# Discretization of Commuting Ordinary Differential Operators.

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We denote by $L_k,\ L_s$ the operators of orders $k = N_- + N_+$ and $s = M_- + M_+$

$$L_k = \sum_{j=-N_-}^{N_+} u_j(n)T^j, \qquad L_s = \sum_{j=-M_-}^{M_+} v_j(n)T^j,$$

where $n \in \mathbb{Z},\ N_\pm, M_\pm \geq 0$, $T$ is the shift operator

$$Tf(n) = f(n+1), \qquad f : \mathbb{Z} \to \mathbb{C}.$$

If two difference operators $L_k$ and $L_s$ commute, then there is a nonzero polynomial $F(z,w)$ such that $F(L_k,L_s)=0$. The polynomial $F$ defines the *spectral curve* of the pair $L_k,\ L_s$

$$\Gamma=\{(z,w)\in\mathbb{C}^2|F(z,w)=0\}.$$

The common eigenvalues are parametrized by the spectral curve

$$L_k\psi = z\psi, \quad L_s\psi = w\psi, \quad (z,w) \in \Gamma.$$

The dimension of the space of common eigenfunctions of the pair $L_k,\ L_s$ for fixed eigenvalues is called the *rank* of $L_k,\ L_s$

$$l = \dim\{\psi : L_k\psi = z\psi, \quad L_s\psi = w\psi, \quad (z,w) \in \Gamma.\}$$

Any commutative ring of difference operators in one discrete variable is isomorphic to the ring of meromorphic functions on a spectral curve with $s$ fixed poles (I. M. Krichever, S. P. Novikov). Such operators are said to be $s$–point.

Spectral data were found:

- $l = 1,\ m = 2,\ \forall g$ — I. M. Krichever.
- $l = 1,\ m = 1,\ \forall g$ — G. S. Mauleshova, A. E. Mironov.
- $l > 1,\ m = 1,\ \forall g$ — I. M. Krichever, S. P. Novikov.

Examples were found:

- $l = 1,\ m = 2,\ \forall g$ — I. M. Krichever, D. Mumford.
- $l = 1,\ m = 1,\ \forall g$ — G. S. Mauleshova, A. E. Mironov.
- $l = 2,\ m = 1,\ g = 1$ — I. M. Krichever, S. P. Novikov.
- $l = 2,\ m = 1,\ \forall g$ — G. S. Mauleshova, A. E. Mironov.

Consider the one–point operators of rank two

$$L_4 = \sum_{i=-2}^{2} u_i(n)T^i, \qquad L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_i(n)T^i, \qquad u_2 = v_{2g+1} = 1,$$

with hyperelliptic spectral curve $\Gamma$ genus $g$

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + c_{2g-1}z^{2g-1} + ... + c_0, \tag{1}$$

herewith

$$L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, \quad \psi = \psi(n, P), \quad P = (z, w) \in \Gamma.$$

Common eigenfunctions of $L_4$ and $L_{4g+2}$ satisfy the equation

$$\psi(n+1,P) = \chi_1(n,P)\psi(n-1,P) + \chi_2(n,P)\psi(n,P), \tag{2}$$

where $\chi_1(n,P)$ and $\chi_2(n,P)$ are rational functions on $\Gamma$ having $2g$ simple poles, depending on $n$ (I.M. Krichever, S.P. Novikov). The function $\chi_2(n,P)$ additionally has a simple pole at $q=\infty$. To find $L_4$ and $L_{4g+2}$ it is sufficient to find $\chi_1$ and $\chi_2$. Let $\sigma$ be the holomorphic involution on $\Gamma$,

$$\sigma(z,w) = (z,-w).$$

## Theorem 1

If
$$\chi_1(n,P)=\chi_1(n,\sigma(P)),$$
then $L_4$ has the form
$$L_4=(T+U_n+V_nT^{-1})^2+W_n,$$
where
$$\chi_1=-V_n\frac{Q_{n+1}}{Q_n},\quad \chi_2=\frac{w}{Q_n}+\frac{S_n}{Q_n},$$
$$Q_n=-\frac{S_{n-1}+S_n}{U_{n-1}+U_n},\quad S_n(z)=-U_nz^g+\delta_{g-1}(n)z^{g-1}+\ldots+\delta_0(n).$$
Functions $V_n,U_n,W_n,Q_n$ satisfy the equation
$$F_g(z)=S_n^2+V_nQ_{n-1}Q_{n+1}+V_{n+1}Q_nQ_{n+2}+ \tag{3}$$
$$(z-U_n^2-V_n-V_{n+1}-W_n)Q_nQ_{n+1}.$$

**Corollary 1**

Functions $S_n(z), V_n, U_n, W_n$ satisfy

$$(U_n + U_{n+1})(S_n - S_{n+1}) - V_n Q_{n-1} + V_{n+2} Q_{n+3} -$$
$$(z - U_n^2 - V_n - V_{n+1} - W_n) Q_n +$$
$$(z - U_{n+1}^2 - V_{n+1} - V_{n+2} - W_{n+1}) Q_{n+2} = 0.$$

## Theorem 2

In the case of an elliptic spectral curve $\Gamma$, given by the equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0,$$

operator $L_4$ type

$$L_4 = (T + U_n + V_n T^{-1})^2 + W_n,$$

where

$$U_n = -\frac{\nu_n + \nu_{n+1}}{\gamma_n - \gamma_{n+1}}, \quad V_n = \frac{\nu_n^2 - F_1(\gamma_n)}{(\gamma_n - \gamma_{n-1})(\gamma_{n+1} - \gamma_n)},$$

$$W_n = -c_2 - \gamma_n - \gamma_{n+1},$$

$\gamma_n, \ \nu_n$ — arbitrary function parametres, commute with operator

$$L_6 = L_4\big(T + U_n + V_n T^{-1}\big) - \gamma_{n+2} T - (\nu_n + U_n \gamma_n) - V_n \gamma_{n-1} T^{-1}.$$

## Theorem 3

The operator

$$L_4 = (T + a + (r_3n^3 + r_2n^2 + r_1n + r_0)T^{-1})^2 + g(g+1)r_3n, \qquad r_3 \neq 0$$

commutes with a difference operator $L_{4g+2}$.

If $\nu_n = 0$, then $U_n = 0$. Therefore, we have

$$L_4 = (T + V_n T^{-1})^2 + W_n$$

— difference operator of rank two.

If $\nu_n = \pm\sqrt{F_1(\gamma_n)}$, then $V_n = 0$. Therefore, we have

$$L_2 = (T + U_n)^2 + W_n$$

— difference operator of rank one.

Formally Self–Adjoint Operators

$$\mathcal{L}_4 = (\partial_x^2 + \mathcal{U}(x))^2 + \mathcal{W}(x) \tag{4}$$

где

$$\mathcal{U}(x) = \frac{-16F_1(-(c_2+\mathcal{W})/2) + \mathcal{W}_{xx}^2 - 2\mathcal{W}_x\mathcal{W}_{xxx}}{4\mathcal{W}_x^2}.$$

commute with operator

$$\mathcal{L}_6 = (\partial_x^2 + \mathcal{U})^3 + \frac{1}{2}(c_2 + 3\mathcal{W})(\partial_x^2 + \mathcal{U}) + \frac{3}{2}\mathcal{W}_x\partial_x + \frac{5}{4}\mathcal{W}_{xx}. \tag{5}$$

Let $T_\varepsilon$ is the shift operator on $\varepsilon$, $T_\varepsilon f(x) = f(x+\varepsilon)$.

## Theorem 4

Operator

$$L_4^\flat = (\frac{T_\varepsilon}{\varepsilon^2} + U(x,\varepsilon) + \varepsilon^2 V(x,\varepsilon) T_\varepsilon^{-1})^2 + W(x,\varepsilon),$$

where

$$U(x,\varepsilon) = -\frac{\nu(x,\varepsilon)+\nu(x+\varepsilon,\varepsilon)}{\gamma(x,\varepsilon)-\gamma(x+\varepsilon,\varepsilon)}, \quad W(x,\varepsilon) = -c_2 - \gamma(x,\varepsilon) - \gamma(x+\varepsilon,\varepsilon),$$

$$V(x,\varepsilon) = \frac{\nu^2(x,\varepsilon) - F_1(\gamma(x,\varepsilon))}{(\gamma(x,\varepsilon)-\gamma(x-\varepsilon,\varepsilon))(\gamma(x+\varepsilon,\varepsilon)-\gamma(x,\varepsilon))}, \tag{6}$$

$\gamma(x)$, $\nu(x)$ — arbitrary functions, commute with operator

$$L_6^\flat = L_4^\flat(\frac{T_\varepsilon}{\varepsilon^2} + U(x,\varepsilon) + \varepsilon^2 V(x,\varepsilon) T_\varepsilon^{-1}) -$$

$$\gamma(x+2\varepsilon,\varepsilon)\frac{T_\varepsilon}{\varepsilon^2} - (\nu(x,\varepsilon) + U(x,\varepsilon)\gamma(x,\varepsilon)) - \varepsilon^2 V(x,\varepsilon)\gamma(x-\varepsilon,\varepsilon) T_\varepsilon^{-1}.$$

**Example 1.** Spectral curve of operators $L_4^\flat, L_6^\flat$ is given by the equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

Let

$$\gamma(x,\varepsilon) = -\frac{1}{2}(c_2 + \mathcal{W}(x)), \qquad \nu(x,\varepsilon) = \frac{\mathcal{W}_x}{2\varepsilon}.$$

Then

$$L_4^\flat = \mathcal{L}_4 + O(\varepsilon), \qquad L_6^\flat = \mathcal{L}_6 + O(\varepsilon)$$

Moreover, the spectral curves of the pairs $L_4^\flat,\ L_6^\flat$ and $\mathcal{L}_4,\ \mathcal{L}_6$ coincide.

By analogy with Theorem 3, we can prove that $\varepsilon$–difference operator

$$\tilde{L}_4 = \left(\frac{T_\varepsilon}{\varepsilon^2} - \frac{2}{\varepsilon^2} + (\frac{1}{\varepsilon^2} + r_3x^3 + r_2x^2 + r_1x + r_0)T^{-1}\right)^2 + r_3(g(g+1)x+\varepsilon),\ r_3 \neq 0$$

commute with some operator $\tilde{L}_{4g+2}$. In the limit of $\varepsilon \to 0$, this operator goes to the next differential operator

$$L_4^\sharp = (\partial_x^2 + r_3x^3 + r_2x^2 + r_1x + r_0)^2 + g(g+1)r_3x,$$

which commute with $L_{4g+2}^\sharp$. The operators $L_4^\sharp, L_{4g+2}^\sharp$ define a commutative subalgebra in the first Weil algebra with a spectral curve of genus $g$. For $g = 1$, the spectral curve of the pair $L_4^\sharp, L_6^\sharp$ coincides with the spectral curve of the operators $\tilde{L}_4, \tilde{L}_6$. For $g > 1$, the spectral curve of $\tilde{L}_4, \tilde{L}_{4d+2}$ depends on $\varepsilon$. In order to obtain operators whose spectral curve does not depend on $\varepsilon$, instead of the constants $r_j$, you need to find some functions of$\varepsilon$.

$$[H,\ M]=0,$$

where $H=\partial_x^2+u(x)$, $M=\partial_x^{2g+1}+v_{2g}(x)\partial_x^{2g}+\ldots+v_0(x)$. The spectral curve $\Gamma$ of the pair $H,\ M$ is given by an equation of the form $w^2=F_g(z)$, and if $\psi(x)$ is a common eigenfunction, i.e.,

$$H\psi(x)=z\psi(x),\qquad M\psi(x)=w\psi(x),$$

then $(z,w)\in\Gamma$.

**Example.**

$$\psi=e^{-x\zeta(z)}\frac{\sigma(z+x)}{\sigma(x)\sigma(z)},$$

$$(\partial_x^2-2\wp(x))\psi(x,z)=\wp(z)\psi(x,z),$$

$$(\partial_x^3-3\wp(x)\partial_x-\frac{3}{2}\wp'(x))\psi(x,z)=\frac{1}{2}\wp'(z)\psi(x,z).$$

Lame operators

$$\partial_x^2-g(g+1)\wp(x).$$

**Example 2.** If $\gamma(x,\varepsilon)=\wp(x-\varepsilon),\ \nu(x,\varepsilon)=\wp'(x-\varepsilon)$ in Theorem 4, then

$$L_4^\flat=\frac{T_\varepsilon^2}{\varepsilon^2}+(-2\zeta(\varepsilon)-\zeta(x-\varepsilon)+\zeta(x+\varepsilon))\frac{T_\varepsilon}{\varepsilon}+\wp(\varepsilon).$$

$$L_6^\flat=\frac{T_\varepsilon^3}{\varepsilon^3}-\big(3\zeta(\varepsilon)+\zeta(x-\varepsilon)-\zeta(x+2\varepsilon)\big)\frac{T_\varepsilon^2}{\varepsilon^2}+$$

$$\big((\zeta(\varepsilon)+\zeta(x-\varepsilon)-\zeta(x))(\zeta(\varepsilon)+\zeta(x)-\zeta(x+\varepsilon))+2\wp(\varepsilon)+\wp(x)\big)\frac{T_\varepsilon}{\varepsilon}+\frac{1}{2}\wp'(\varepsilon).$$

Moreover,

$$L_4^\flat=\big(\partial_x^2-2\wp(x)\big)+O(\varepsilon).$$

$$L_6^\flat=\big(\partial_x^3-3\wp(x)\partial_x-\frac{3}{2}\wp'(x)\big)+O(\varepsilon).$$

Consider the function $A_g(x,\varepsilon)$ defined as follows. We put

$$A_1 = -2\zeta(\varepsilon) - \zeta(x-\varepsilon) + \zeta(x+\varepsilon)$$

and

$$A_2 = -\frac{3}{2}\big(\zeta(\varepsilon) + \zeta(3\varepsilon) + \zeta(x-2\varepsilon) - \zeta(x+2\varepsilon)\big),$$

where $\zeta(x)$ is the Weierstrass function. Next, for odd $g = 2g_1 + 1$, we put

$$A_g = A_1 \prod_{k=1}^{g_1} \left(1 + \frac{\zeta(x-(2k+1)\varepsilon) - \zeta(x+(2k+1)\varepsilon)}{\zeta(\varepsilon) + \zeta((4k+1)\varepsilon)}\right),$$

and for even $g = 2g_1$, we put

$$A_g = A_2 \prod_{k=2}^{g_1} \left(1 + \frac{\zeta(x-2k\varepsilon) - \zeta(x+2k\varepsilon)}{\zeta(\varepsilon) + \zeta((4k-1)\varepsilon)}\right).$$

## Theorem 5

The operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A_g(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon)$$

commutes with $L_{2g+1}$. Moreover,

$$L_2 = \partial_x^2 - g(g+1)\wp(x) + O(\varepsilon).$$

Treibich–Verdier operator

$$-\partial_x^2 + \sum_{i=0}^{3} a_i(a_i+1)\wp(x+\omega_i),$$

were $\omega_i$ is a half periods and

$$(\wp'(x))^2 = 4\wp^3(x) + g_2\wp^2(x) + g_1\wp(x) + g_0.$$

Let

$$\left((\frac{1}{\sin^2(x)})'\right)^2 = 4\left(\frac{1}{\sin^2(x)}\right)^2\left(\frac{1}{\sin^2(x)} - 1\right).$$

Special case

$$\partial_x^2 - \left(\frac{6}{\sin^2(x)} + \frac{2}{\cos^2(x)}\right).$$

Let

$$A_2 = (-\frac{3}{4}\varepsilon - \frac{9}{8}\varepsilon^3)\big(2\cot(\varepsilon) + \tan(\varepsilon - x) + \tan(\varepsilon + x)\big)\times$$
$$\big(\cot(\varepsilon) + \cot(3\varepsilon) - \cot(2\varepsilon - x) - \cot(2\varepsilon + x)\big).$$

**Example 3.** The operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A_2(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon)$$

commutes with $L_5$. Moreover,

$$L_2 = \partial_x^2 - \left(\frac{6}{\sin^2(x)} + \frac{2}{\cos^2(x)}\right) + O(\varepsilon).$$

Thank you for attention!