On hyperelliptic Euclidean 3-manifolds

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Dec 14, 2020

Hyperelliptic involution

Let S_g be a Riemann surface of genus $g, g > 1$. An involution $\tau \in \mathrm{Iso}(S_g)$ is said to be *hyperelliptic* if the quotient space $S_g/\langle\tau\rangle$ is homeomorphic to the 2-dimensional sphere S^2.

A Riemann surface is said to be *hyperelliptic* if it admits a hyperelliptic involution

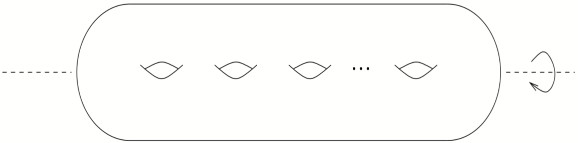


Fig.: Rotation by π about the indicated axis is a hyperelliptic involution

In higher dimensions

Let M be an n-dimensional manifold. Suppose that there exists an involution $\tau : M \to M$ such that the quotient space $M/\langle\tau\rangle$ is homeomorphic to the n-dimensional sphere S^n. Then, τ is said to be a *hyperelliptic involution* and M is said to be a *hyperelliptic manifold*. If M admits a geometric structure then we assume in the definition that τ is an isometry.

Fact: If M is a 3-dimensional hyperelliptic manifold, with a hyperelliptic involution τ, then M is the 2-fold branched covering of S^3 branched over some link (in particular, a knot) L. The covering is given by the action of τ and each point of L has branching index 2.

In this situation, M is the 2-fold covering of a π-orbifold $O^3 = S^3(L)$ with underling space S^3 and singular set L with singular angle π at each point of L.

Thurston eight geometries

$$\mathbb{H}^2 \times \mathbb{R}, \quad \mathbb{S}^2 \times \mathbb{R}, \quad E^3, \quad \mathrm{Sol}, \quad \mathrm{Nil}, \quad \mathbb{S}^3, \quad \widetilde{SL_2 R}, \quad \mathbb{H}^3$$

Survey paper titled The geometries of 3-manifolds by Peter Scott
http://www.math.lsa.umich.edu/~pscott/
A nice post about picturing these geometries.
https://mathoverflow.net/questions/24572/drawing-of-the-eight-thurston-geometries

William Thurston

William Thurston in 1991

Existence of hyperelliptic manifolds in each of the eight Thurston geometries was shown by A. D. Mednykh [1]

There are 6 closed orientable Euclidean manifolds. In notations of J.A. Wolf [2] they are $\mathcal{G}_i,\ i = 1, 2, 3, 4, 5, 6$.

The first one is the three-dimensional torus. R. H. Fox [3] showed that the n-torus is not a double branched covering of S^n for $n > 2$. So, the three dimensional torus is not a hyperelliptic manifold.

W. Dunbar in his Ph.D. thesis classified all oriented Euclidean fibered orbifolds with underlying space S^3

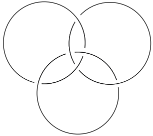
Only eight of them are π-orbifolds.

[1] A. D. Mednykh, Three-dimensional hyperelliptic manifolds, Ann. Global Anal. Geom. 8 (1990) 13–19.

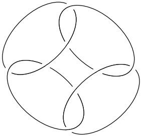
[2] J. A. Wolf, Spaces of Constant Curvature (Publish or Perish, Houston, 1974)

[3] R. H. Fox, A note on branched cyclic covering of spheres, Rev. Mat. Hisp.-Amer. 4(32) (1972) 158–166.

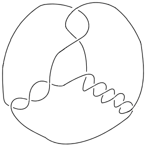
Eight oriented Euclidean fibered π-orbifolds



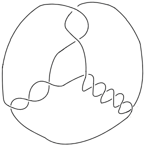
$I2_1 2_1 2_1$



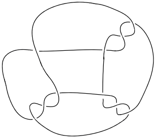
$P222_1$



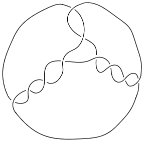
$P6_1 22$



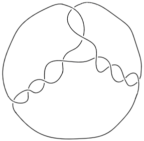
$P6_5 22$



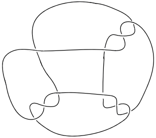
$P3_1 12$



$P4_1 22$



$P4_3 22$



$P3_2 12$

Theorem (Mednykh, V., 2020)

Each of the manifolds $\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6$ is a double branched covering of a Euclidean π-orbifold O^3 with underlying space S^3 and singular set which is a link.

In Table below for each orbifold O^3 we point out which Euclidean manifold M^3 is obtained as its double branched covering.

Table 1. One-to-one correspondence between Euclidean manifolds M^3 and π-orbifolds O^3.

Manifold M^3	$H_1(M^3, \mathbb{Z})$	Orbifold O^3	Singular set of O^3
$\mathcal{G}_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$P222_1$	$P(-2,2,-2,2)$
$\mathcal{G}_3$	$\mathbb{Z} \times \mathbb{Z}_3$	$P3_112$, $P3_212$	$P(1,-3,-3,-3)$, $P(-1,3,3,3)$
$\mathcal{G}_4$	$\mathbb{Z} \times \mathbb{Z}_2$	$P4_322$, $P4_122$	$P(2,-4,-4)$, $P(-2,4,4)$
$\mathcal{G}_5$	$\mathbb{Z}$	$P6_122$, $P6_522$	$P(2,-3,-6)$, $P(-2,3,6)$
$\mathcal{G}_6$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$I2_12_12_1$	Borromean rings

Alexander Mednykh and Bao Vuong, *On hyperelliptic Euclidean 3-manifolds* // Journal of Knot Theory and Its Ramifications, 8 pp. (accepted)

Thank you for your attention!