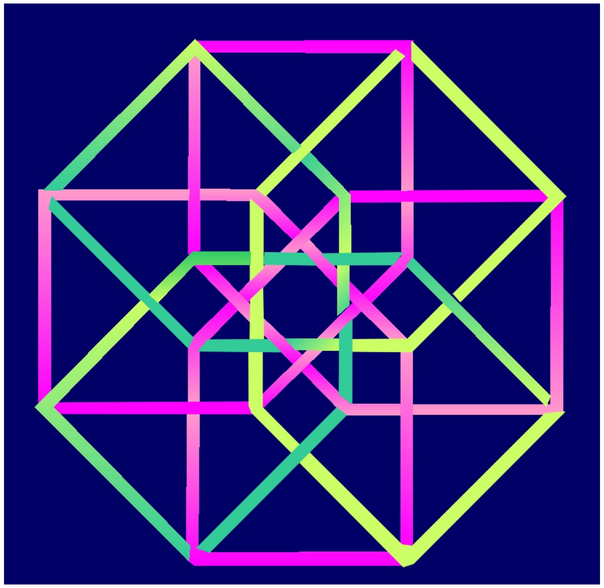
Majorana Fermions, Braiding and The Dirac Equation

Louis H. Kauffman (kauffman@uic.edu) UIC and NSU

(and containing joint work with Peter Rowlands
on a nilpotent Majorana-Dirac Equation)

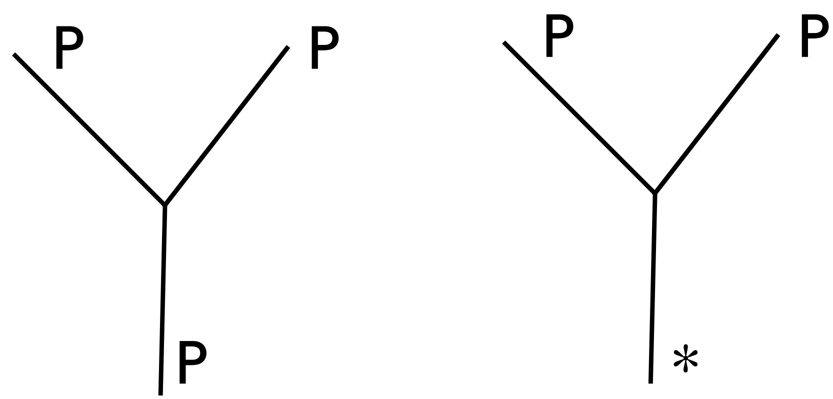


A possible sighting of Majorana states

Nearly 80 years ago, the Italian physicist Ettore Majorana proposed the existence of an unusual type of particle that is its own antiparticle, the so-called Majorana fermion. The search for a free Majorana fermion has so far been unsuccessful, but bound Majorana-like collective excitations may exist in certain exotic superconductors. Nadj-Perge *et al*. created such a topological superconductor by depositing iron atoms onto the surface of superconducting lead, forming atomic chains (see the Perspective by Lee). They then used a scanning tunneling microscope to observe enhanced conductance at the ends of these chains at zero energy, where theory predicts Majorana states should appear.

A Very Elementary Particle -

Fusion Rules for a Majorana Fermion



The "particle" P interacts with P
to produce either P or *.
* is neutral.

For a Standard Fermion there is a
an annihilation operator F
and a creation operator F*.

These correspond to the fact that
the antiparticle is distinct from the
particle.

We have FF = F*F* =0 (Pauli Exclusion)
and
FF* + F*F = 1.
Later in the talk we will see
much more about this relation.

Majorana and Clifford Algebra

The Creation/Annihilation algebra for a Majorana Fermion is very simple.
Just an element a with aa =1.
If there are two Majorana Fermions, we have
a,b
with aa = 1, bb=1 and
ab +ba = 0.

A Clifford Algebra.

Algebraic Justification of this Statement Follows...

Majorana Fermions are their own antiparticles.

An Electron's creation and annihilation operators are combinations of Majorana Fermion operators:

$$U = (a + ib)/2 \quad \text{and} \quad U^* = (a - ib)/2$$

where $ab+ba = 0$ and $aa = bb = 1$.

$$U = (a + ib)/2 \quad \text{and} \quad U^* = (a - ib)/2$$

$$4UU = (a+ib)(a+ib)$$
$$= aa - bb + i(ab + ba)$$

$$UU = 0$$
$$U^*U^* = 0.$$

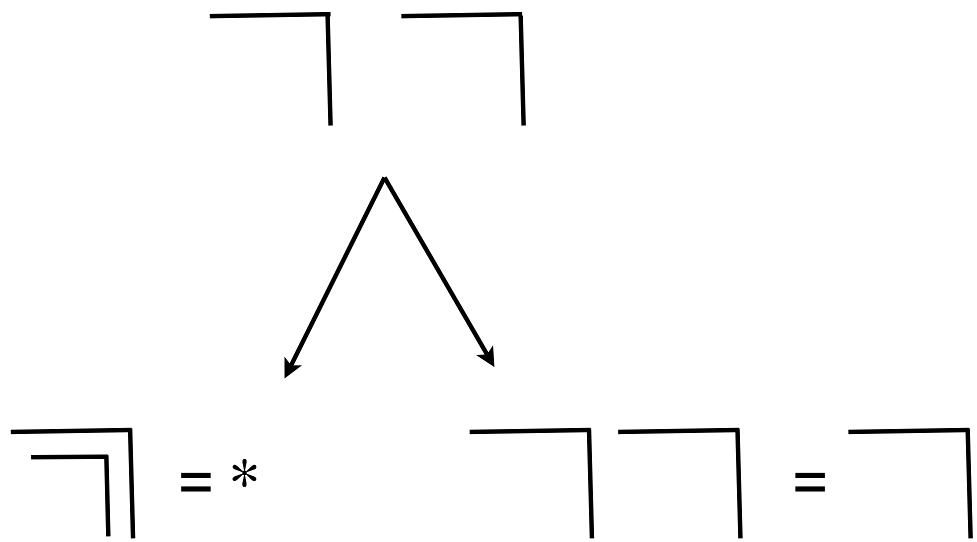
$$UU^* + U^*U = (U + U^*)(U+U^*) = aa = 1$$

This is the creation/annihilation algebra for an electron.

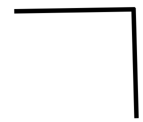
Iconics

This next part is motivated by G. Spencer-Brown's invention of a 'logical particle' that interacts with itself to either confirm itself or to cancel itself. This interaction, combined with recursion, leads both to matrix algebra and the very elementary mathematics of a Majorana Fermion.

The Mark is a logical particle
(Laws of Form by G. Spencer-Brown)
that interacts with itself either to annihilate itself, or to produce itself.



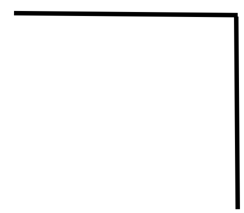
The Mark is a "logical particle" for a level of logic deeper than Boolean Logic.



In this formalism the mark is seen
to make a distinction in the plane.
The formal language of the
calculus of indications refers to the mark and is
built from the mark.

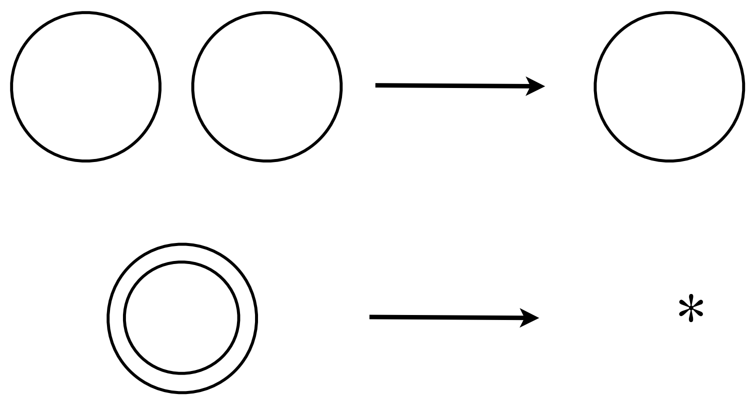
The language using the mark is inherently self-referential.

The Calculus writes itself in terms of
itself.

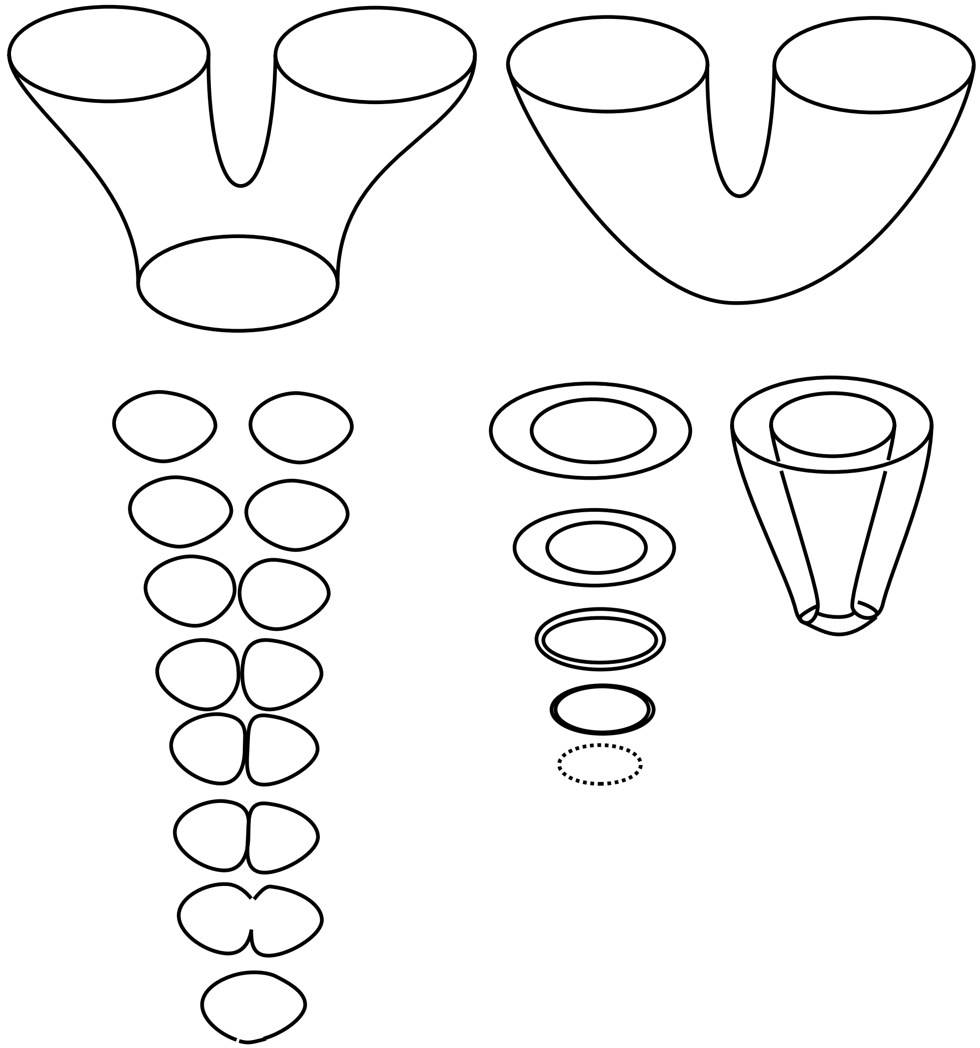


The first distinction, the mark, and the observer are not only interchangeable, but, in the form, identical.

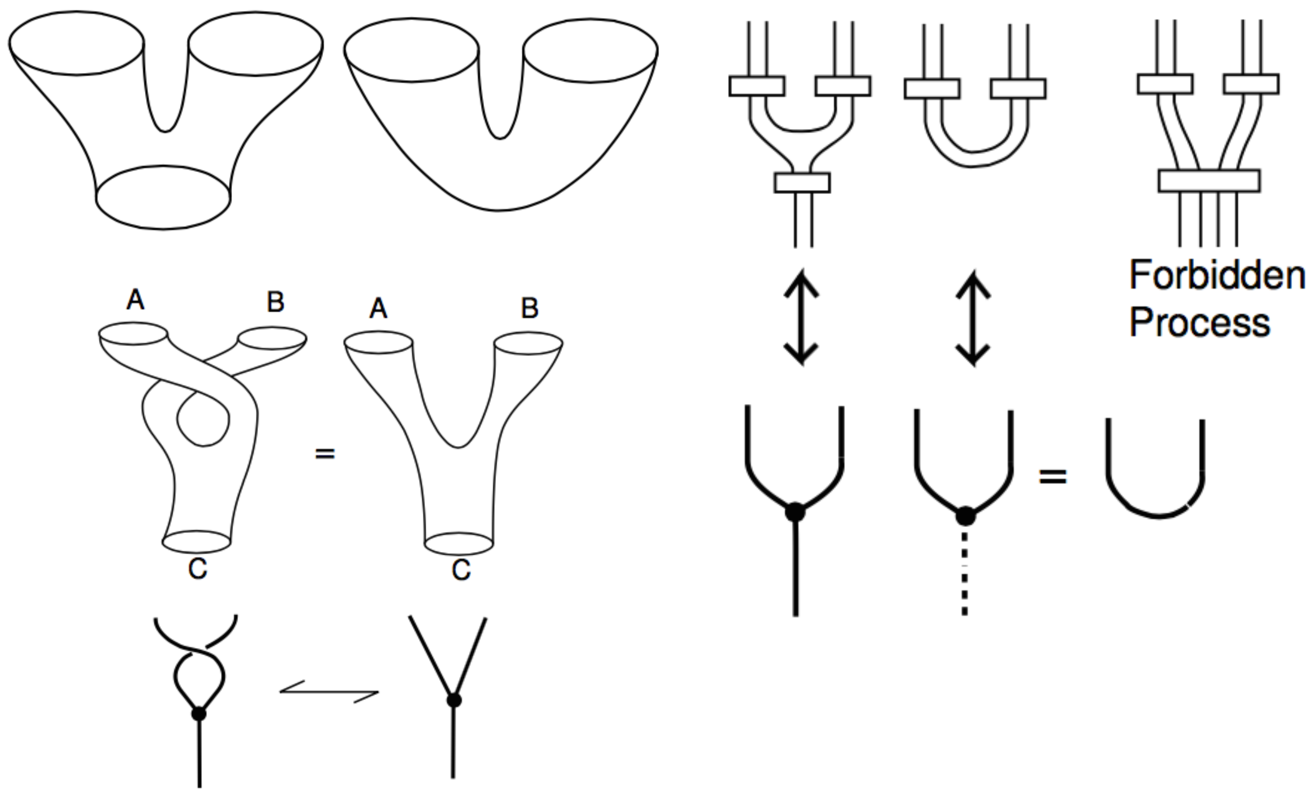
Formally, we can distinguish the two interactions via adjacency and concentricity.



PP = * + P

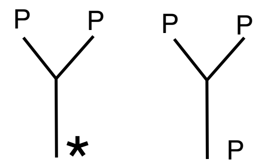


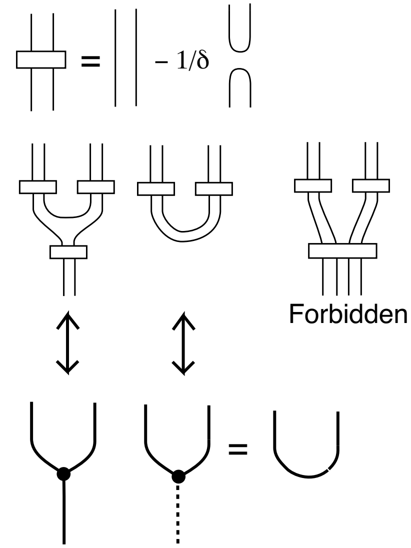
Flattening String Iconics



Fibonacci Model

$$A = e^{3\pi i/5}.$$





$$\delta = -A^2 - A^{-2}$$

$$\Delta = \delta = (1 + \sqrt{5})/2.$$

$$F = \begin{pmatrix} 1/\Delta & 1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & -1/\Delta \end{pmatrix} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

$$R = \begin{pmatrix} -A^4 & 0 \\ 0 & A^8 \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}.$$

Braid Representations Dense in Unitary Groups

Temperley Lieb Representation of Fibonacci Model

The Majorana P is a Fibonacci Particle

P^2 = * + P

P^3 = P* + PP = P + * + P = * + 2P

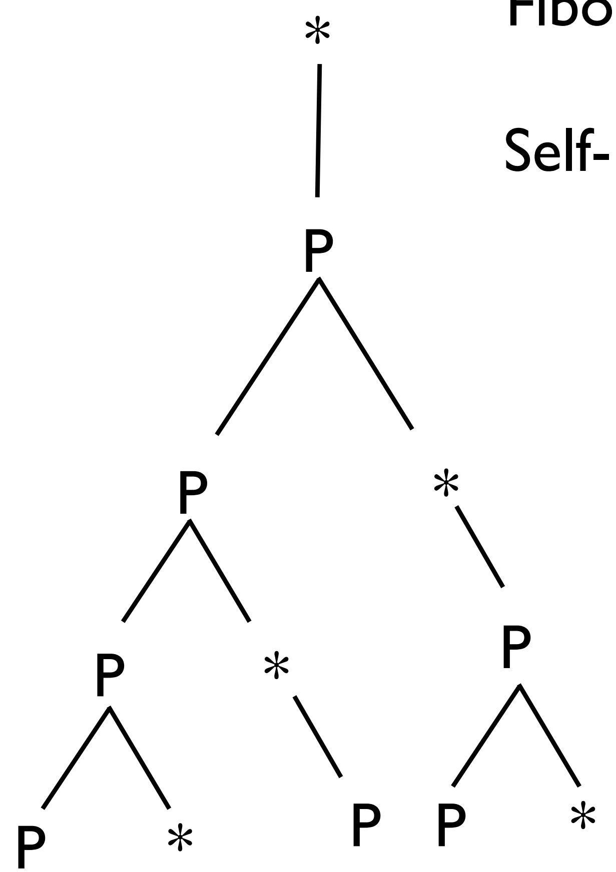
P^4 = P + 2* + 2P = 2* + 3P

P^5 = 3* + 5P

P^6 = 5* + 3P

P^7 = 8* + 5P

Fibonacci Processes from Self-Interaction of P.



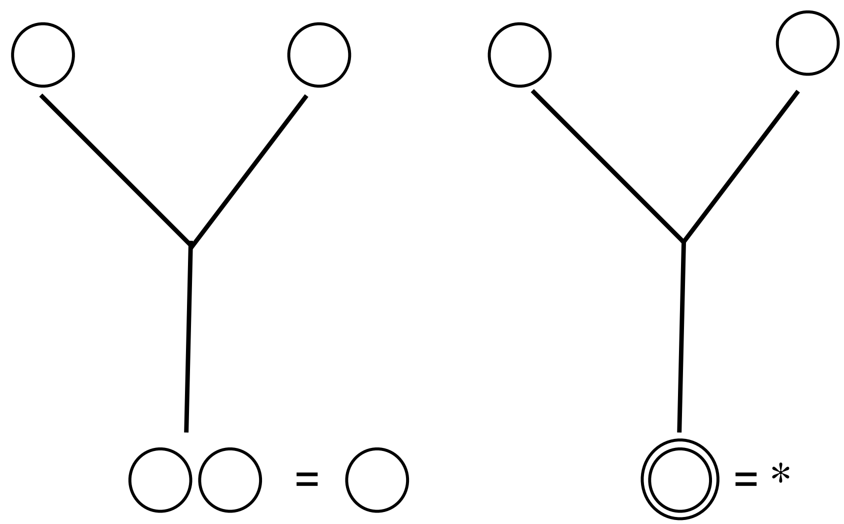
P* = P

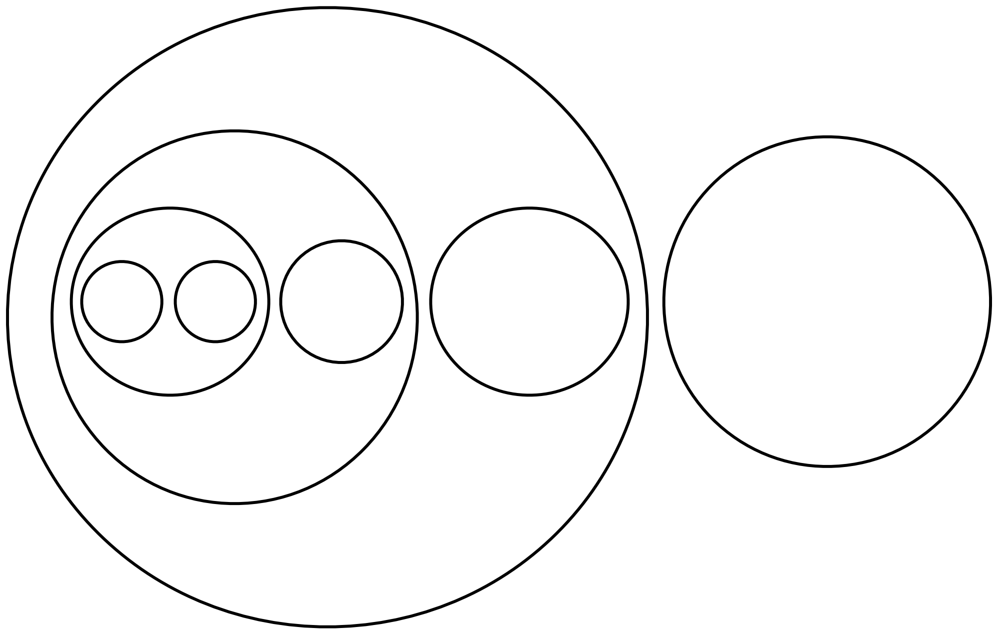
PP = P + *

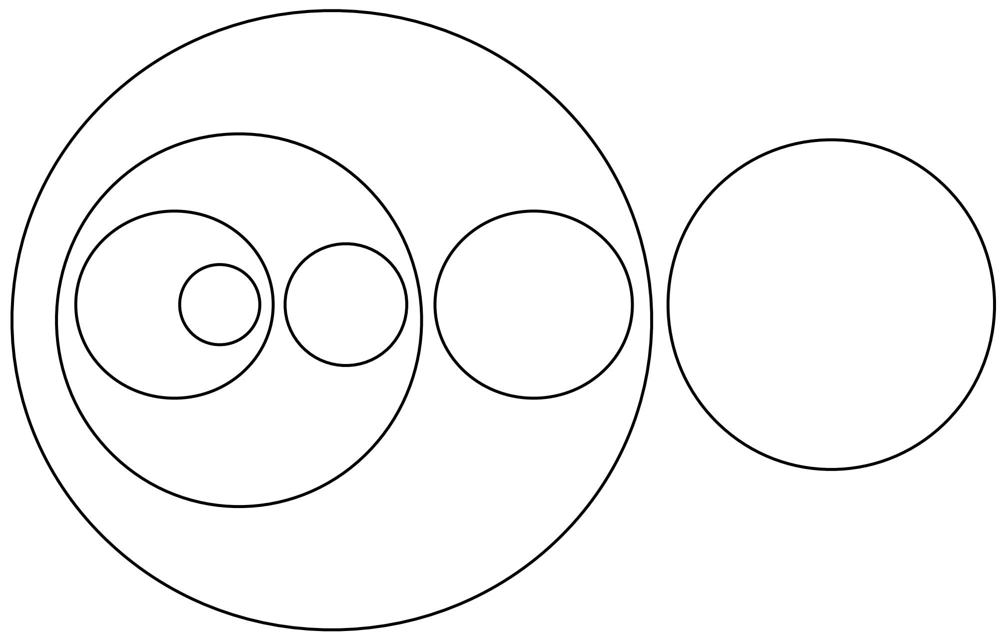
PPP= 2P + *

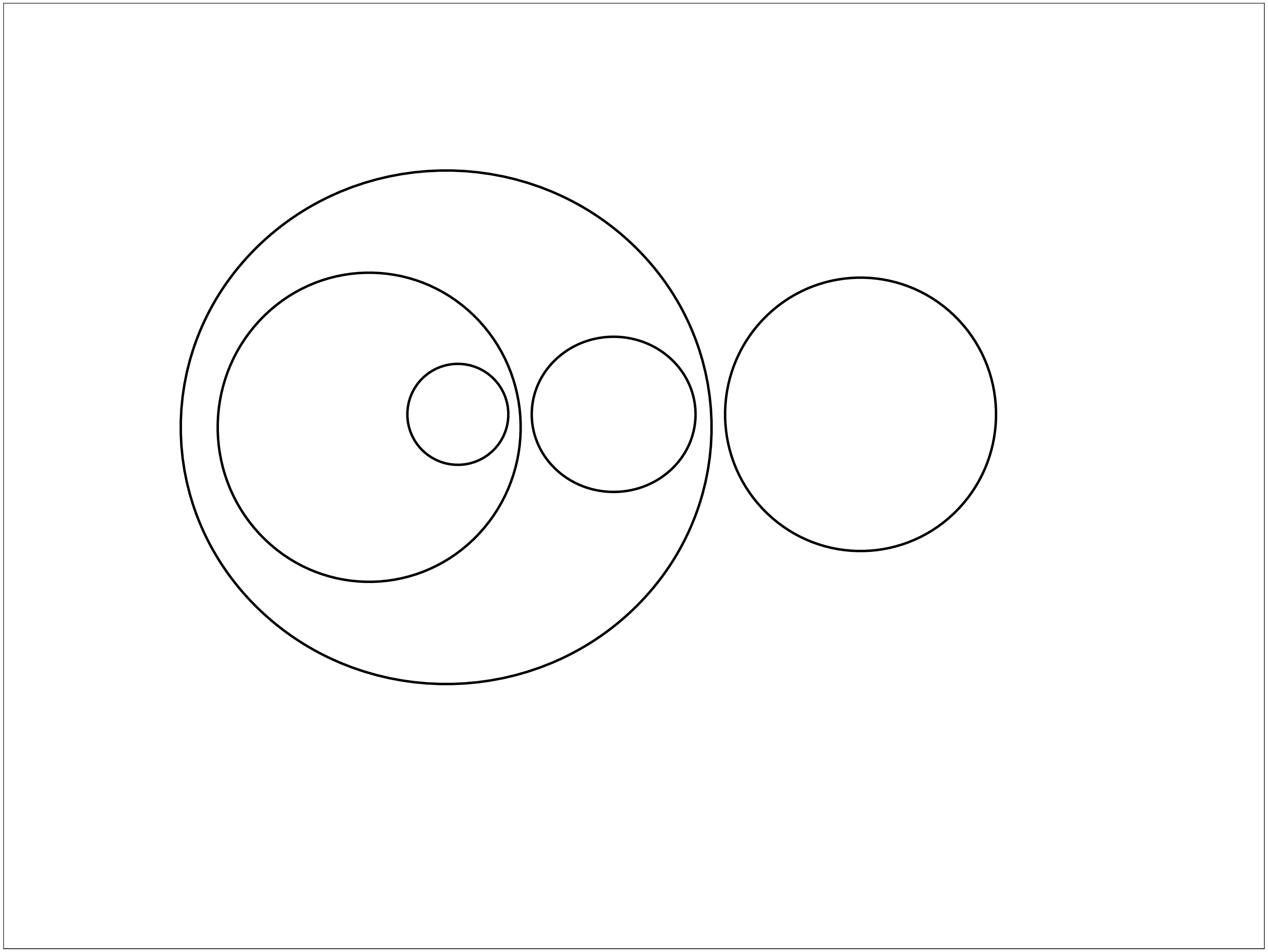
PPPP = 3P + 2*

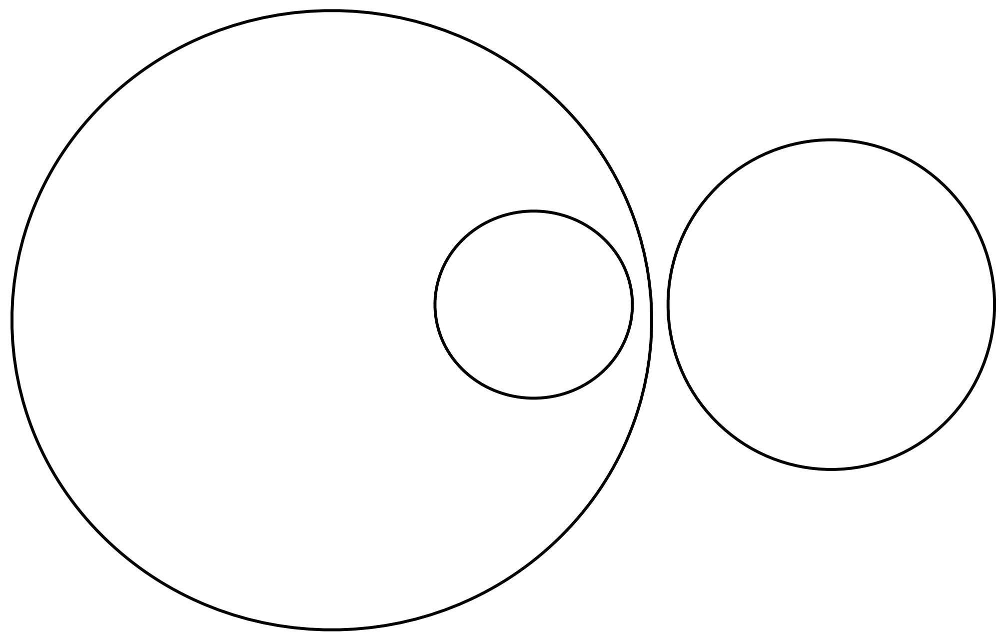
Note that we have shown how the formalism of the mark, as logical particle is coherent with its interpretation as a Majorana Fermion.

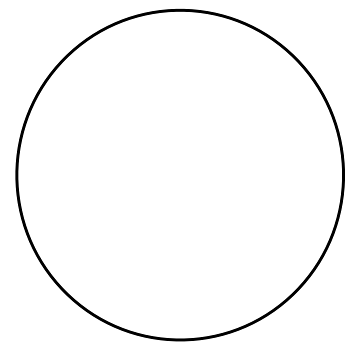


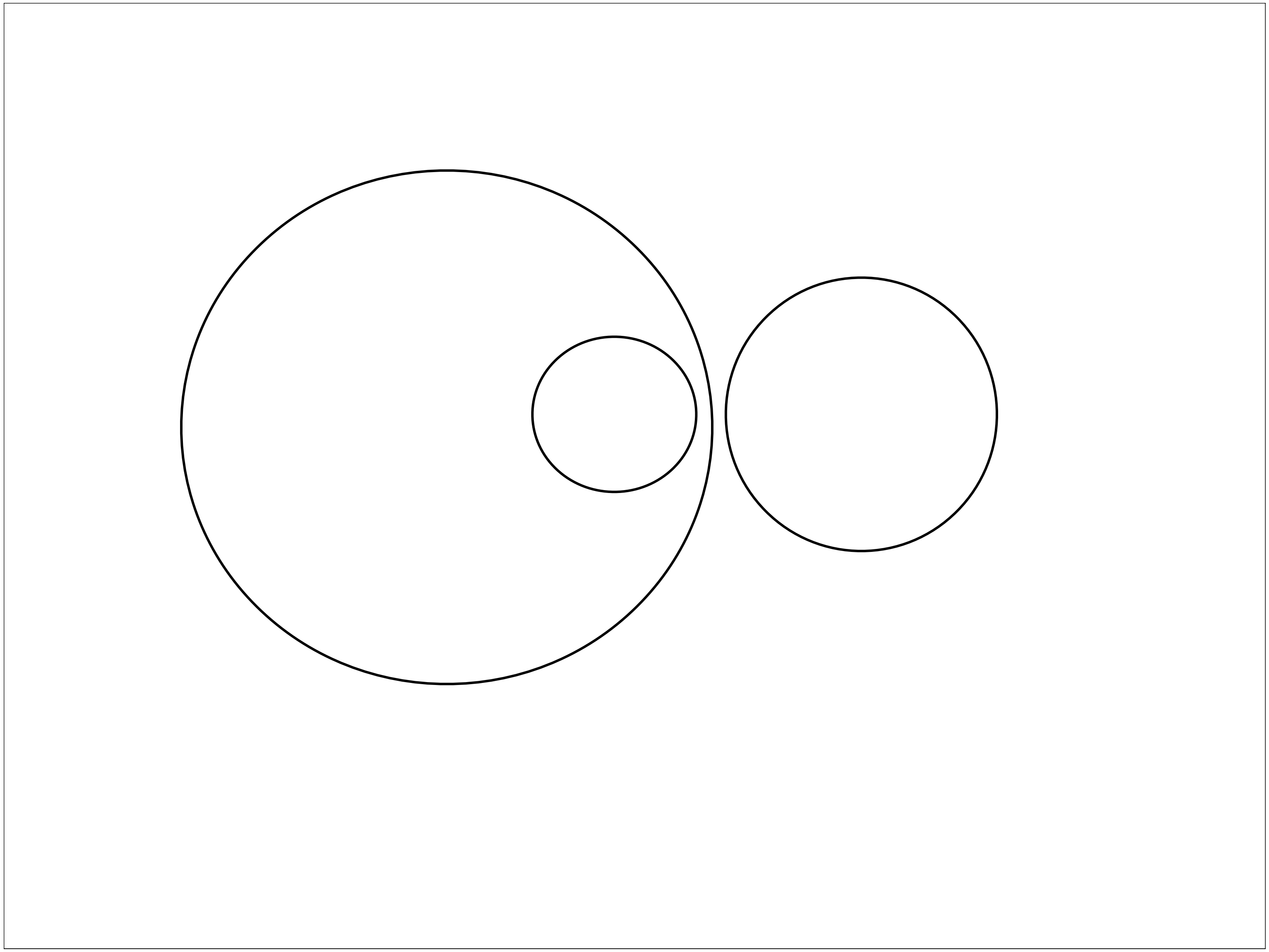


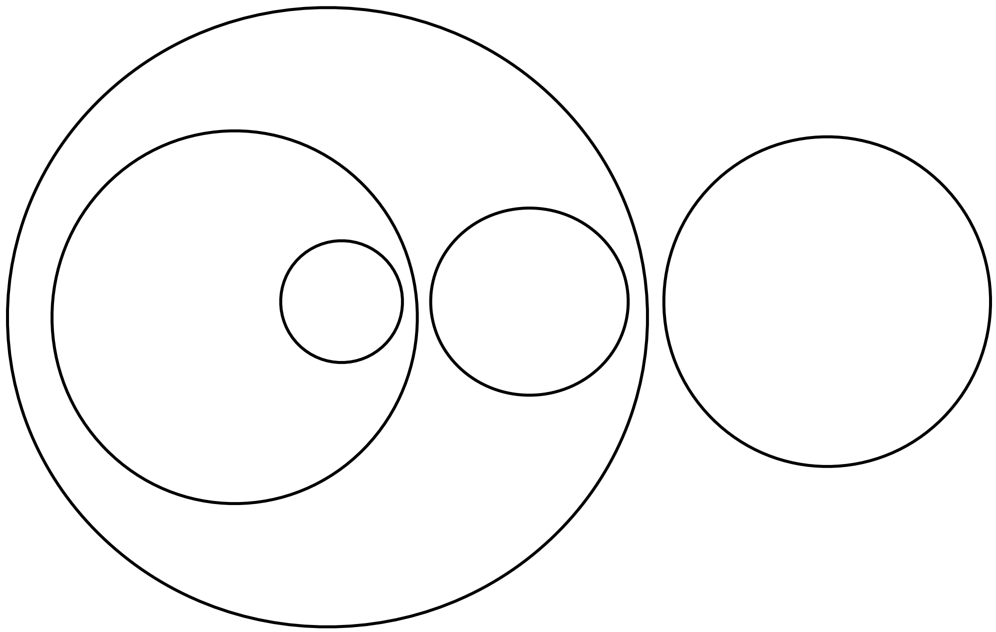


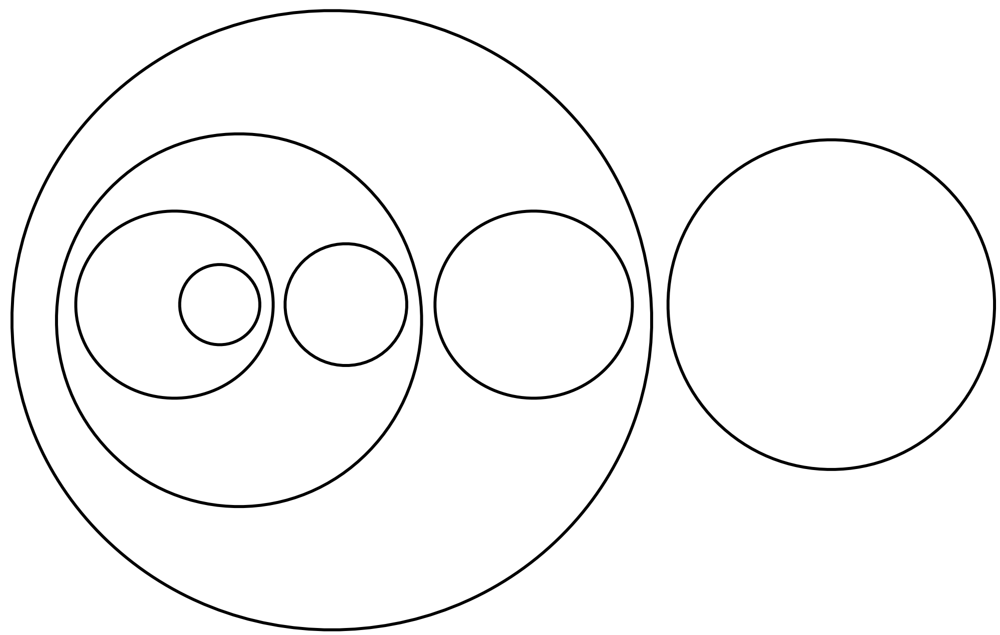


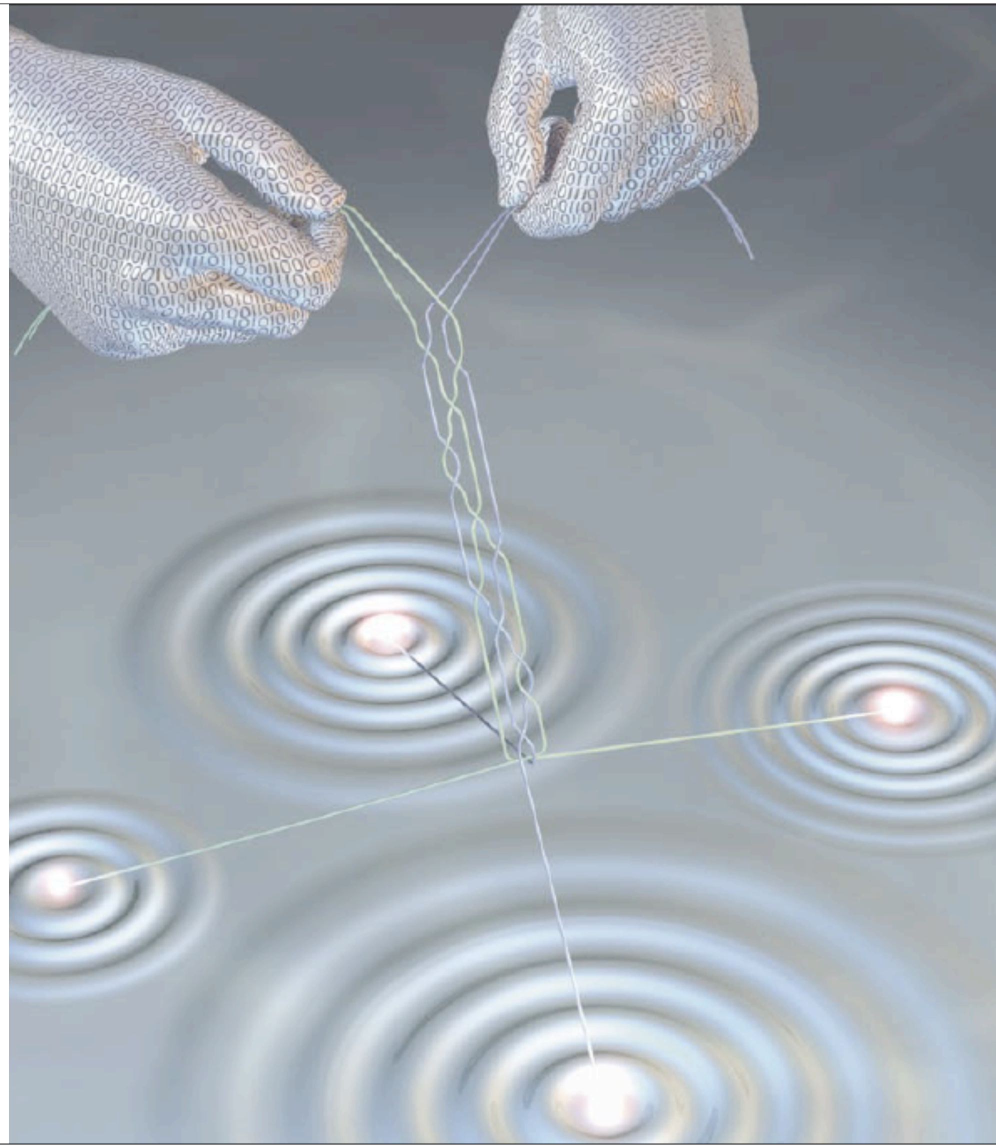












Fermion Algebra Directly From Re-entering Mark

$$[a,b][c,d] = [ac,bd]$$

$$\cdots 01010101010101 \cdots$$

$$[0,1] \quad [1,0]$$

$$p \quad q$$

$$\eta[0,1]\eta = [1,0]$$

$$[0,1]\eta = \eta[1,0]$$

$$\eta^2 = 1$$

$$p^2 = p \quad pq = 0 \quad p + q = 1 \quad q^2 = q$$

$$u = p\eta, \; u^{+} = q\eta$$

$$u^2 = p\eta p\eta = pq = 0$$

$$u^{+2} = q\eta q\eta = qp = 0$$

$$u + u^{+} = \eta \Rightarrow uu^{+} + u^{+}u = 1$$

Fermion algebra arises <u>directly</u> from the re-entering mark.

We introduce a *temporal shift operator* η such that

$$[a, b]\eta = \eta[b, a]$$

and

$$\eta\eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\cdots ababab \cdots.$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define i by the equation

$$i = [1, -1]\eta.$$

This makes i both a value and an operator that takes into account a step in time.

We calculate

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

$$e = [1, -1].$$

$$e^2 = [1, 1] = 1$$

$$e\eta = [1, -1]\eta = [-1, 1]\eta = -e\eta.$$

$$e^2 = 1,$$

$$\eta^2 = 1,$$

$$e\eta = -\eta e.$$

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

At this point we see that it is not just the square root of minus one that has emerged from the structure of the oscillation, but the simple non-commutative algebra of the split quaternions.

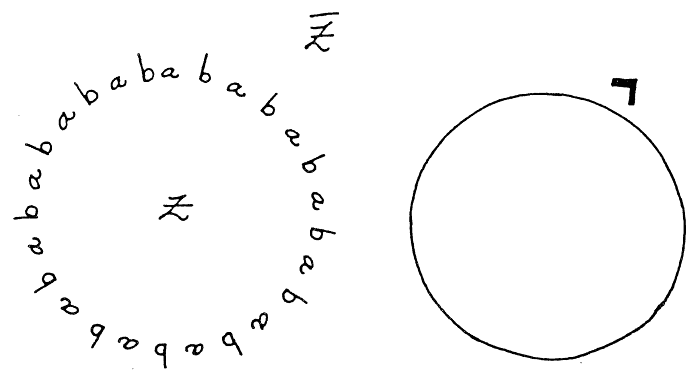
In fact, we have uncovered matrix algebra.

But note that we have shown that the Fermion algebra is primordial, related just to the oscillation of a distinction. The Clifford algebra occurs at the Arithmetic level with its extended polarity of -1, 0 , +1.

Note also that -0 = 0 so that at the arithmetic level, 0 becomes re-entrant.

SpaceTime

Temporality is the tempo of pattern.



$$(T, X, Y, Z) \leftrightarrow \begin{bmatrix} T + iSX \\ Y + iZ \end{bmatrix} \in \mathbb{D} \times \mathbb{C}$$

$$ss = 1$$
$$ii = -1$$

$$\updownarrow$$

$$\begin{bmatrix} T + X & Y + iZ \\ Y - iZ & T - X \end{bmatrix} = \begin{bmatrix} a & \bar{z} \\ z & b \end{bmatrix} = E \, .$$

At this stage we have shown how the Clifford algebra generated by e and eta (the split quaternions) emerges naturally from the discrete dynamics of the square root of minus one. Dynamics of the reentering mark.

The mark embodies the fusion algebra for a Majorana Fermion. It can interact with itself to either produce itself or annihilate itself.

A row of n electrons can be regarded as a row of 2n Majorana Fermions.

Recent work suggests that Majorana Fermions can be detected in nanowires.

Unpaired Majorana fermions in quantum wires

Alexei Yu. Kitaev*

Microsoft Research

Microsoft, #113/2032, One Microsoft Way,

Redmond, WA 98052, U.S.A.

kitaev@microsoft.com

27 October 2000

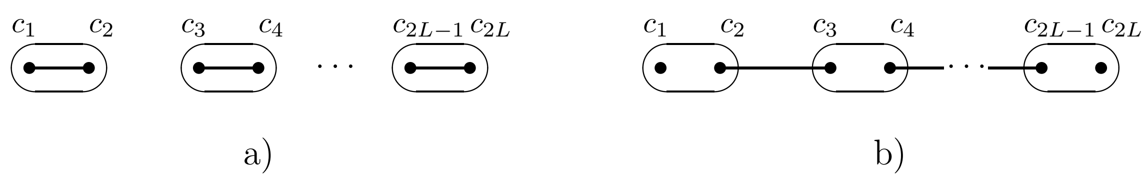


Figure 2: Two types of pairing.

Note that the state $|\psi_0\rangle$ has an even fermionic parity (i. e. it is a superposition of states with even number of electrons) while $|\psi_1\rangle$ has an odd parity. The parity is measured by the operator

$$P = \prod_j (-ic_{2j-1}c_{2j}). \tag{9}$$

Non-abelian statistics of half-quantum vortices in p-wave superconductors

D. A. Ivanov
Institut für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
(May 11, 2000)

Excitation spectrum of a half-quantum vortex in a p-wave superconductor contains a zero-energy Majorana fermion. This results in a degeneracy of the ground state of the system of several vortices. From the properties of the solutions to Bogoliubov-de-Gennes equations in the vortex core we derive the non-abelian statistics of vortices identical to that for the Moore-Read (Pfaffian) quantum Hall state.

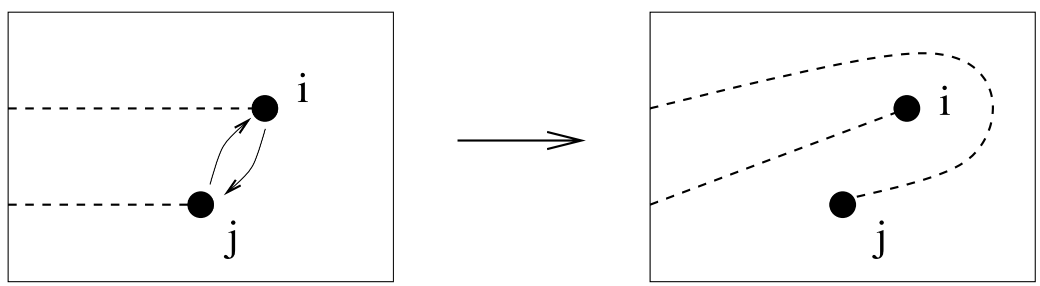


FIG. 3. Elementary braid interchange of two vortices.

On the Dynamical detection of Majorana fermions in current-biased nanowires

Fernando Domínguez,[1] Fabian Hassler,[2] and Gloria Platero[1]

[1] *Instituto de Ciencia de Materiales, CSIC, Cantoblanco, E-28049 Madrid, Spain*
[2] *Institute for Quantum Information, RWTH Aachen University, 52056 Aachen, Germany*
(Dated: October 29, 2012)

We analyze the current-biased Shapiro experiment in a Josephson junction formed by two one-dimensional nanowires featuring Majorana fermions. Ideally, these junctions are predicted to have an unconventional 4π-periodic Josephson effect and thus only Shapiro steps at even multiples of the driving frequency. Taking additionally into account overlap between the Majorana fermions, due to the finite length of the wire, renders the Josephson junction conventional for any dc-experiments. We show that probing the current-phase relation in a current biased setup dynamically decouples the Majorana fermions. We find that besides the even integer Shapiro steps there are additional steps at odd and fractional values. However, different from the voltage biased case, the even steps dominate for a wide range of parameters even in the case of multiple modes thus giving a clear experimental signature of the presence of Majorana fermions.

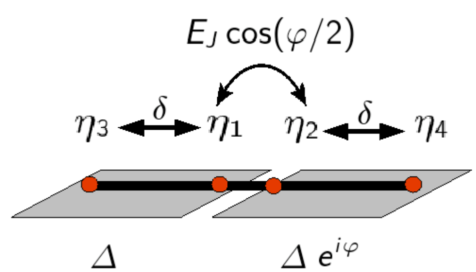


FIG. 1: Josephson junction with a nanowire on top. Red spots (color online) represent Majorana fermions. Double arrows represent the overlap between the Majorana fermions.

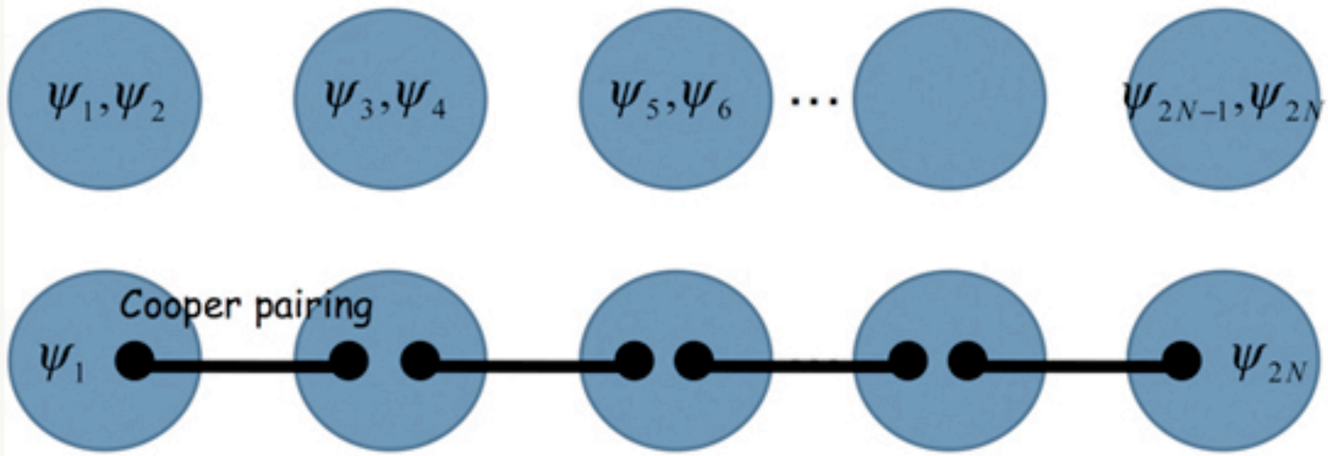
Majorana (real) Fermions

f^+, f Usual (complex) fermions

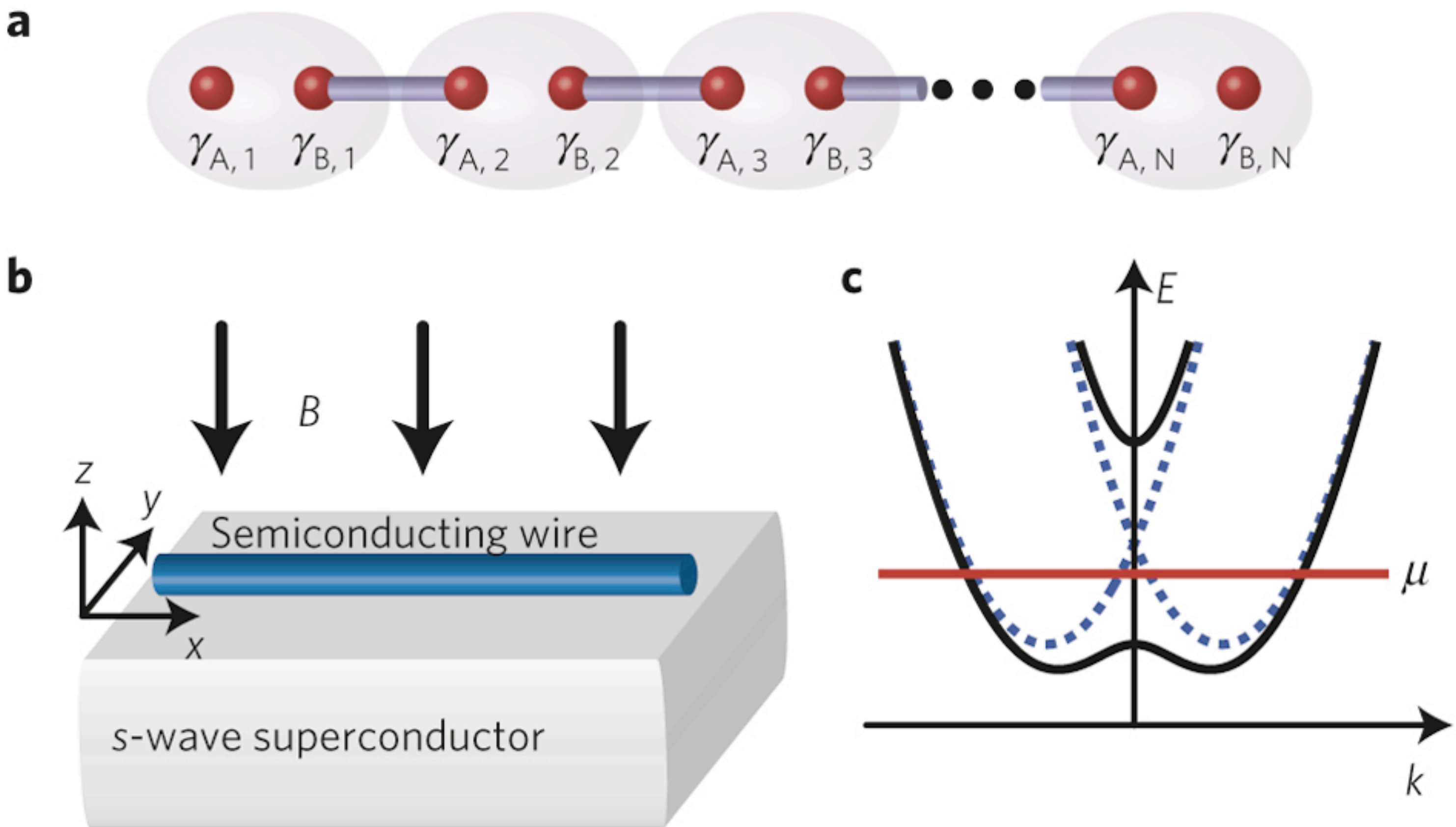
$\psi = (f^+ + f)/\sqrt{2}$ ➡ $\psi = \psi^+ \quad \psi^2 = 1$

$f = (\psi_1 + i\psi_2)/\sqrt{2}$

"half" of the usual (complex) fermion
"real" fermion

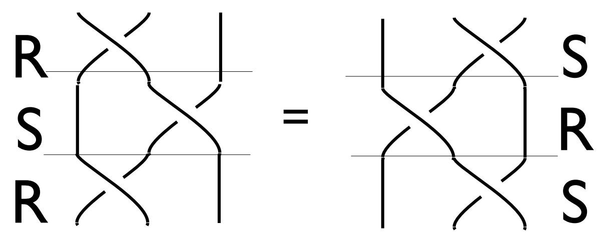


ψ_1, ψ_{2N} Single fermion → 1 q-bit



a, Pictorial representation of the ground state of equation (1) in the limit μ=0, t=|Δ|. Each spinless fermion in the chain is decomposed in terms of two Majorana fermions $\gamma_{A,x}$ and $\gamma_{B,x}$. Majoranas $\gamma_{B,x}$ and $\gamma_{A,x+1}$ combine to form an ordinary, finite-energy fermion, leaving two zero-energy end Majoranas $\gamma_{A,1}$ and $\gamma_{B,N}$ as shown[23]. **b**, A spin–orbit-coupled semiconducting wire deposited on an s-wave superconductor can be driven into a topological superconducting state exhibiting such end Majorana modes by applying an external magnetic field[21, 22]. **c**, Band structure of the semiconducting wire when B=0 (dashed lines) and $\mathbf{B} \neq 0$ (solid lines). When μ lies in the band gap generated by the field, pairing inherited from the proximate superconductor drives the wire into the topological state.

Braiding Majorana Fermions



RSR = SRS

Non-abelian statistics of half-quantum vortices in p-wave superconductors

D. A. Ivanov
Institut für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland

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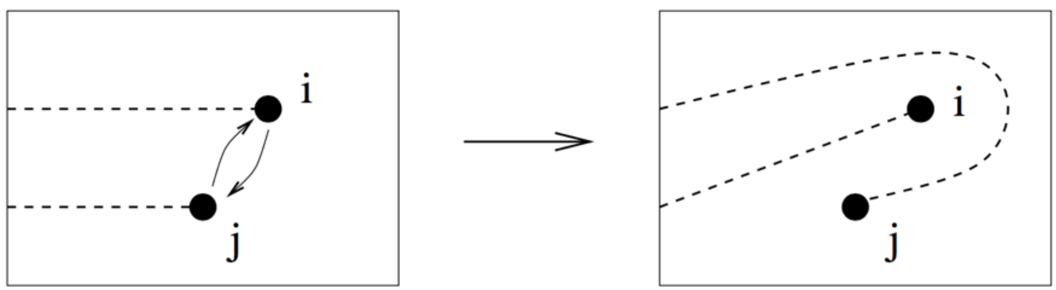


FIG. 3. Elementary braid interchange of two vortices.

Ivanov's Braiding Operators

Let a1, a2, a3, ..., a2n be Majorana Fermions.

S(i) = (1 + a(i+1)a(i))/Sqrt(2)

S^(i) = (1 - a(i+1)a(i))/Sqrt(2)

The operators S(i) form a unitary representation of the Artin Braid Group:

S(i)S(i+1)S(i) = S(i+1)S(i)S(i+1)

S(i)S(j) = S(j)S(i) when |i=j|>2.

Operators T(i) act on the space of MF's via T(i)x = S(i) x S^(i).

T(i)a(i) = a(i+1)

T(i)a(i+1) = - a(i).

$$\boxed{\begin{array}{l} AB = -BA \\ A^2 = B^2 = -1 \end{array}} \Rightarrow \begin{array}{l} ABA = B \\ BAB = A \end{array}$$

$$\sigma_A = \frac{1+A}{\sqrt{2}}, \quad \sigma_B = \frac{1+B}{\sqrt{2}}$$

$$\sigma_A \sigma_B \sigma_A = \left(\frac{\sqrt{2}}{4}\right)(1+A)(1+B)(1+A)$$

$$= \left(\frac{\sqrt{2}}{4}\right)(1+A+B+AB)(1+A)$$

$$= \left(\frac{\sqrt{2}}{4}\right)(1+A+B+AB + A + \underbrace{A^2}_{-1} + BA + \underbrace{ABA}_{B})$$

$$= \left(\frac{\sqrt{2}}{4}\right)(2A+2B)$$

$$= \left(\frac{1}{\sqrt{2}}\right)(A+B).$$

Since this expression is symmetric in A and B, we have

$$\sigma_A \sigma_B \sigma_A = \sigma_B \sigma_A \sigma_B. \quad //$$

Clifford Braiding Theorem. Let C be the Clifford algebra over the real numbers generated by linearly independent elements $\{c_1, c_2, \ldots c_n\}$ with $c_k^2 = 1$ for all k and $c_k c_l = -c_l c_k$ for $k \neq l$. Then the algebra elements $\tau_k = (1 + c_{k+1} c_k)/\sqrt{2}$, form a representation of the (circular) Artin braid group. That is, we have $\{\tau_1, \tau_2, \ldots \tau_{n-1}, \tau_n\}$ where $\tau_k = (1 + c_{k+1} c_k)/\sqrt{2}$ for $1 \leq k < n$ and $\tau_n = (1 + c_1 c_n)/\sqrt{2}$, and $\tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1}$ for all k and $\tau_i \tau_j = \tau_j \tau_i$ when $|i - j| > 2$. Note that each braiding generator τ_k has order 8.

It is worth noting that a triple of Majorana fermions say a, b, c gives rise to a representation of the quaternion group. This is a generalization of the well-known association of Pauli matrices and quaternions. We have $a^2 = b^2 = c^2 = 1$ and they anticommute. Let $I = ba$, $J = cb$, $K = ac$. Then

$$I^2 = J^2 = K^2 = IJK = -1,$$

giving the quaternions. The operators

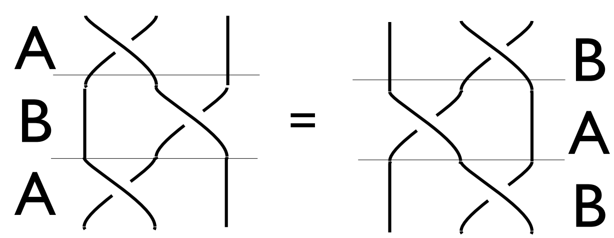
$$A = (1/\sqrt{2})(1 + I)$$

$$B = (1/\sqrt{2})(1 + J)$$

$$C = (1/\sqrt{2})(1 + K)$$

braid one another:

$$ABA = BAB, BCB = CBC, ACA = CAC.$$



Majoranas are related to standard fermions as follows: The algebra for Majoranas is $c = c^\dagger$ and $cc' = -c'c$ if c and c' are distinct Majorana fermions with $c^2 = 1$ and $c'^2 = 1$. One can make a standard fermion from two Majoranas via

$$\psi = (c + ic')/2,$$

$$\psi^\dagger = (c - ic')/2.$$

Similarly one can mathematically make two Majoranas from any single fermion. Now if you take a set of Majoranas

$$\{c_1, c_2, c_3, \cdots, c_n\}$$

then there are natural braiding operators that act on the vector space with these c_k as the basis. The operators are mediated by algebra elements

$$\tau_k = (1 + c_{k+1}c_k)/\sqrt{2},$$

$$\tau_k^{-1} = (1 - c_{k+1}c_k)/\sqrt{2}.$$

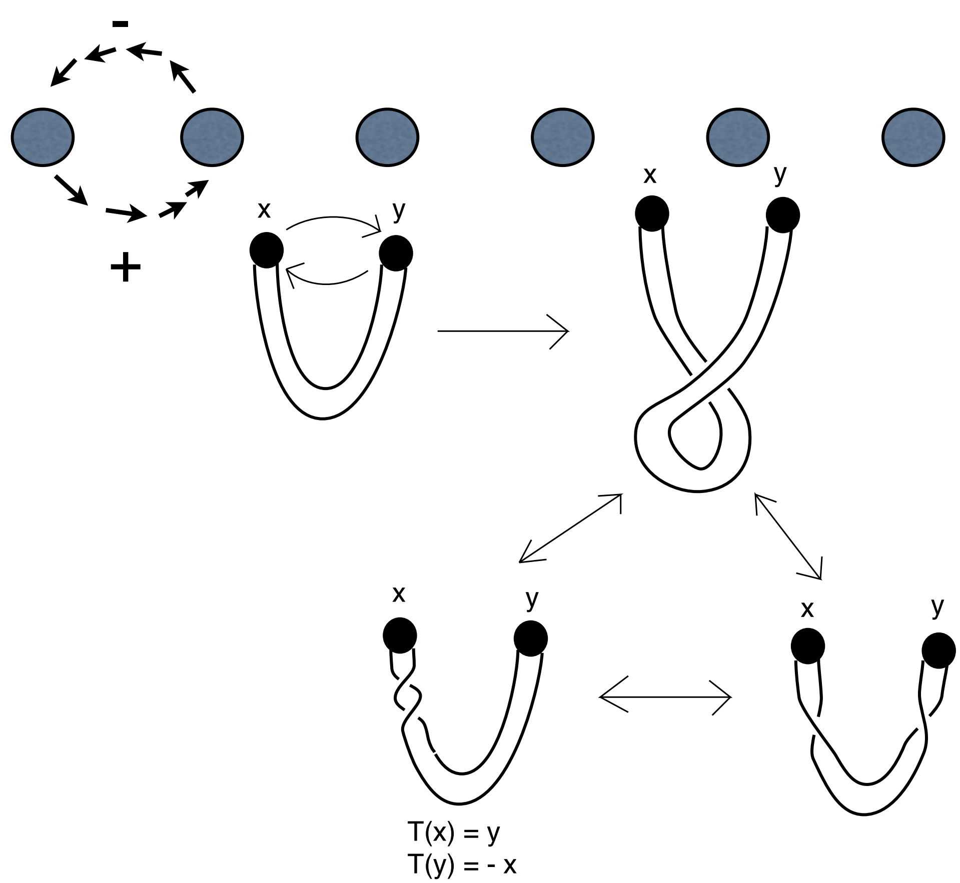
Then the braiding operators are

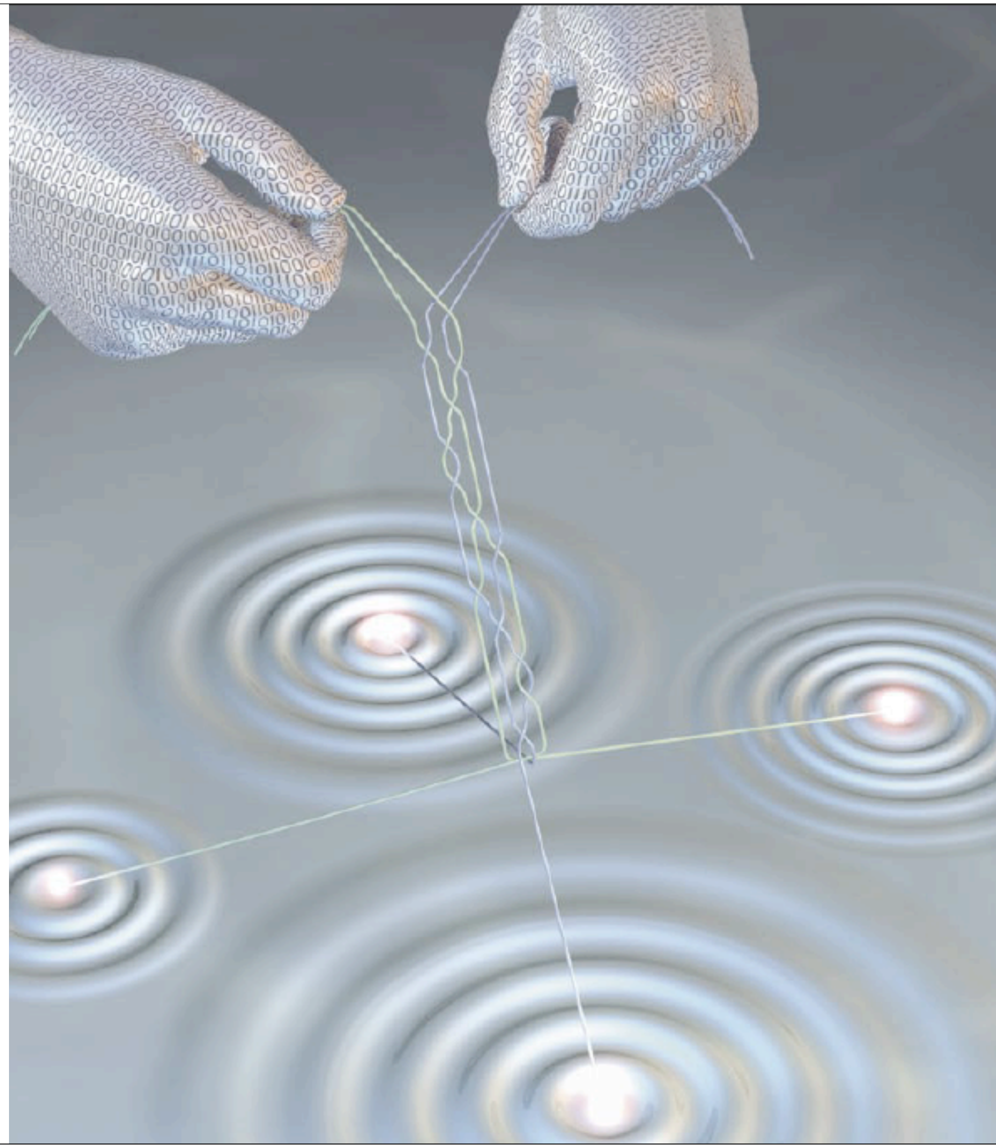
$$T_k : Span\{c_1, c_2, \cdots, , c_n\} \longrightarrow Span\{c_1, c_2, \cdots, , c_n\}$$

via

$$T_k(x) = \tau_k x \tau_k^{-1}.$$

Braiding Majorana Fermions





Now we show how the Majorana Fermion algebra is at the base of the Dirac equation, and how nilpotent operators (representing Fermions) arise naturally in relation to plane wave solutions to the Dirac equation.

5.2 Relativity and the Dirac Equation

Starting with the algebra structure of e and η and adding a commuting square root of -1, i, we have constructed fermion algebra and quaternion algebra. We can now go further and construct the Dirac equation. This may sound circular, in that the fermions arise from solving the Dirac equation, but in fact the algebra underlying this equation has the same properties as the creation and annihilation algebra for fermions, so it is by way of this algebra that we will come to the Dirac equation. If the speed of light is equal to 1 (by convention), then energy E, momentum p and mass m are related by the (Einstein) equation

$$E^2 = p^2 + m^2.$$

Dirac constructed his equation by looking for an algebraic square root of p^2+m^2 so that he could have a linear operator for E that would take the same role as the Hamiltonian in the Schrodinger equation. We will get to this operator by first taking the case where p is a scalar (we use one dimension of space and one dimension of time. Let $E = \alpha p + \beta m$ where α and β are elements of a a possibly non-commutative, associative algebra. Then

$$E^2 = \alpha^2 p^2 + \beta^2 m^2 + pm(\alpha\beta + \beta\alpha).$$

Hence we will satisfiy $E^2 = p^2 + m^2$ if $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$. This is our familiar Clifford algebra pattern and we can use the iterant algebra generated by e and η if we wish. Then, because the quantum operator for momentum is $-i\partial/\partial x$ and the operator for energy is $i\partial/\partial t$, we have the Dirac equation

$$i\partial\psi/\partial t = -i\alpha\partial\psi/\partial x + \beta m\psi.$$

Let

$$\mathcal{O} = i\partial/\partial t + i\alpha\partial/\partial x - \beta m$$

so that the Dirac equation takes the form

$$\mathcal{O}\psi(x,t) = 0.$$

Now note that

$$\mathcal{O}e^{i(px-Et)} = (E - \alpha p - \beta m)e^{i(px-Et)}.$$

We let

$$\Delta = (E - \alpha p - \beta m)$$

and let

$$U = \Delta\beta\alpha = (E - \alpha p - \beta m)\beta\alpha = \beta\alpha E + \beta p + \alpha m,$$

then

$$U^2 = -E^2 + p^2 + m^2 = 0.$$

This nilpotent element leads to a (plane wave) solution to the Dirac equation as follows: We have shown that

$$\mathcal{O}\psi = \Delta\psi$$

for $\psi = e^{i(px-Et)}$. It then follows that

$$\mathcal{O}(\beta\alpha\Delta\beta\alpha\psi) = \Delta\beta\alpha\Delta\beta\alpha\psi = U^2\psi = 0,$$

from which it follows that

$$\psi = \beta\alpha U e^{i(px-Et)}$$

is a (plane wave) solution to the Dirac equation.

Recapitulation and One More

We start with $\psi = e^{i(px-Et)}$ and the operators

$$\hat{E} = i\partial/\partial_t$$

and

$$\hat{p} = -i\partial/\partial_x$$

so that

$$\hat{E}\psi = E\psi$$

and

$$\hat{p}\psi = p\psi.$$

The Dirac operator is

$$\mathcal{O} = \hat{E} - \alpha\hat{p} - \beta m$$

and the modified Dirac operator is

$$\mathcal{D} = \mathcal{O}\beta\alpha = \beta\alpha\hat{E} + \beta\hat{p} - \alpha m,$$

so that

$$\mathcal{D}\psi = (\beta\alpha E + \beta p - \alpha m)\psi = U\psi.$$

$$\mathcal{D}(Ue^{i(px-Et)}) = U^2e^{i(px-Et)} = 0.$$

We have arrived at
Peter Rowland's Nilpotent Solutions
to the Dirac Equation.

If we let

$$\tilde{\psi} = e^{i(px+Et)}$$

(reversing time), then we have

$$\mathcal{D}\tilde{\psi} = (-\beta\alpha E + \beta p - \alpha m)\psi = U^{\dagger}\tilde{\psi},$$

giving a definition of $U^{\dagger}$ corresponding to the anti-particle for $U\psi$.

We have that

$$U^2 = U^{\dagger 2} = 0$$

and

$$UU^{\dagger} + U^{\dagger}U = 4E^2.$$

Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation.

Note that we get different decompositions of the Fermion into Majorana operators according to what is reversed.

{E} and {p,m} for time reversal.
{p} and {E,m} for spin reversal.
{E,p} and {m} for spin and time reversal.

In analogy to our previous discussion we let

$$\psi(x,t) = e^{i(p \bullet x - Et)}$$

and construct solutions by first applying the Dirac operator to this ψ. The two Clifford algebras interact to generalize directly the nilpotent solutions and Fermion algebra that we have detailed for one spatial dimension to this three dimensional case. To this purpose the modified Dirac operator is

$$\mathcal{D} = i\beta\alpha\partial/\partial t + \beta\nabla \bullet \sigma - \alpha m.$$

And we have that

$$\mathcal{D}\psi = U\psi$$

where

$$U = \beta\alpha E + \beta p \bullet \sigma - \alpha m.$$

We have that $U^2 = 0$ and $U\psi$ is a solution to the modified Dirac Equation, just as before. And just as before, we can articulate the structure of the Fermion operators and locate the corresponding Majorana Fermion operators. We leave these details to the reader.

The Majorana-Dirac Equation

There is more to do. We will end with a brief discussion making Dirac algebra distinct from the one generated by $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3$ to obtain an equation that can have real solutions. This was the strategy that Majorana [7] followed to construct his Majorana Fermions. A real equation can have solutions that are invariant under complex conjugation and so can correspond to particles that are their own anti-particles. We will describe this Majorana algebra in terms of the split quaternions ϵ and η. For convenience we use the matrix representation given below. The reader of this paper can substitute the corresponding iterants.

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\hat{\epsilon}$ and $\hat{\eta}$ generate another, independent algebra of split quaternions, commuting with the first algebra generated by ϵ and η. Then a totally real Majorana Dirac equation can be written as follows:

$$(\partial/\partial t + \hat{\eta}\eta\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\eta\partial/\partial z - \hat{\epsilon}\hat{\eta}\eta m)\psi = 0.$$

To see that this is a correct Dirac equation, note that

$$\hat{E} = \alpha_x \hat{p_x} + \alpha_y \hat{p_y} + \alpha_z \hat{p_z} + \beta m$$

(Here the "hats" denote the quantum differential operators corresponding to the energy and momentum.) will satisfy

$$\hat{E}^2 = \hat{p_x}^2 + \hat{p_y}^2 + \hat{p_z}^2 + m^2$$

if the algebra generated by $\alpha_x, \alpha_y, \alpha_z, \beta$ has each generator of square one and each distinct pair of generators anti-commuting. From there we obtain the general Dirac equation by replacing $\hat{E}$ by $i\partial/\partial t$, and $\hat{p_x}$ with $-i\partial/\partial x$ (and same for y, z).

$$(i\partial/\partial t + i\alpha_x\partial/\partial x + i\alpha_y\partial/\partial y + i\alpha_z\partial/\partial y - \beta m)\psi = 0.$$

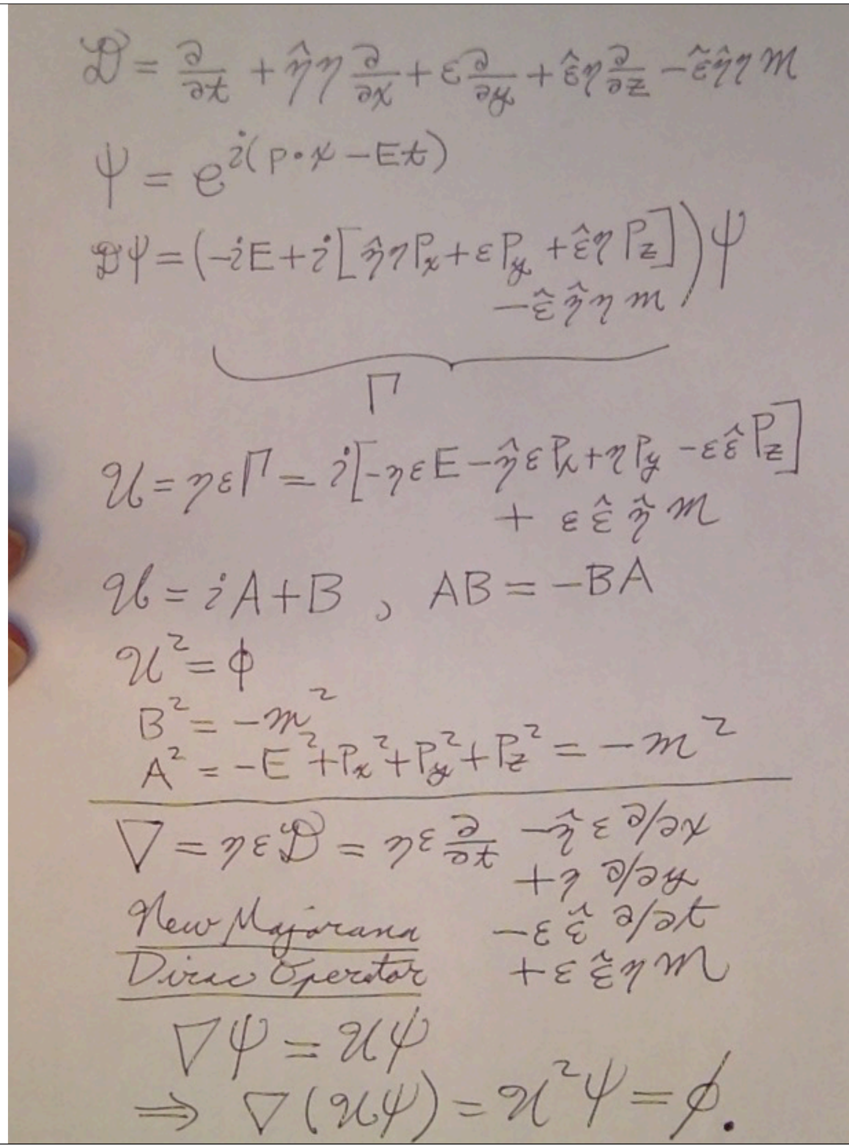
This is equivalent to

$$(\partial/\partial t + \alpha_x\partial/\partial x + \alpha_y\partial/\partial y + \alpha_z\partial/\partial y + i\beta m)\psi = 0.$$

Thus, here we take

$$\alpha_x = \hat{\eta}\eta, \alpha_y = \epsilon, \alpha_z = \hat{\epsilon}\eta, \beta = i\hat{\epsilon}\hat{\eta}\eta,$$

and observe that these elements satisfy the requirements for the Dirac algebra. Note how we have a significant interaction between the commuting square root of minus one (i) and the element $\hat{\epsilon}\hat{\eta}$ of square minus one in the split quaternions. This brings us back to our original considerations about the source of the square root of minus one. Both viewpoints combine in the element $\beta = i\hat{\epsilon}\hat{\eta}\eta$ that makes this Majorana algebra work. Since the algebra appearing in the Majorana Dirac operator is constructed entirely from two commuting copies of the split quaternions, there is no appearance of the complex numbers, and when written out in 2×2 matrices we obtain coupled real differential equations to be solved. Clearly this ending is actually a beginning of a new study of Majorana Fermions. That will begin in a sequel to the present paper.



Core of our joint work with Peter Rowlands. Forming a nilpotent version of the Majorana-Dirac Equation.

Letting $\Omega = (p \bullet x - Et)$, we have

$$U\psi = (A + Bi)e^{i\Omega} = (A + Bi)(Cos(\Omega) + iSin(\Omega)) =$$

$$ACos(\gamma) - BSin(\Omega)) + i(BCos(\Omega) + ASin(\Omega)).$$

Thus we have found two real solutions to the Majorana Dirac Equation:

$$ACos(\Omega) - BSin(\Omega)$$

and

$$BCos(\Omega) + ASin(\Omega)$$

with

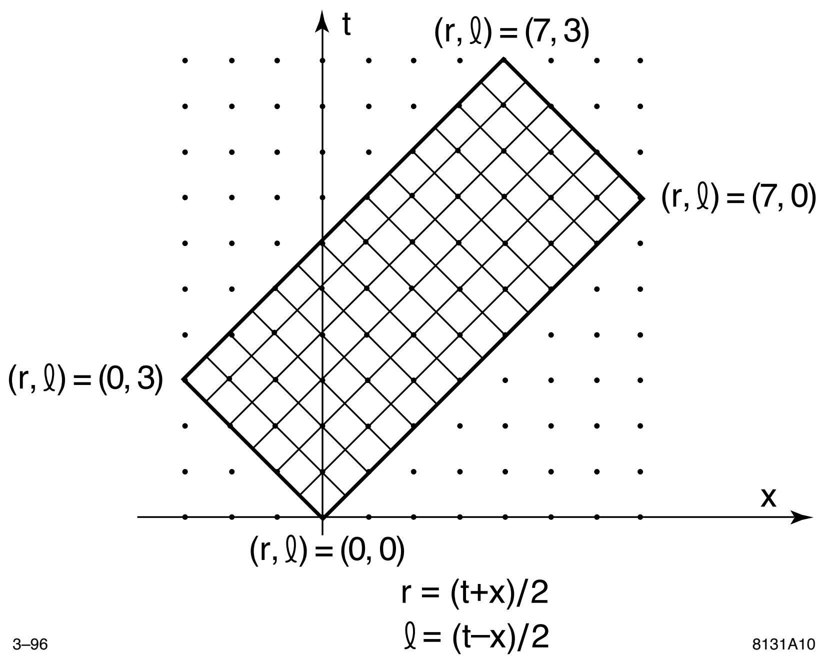
$$\Omega = (p \bullet x - Et).$$

Using the Majorana Operators A and B with AB = -BA and A^2 = B^2 = - m^2.

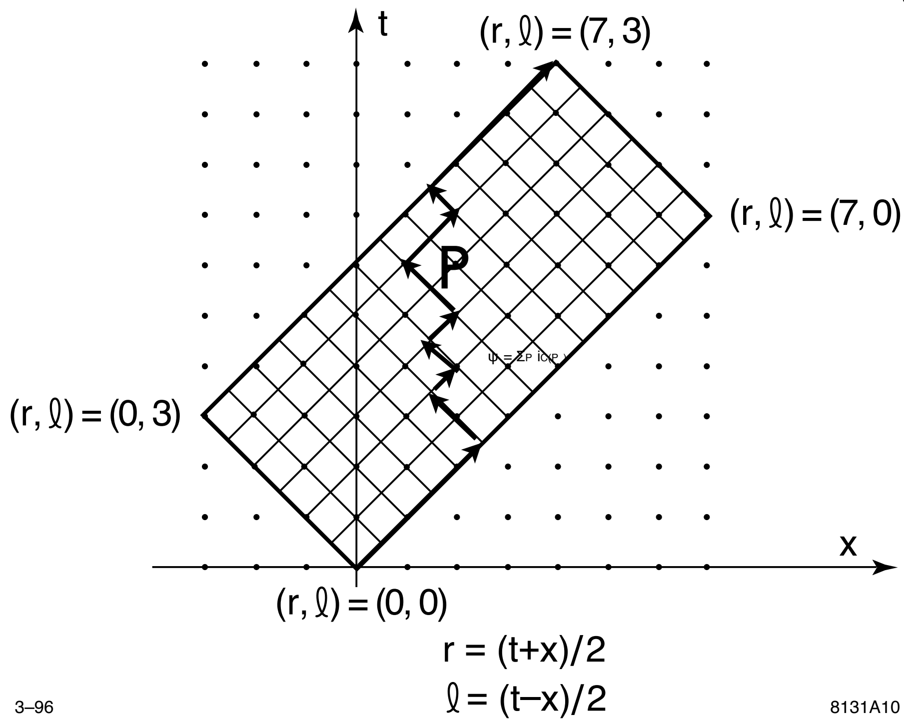
Thus we now see, via the nilpotent approach to the Dirac equation, how the Majorana Operators are directly related to real solutions to the Majorana-Dirac Equation.

Feynman Checkerboard

$$\text{RII:} \begin{pmatrix} -i\psi_2 \\ -i\psi_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial l} \\ \frac{\partial\psi_2}{\partial r} \end{pmatrix}$$



The Feynman Checkerboard



$$\psi = \Sigma_P i^{C(P)}$$

Dirac Amplitude

C(P) =
number
of corners in
path P

In the RII, Majorana Fermion case we have

$$\partial\psi_2/\partial r = \psi_2$$

$$\partial\psi_1/\partial l = -\psi_1$$

Thus the Checkerboard works with plus/minus cornering.
This model gives a picture of how Majorana's equations can look in an explicit case and show how discrete quantum physics may avoid complex numbers.

3.1 Spacetime in $1+1$ dimensions.

Using the method of this section and spacetime with one dimensic
of space (x), we can write a real Majorana Dirac operator in tl
form

$$\partial/\partial t + \epsilon\partial/\partial x + \epsilon\eta m$$

where, the matrix representation is now two dimensional with

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \epsilon\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We obtain a nilpotent operator, $\mathcal{D}$ by multiplying by $i\eta$:

$$\mathcal{D} = i\eta\partial/\partial t + i\eta\epsilon\partial/\partial x - i\epsilon m.$$

Letting $\psi = e^{i(px-Et)}$, we have

$$\mathcal{D}\psi = (A + iB)\psi$$

where

$$A = \eta E + \epsilon\eta p$$

and

$$B = -\epsilon m.$$

$$A = \eta E + \epsilon\eta p$$

and

$$B = -\epsilon m.$$

Note that $A^2 = E^2 - p^2 = m^2$ and $B^2 = m^2$, from which it is easy to see that $A + iB$ is nilpotent. A and B are the Majorana operators for this decomposition. Multiplying out, we find

$$(A + iB)\psi = (A + iB)(cos(\theta) + isin(\theta)) =$$

$$(Acos(\theta) - Bsin(\theta)) + i(Bcos(\theta) + Asin(\theta))$$

where $\theta = px - Et$. We now examine the real part of this expression, as it will be a real solution to the Dirac equation. The real part is

$$Acos(\theta) - Bsin(\theta) = (\eta E + \epsilon\eta p)cos(\theta) + emsin(\theta)$$

$$= \begin{pmatrix} -msin(\theta) & (E - p)cos(\theta) \\ (E + p)cos(\theta) & msin(\theta) \end{pmatrix}.$$

Each column vector is a solution to the original Dirac equation corresponding to the operator

$$\nabla = \partial/\partial t + \epsilon\partial/\partial x + \epsilon\eta m$$

written as a 2×2 matrix differential operator. We can see this in an elegant way by changing to light-cone coordinates:

$$r = \frac{1}{2}(t + x), l = \frac{1}{2}(t - x).$$

(Recall that we take the speed of light to be equal to 1 in this discussion.) Then

$$\theta = px - Et = -(E - p)r - (E + p)l.$$

and the Dirac equation

$$(\partial/\partial t + \epsilon\partial/\partial x + \epsilon\eta m)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

becomes the pair of equations

$$\partial\psi_1/\partial l = m\psi_2,$$

$$\partial\psi_2/\partial r = -m\psi_1.$$

$$\partial\psi_1/\partial l = m\psi_2,$$

$$\partial\psi_2/\partial r = -m\psi_1.$$

Note that these equations are satisfied by

$$\psi_1 = -msin(-(E-p)r - (E+p)l),$$

$$\psi_2 = (E+p)cos(-(E-p)r - (E+p)l)$$

exactly when $E^2 = p^2 + m^2$ as we have assumed. It is quite interesting to see these direct solutions to the Dirac equation emerge in this 1+1 case. The solutions are fundamental and they are distinct from the usual solutions that emerge from the Feynman Checkerbooad Model [2, 6]. It is the above equations that form the basis for the Feynman Checkerboard model that is obtained by examining paths in a discrete Minkowski plane generating a path integral

In fact, this solution to the Feynmann Checkerboard model is different from the solutions from the path sum, suggesting new approaches to the original problem of path integrals for the Dirac equation.

4 Spacetime Algebra

Another way to put the Dirac equation is to formulate it in terms of a *spacetime algebra.* By a spacetime algebra we mean a Clifford algebra with generators $\{e_1, e_2, e_3, e_4\}$ such that $e_1^2 = e_2^2 = e_3^2 = 1$, $e_4^2 = -1$ and $e_i e_j + e_j e_i = 0$ for $i \neq j$. Thus the generators of the algebra fit the Minkowski metric and we can represent a point in space time by $p = xe_1 + ye_2 + ze_3 + te_4$ so that $p^2 = x^2 + y^2 + z^2 - t^2$ corresponds to the spacetime metric with the speed of light $c = 1$. (The reader may wish to compare this approach with Hestenes [15].)

In spacetime algebra terms the Dirac operator becomes

$$\mathcal{O}' = \partial/\partial t + e_1\partial/\partial x + e_2\partial/\partial y + e_3\partial/\partial z + e_4 m.$$

This point of view makes it clear how to search for Majorana algebra since we can search for a spacetime algebra of real matrices. Then the Dirac equation in the form

$$\mathcal{O}'\psi = 0$$

will be an equation over the real numbers. In fact the algebra that we have already written for Majorana is a spacetime algebra:

$$e_1 = \hat{\eta}\eta, e_2 = \epsilon, e_3 = \hat{\epsilon}\eta, e_4 = \hat{\epsilon}\hat{\eta}\eta.$$

Furthermore, we can see that the following lemma gives us a guide to constructing nilpotent formulations of the Dirac equation.

Definition 1. Suppose that $\{e'_1, e'_2, e'_3, e'_4\}$ generates a spacetime algebra $\mathcal{A}$ and that μ is an element of $\mathcal{A}$ with $\mu^2 = -1$ and so that $\{e_1 = \mu e'_1, e_2 = \mu e'_2, e_3 = \mu e'_3, e_4 = \mu e'_4\}$ is also a spacetime algebra with $e_1^2 = e_2^2 = e_3^2 = 1$, $e_4^2 = -1$ and $e_i e_j + e_j e_i = 0$ for $i \neq j$. Under these circumstances, we call the spacetime algebra $\mathcal{A}$ *nilpotent.*

Lemma. Let $\mathcal{A}$ be a nilpotent spacetime algebra, with notation as in Definition1 above. Then the operator

$$\mathcal{D} = \mu\partial/\partial t + e_1\partial/\partial x + e_2\partial/\partial y + e_3\partial/\partial z + e_4 m$$

generates a nilpotent Dirac equation.

Proof. We wish to show that if $\psi = e^{i(p\bullet(x,y,z)-Et)}$ and $\mathcal{D}\psi = U\psi$ then $U^2 = 0$. Calculating, we find that

$$U = i(-\mu E + p \bullet (e_1, e_2, e_3)) + e_4 m.$$

It ifollows that

$$U^2 = -(-E^2 + p_x^2 + p_x^2 + p_x^2) - m^2 = E^2 - p_x^2 - p_y^2 - p_z^2 - m^2 = 0.$$

This completes the proof. $\square$

$$I^2 = J^2 = i^2 = j^2 = 1 \text{ and } IJ + JI = 0 \text{ and } ij + ji = 0.$$

Theorem. *All real Majorana spacetime algebras are nilpotent and, up to permutations and substitutions, they are of the following types:*

$$\{i, jI, jJ, ij\},$$
$$\{j, iI, iJ, IJi\},$$
$$\{ijIJ, I, J, IJi\}.$$

This provides a platform for deeper study of the Majorana Dirac Equation.

But we should take a WIDER VIEW.

The universal equation should be about the (state of) the Universe U.

An operator D acts on U to produce Nothing.

D U = 0.

But the universe U is both operator and operand.
So we take D = U and obtain
the Universal Nilpotent Equation.
UU = 0,
of which the Dirac equation is one of the first special cases.

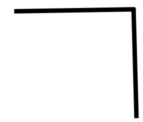
The Simplest example of the Universal Nilpotent Equation is given by the operator

$$Ux = \overline{\;x\;}\big|$$

Here the Universe U is that Universe (self) created by the Mark and taken to Nothing by the crossing from the marked state to the unmarked state.

$$UU = \overline{\overline{\;\;\;}\big|\,}\big| = \qquad .$$

$$UU = 0.$$



In this formalism the mark is seen
to make a distinction.
The formal language of the
calculus of indications refers to the mark and is
built from the mark.

The language using the mark is inherently self-
referential.
The mark and the observer are seen, in the form,
to be identical.

The Calculus writes itself in terms of
itself.

Physical theory is
seen to write itself in terms of the condition
for observation to occur at all.