_i), v_i = \varphi_M(\beta)(v_i), \quad \text{where} \;\; 1 \le i, j \le n \rangle. \\
G(\beta) &:= \langle x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_n \;\mid\; [v_i, v_j] = 1, \\
&\qquad x_i = \phi_M(\beta)(x_i), v_i = \phi_M(\beta)(v_i), \quad \text{where} \;\; 1 \le i, j \le n \rangle. \\
G_M(\beta) &\cong G(\beta).
\end{aligned}
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$$G_M(\beta) \cong G(\beta).$$

Theorem

If $\beta \in VB_n$ and $\beta' \in VB_m$ are two virtual braids such that their closure define the same link L, then $G_M(\beta) \cong G_M(\beta')$, i.e, the group $G_M(\beta)$ is a link invariant.

The diagram approach

Let $D(L)$ be a virtual link diagram with m-components.

- Enumerate all components with integers from 1 to m.
- Label each arc from one classical crossing to another classical crossing with labels $x_1, x_2, \dots, x_n$.
- Define the *virtual link group* $G_M(D(L))$ as

$$\langle x_1, x_2 \dots, x_n, v_1, v_2, \dots, v_m \mid \mathcal{R}, [v_i, v_j] \ \text{ where } 1 \leq i, j \leq m \rangle,$$

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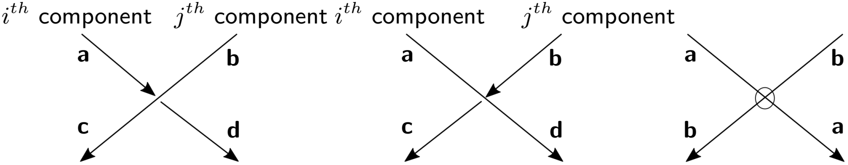

Positive

$$c = b^{v_i},$$
$$d = b^{-v_i v_j^{-1}} a^{v_j^{-1}} b^{v_i v_j^{-1}}.$$

Negative

$$c = a b^{v_i} a^{-1},$$
$$d = a^{v_j^{-1}}.$$

Virtual

Theorem

If $D(L)$ *and* $D(L')$ *are two diagrams of a virtual link* L, *then groups* $G_M(D(L))$ *and* $G_M(D(L'))$ *are isomorphic. Hence* $G_M(D(L))$ *is an invariant of* L.

Theorem

Let L *be a virtual link,* $D(L)$ *be its diagram,* β *be a braid such that its closure* $\hat{\beta}$ *is equivalent to* L, *then* $G_M(D(L)) \cong G_M(\beta)$. *Denote* $G_M(L) := G_M(D(L))$.

The Gauss diagram approach

Let D be a Gauss diagram representing the virtual link L having m-components. Then enumerate the circles in D with integers from 1 to m. Label the arc of circles as $x_1, x_2, \dots, x_n$, after cutting the circles at each extreme points of arrows. Define the group

$$\pi_D := \langle x_1, x_2, \dots, x_n, v_1, v_2, \dots, v_m \mid [v_i, v_j], \mathcal{R} \rangle.$$

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In figure below $a, d \in$ set of arcs in i^{th}-circle and $b, c \in$ set of arc in j^{th}-circle.

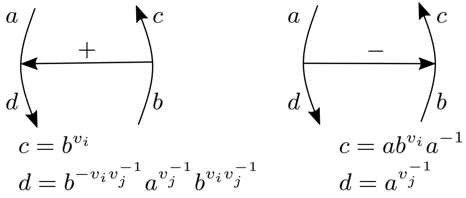


Figure: Relations at crossings.

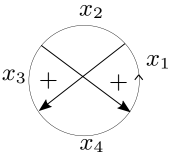
Theorem

Let D be a Gauss diagram representing the link L. Then $G_M(L) \cong \pi_D$.

Remark

If in the group π_D, we put $v_1 = v_2 = \cdots = v_m = 1$, then we get the group $G_K(D)$.

Example



- $$\pi_D = \langle x_1, x_2, x_3, x_4, v \;\; || \;\; x_2 = x_1^v, x_4 = x_1^{-1} x_3^{v^{-1}} x_1, \\ x_3 = x_2^v, x_1 = x_2^{-1} x_4^{v^{-1}} x_2 \rangle.$$

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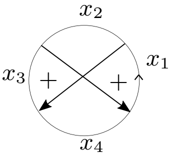

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- $$G_K(D) = \mathbb{Z}.$$

Definition

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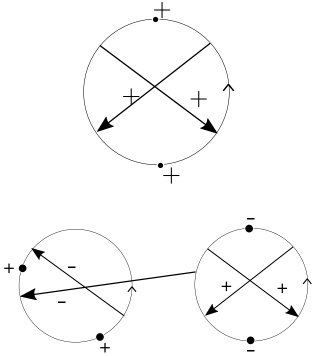

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Two marked Gauss diagrams are said to be equivalent if one can be transformed to another by a finite sequence of *marked Reidemeister moves*.

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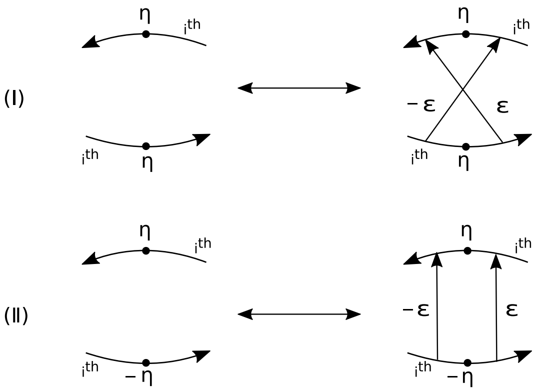

Group of a marked Gauss diagram

Let D be a marked Gauss diagram with m-components. Enumerate the circles in D with integers from 1 to m. Label the arc of circles as $x_1, x_2, \ldots .x_n$ after cutting the circles at nodes and at each each extreme points of arrows. Define the group

$$\Pi_D := \langle x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_m \mid [v_i, v_j], \mathcal{R} \rangle,$$

where $\mathcal{R}$ is the set of relations corresponding to arrows and nodes shown below.

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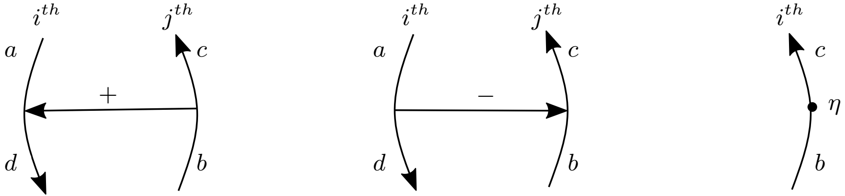

$$c = b^{v_i}$$
$$d = b^{-v_i v_j^{-1}} a^{v_j^{-1}} b^{v_i v_j^{-1}}$$

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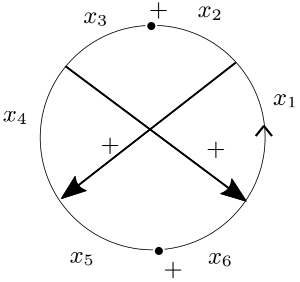

$$\Pi_D = \langle x_1, x_2, x_3, x_4, x_5, x_6, v \;\; || \;\; x_2 = x_1^v, x_5 = x_1^{-1} x_4^{v^{-1}} x_1, x_3 = x_2^v,$$
$$x_4 = x_3^v, x_1 = x_3^{-1} x_6^{v^{-1}} x_3, x_6 = x_5^v \rangle$$
$$\ncong F_2, \text{ thus } D \text{ is a non-trivial marked Gauss diagram.}$$

Let us suppose we are on the k^{th} circle.

- Meridian: Take generator corresponding to any of the arcs in the k^{th}-circle of a given marked Gauss diagram, say x.

Peripheral structure for marked Gauss diagrams

Let us suppose we are on the k^{th} circle.

- Meridian: Take generator corresponding to any of the arcs in the k^{th}-circle of a given marked Gauss diagram, say x.
- Longitude: Start moving from the meridian arc along the circle and write v_t^ϵ when pass the tail of an arrow with sign ϵ and whose head lies on the t^{th}-circle, and when we pass the head of an arrow whose tail is on the n^{th}-circle and is the end point of arc x_i, we use the following rule:
  - if arrow sign is $+1$, write $v_n^{-1} x_i^{v_k v_n^{-1}}$,
  - if arrow sign is -1, write $v_n x_i^{-1}$.

  And when we pass node with sign ϵ, we write v_k^ϵ. On coming back to the meridian arc x, we write $x^{-\alpha}$, where α is the sum of sign of arrows whose head lies on the k^{th}-circle.

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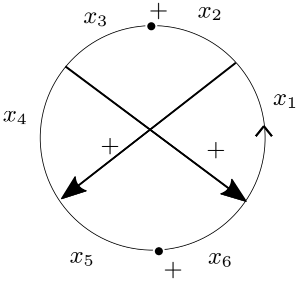
- Peripheral structure: Conjugacy class of peripheral pair.

Example



- Meridian $m = x_1$.
- Longitude $l = vvvv^{-1}x_1vv^{-1}x_3x_1^{-2} = v^2x_1x_3x_1^{-2}$.

Definition

Let m be a non-negative integer. A group G is called a C_m-*group* if it can be defined by a set of generators $Y = X \cup V_m$, where $X = \{x_1, x_2, \dots, x_n\}$, $V_m = \{v_1, v_2, \dots, v_m\}$ and a set of relations $\mathcal{R}$:

$$w_{i,j}^{-1} x_i w_{i,j} = x_j, \quad \text{for some } x_i, x_j \in X \text{ and some words } w_{i,j} \text{ in } Y^{\pm 1};$$
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Definition

A C_m group G, where $m \geq 1$, is said to *irreducible* if its abelianization of is of rank $2m$.

For example, if D is a marked Gauss diagram with m-components, then Π_D is an irreducible C_m-group.

Results

Theorem (Bardakov-Neshchadim-Singh)

Any irreducible C_1-group can be realized as the group associated to a marked Gauss diagrams.

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Neuwirth Problem for marked Gauss diagrams

Let G be a group and $\mu, \nu \in G$. Suppose G is finitely generated by ν and conjugates of μ. An element $\lambda \in G$ is said to be *realizable* if there exists a 1-circle marked Gauss diagram D and an onto homomorphism $\rho : \Pi_D \to G$ such that $\rho(m) = \mu$, $\rho(v) = \nu$ and $\rho(l) = \lambda$. Let Λ_G denotes the set of realizable elements in G. Is Λ_G a non-empty set?

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Let G be a group and $\mu, \nu \in G$. Suppose G is finitely generated by ν and conjugates of μ. An element $\lambda \in G$ is said to be *realizable* if there exists a 1-circle marked Gauss diagram D and an onto homomorphism $\rho : \Pi_D \to G$ such that $\rho(m) = \mu$, $\rho(v) = \nu$ and $\rho(l) = \lambda$. Let Λ_G denotes the set of realizable elements in G. Is Λ_G a non-empty set?

Theorem (Bardakov-Neshchadim-Singh)

The set Λ_G is a non-empty subgroup of G.

Thank you for the attention!