Объемы прямоугольных многогранников в пространстве Лобачевского

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13 декабря, 2020

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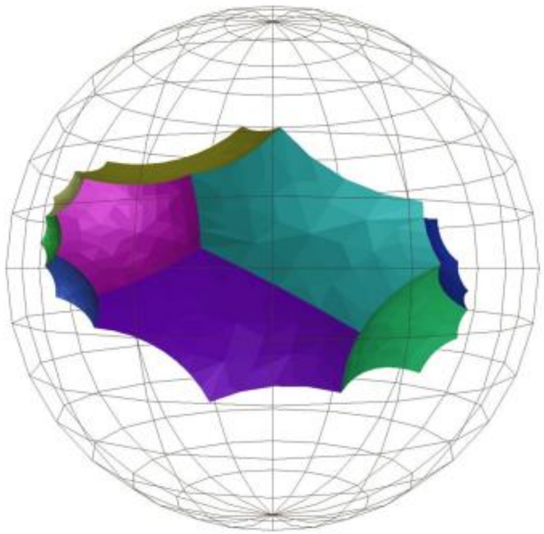
Introduction

In H^3 we consider right-angled hyperbolic polyhedra of two types:

1. Compact (Pogorelov polyhedra) – compact - all vertices are finite,
2. Ideal – with all vertices on the absolute.

Corollory (Andreev's theorem)

A bounded hyperbolic polyhedron in H^n, n ≥ 3, with non-obtuse dihedral angles is uniquely defined by its combinatorics and dihedral angles.



Compact right-angled polyhedra in H^3

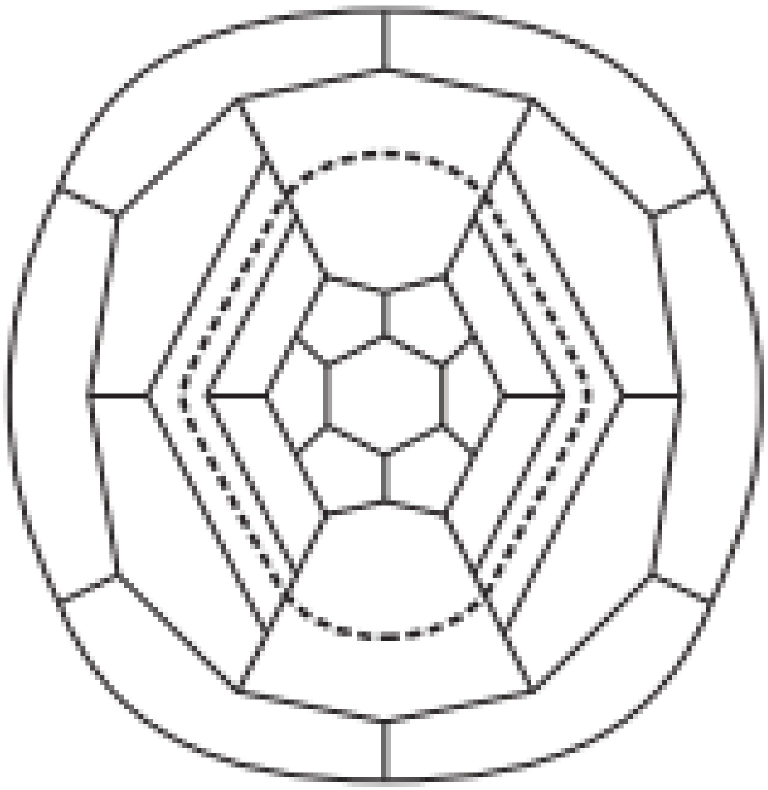
Combinatorial structure

Theorem (Pogorelov 1967, Andreev 1970)

A polyhedral graph P can be realized in H^3 as a bounded right-angled polyhedron if and only if

1. any vertex is incident to 3 edges;
2. any face has at least 5 sides;
3. if a simple closed curve on the surface of the polyhedron separates two faces (prismatic circuit), then it intersects at least 5 edges.

This polyhedron satisfies 1) and 2), but not 3) condition



Reformulation in terms of cyclic connectivity

Definition A graph is called cyclically k-connected if at least k edges have to be removed to split it into two connected components that both have a cycle.

Theorem (Pogorelov, Andreev) A polyhedral graph is realized as a graph of a bounded right-angled polyhedron in H^3 if and only if it is 3-valent and cyclically 5-connected. Moreover, such an implementation is unique.

The program called Plantri, written by G. Brinkmann and B. McKay, can generate such a graphs (in fact, dual to them).

Euler's formula

From Euler's formula V - E + F = 2 and 3-valence, 2E = 3V, it follows that F = (V + 4) / 2.

Let p_k denote the number of k-gonal faces (k ≥ 5) of P. Then

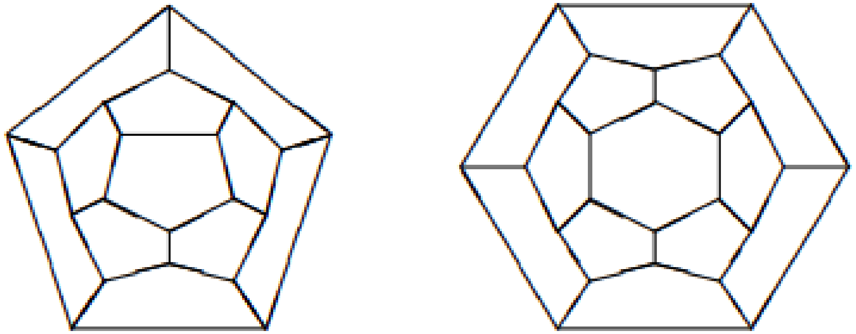
$$p_5 = 12 + \sum_{k>6} p_k (k-6)$$

Consequently, each bounded right-angled polyhedron has at least 12 pentagonal faces.

A dodecahedron is a bounded right-angled polyhedron with a minimum number of faces (12 pentagons).

Löbell polyhedra

For any n≥5, there is a right-angled hyperbolic polyhedron L(n) that has (2n+2) faces. L(5) and L (6) look like this.



Polyhedra L(n) is called Löbell polyhedra.

Theorem (A. Yu. Vesnin, 1987)

$$vol\big(L(n)\big) = \frac{n}{2}\left(2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \; + \Lambda\left(\theta_n - \frac{\pi}{n}\right) + \Lambda\left(\frac{\pi}{2} - 2\theta_n\right)\right),$$

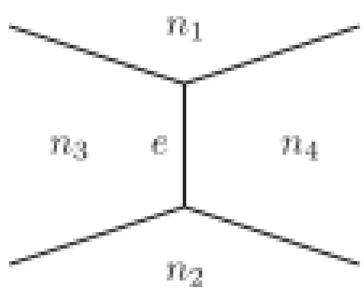
$$where\ \theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos\frac{\pi}{n}}\right).$$

Composition / decomposition

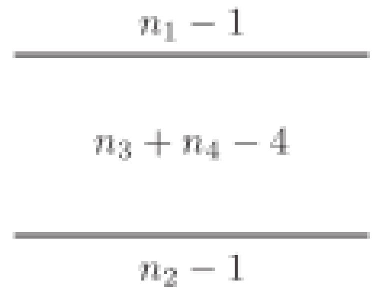
Let there be two combinatorial polyhedra P_1 и P_2 with k-gonal faces $F_1 \subset P_1$ и $F_2 \subset P_2$. Their composition is $P = P_1 \cup_{F_1=F_2} P_2$.

If P_1 and P_2 are realized as compact right-angled polyhedra in H^3, then P is realized as a compact right-angled polyhedron in H^3.

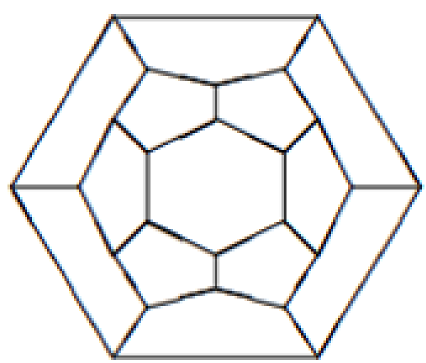
Edge insertion/ edge deleting



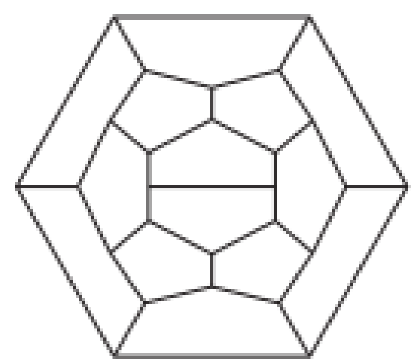
R



$R - e$



L(6)



L(6)+e

We can get any polyhedron from Löbell polyhedra

Theorem (T. Inoue, 2008)

For any compact right-angled hyperbolc polyhedron P there exists a sequence of unions of right-angled hyperbolic polyhedra $P_1, . . . , P_k$ such that each set P_i is obtained from P_{i-1} by decomposition or edge surgery, and P_k consists of Löbell polyhedra. Moreover,

$$\text{vol}(P_0) \geq \text{vol}(P_1) \geq \text{vol}(P_2) \geq \dots \geq \text{vol}(P_k).$$

Atkinson's volume bounds

Theorem (C. Atkinson, 2009) Let P be a compact right-angled hyperbolic polyhedron with N vertices. Then

$$(N-2)\frac{v_8}{32} \leq vol(P) < (N-10)\frac{5v_3}{8}.$$

Constants v_3 and v_8 have a following meaning :

- $v_3 = 3\Lambda(\frac{\pi}{3}) = 1.0149416064096535...,$
- $v_8 = 8\Lambda(\frac{\pi}{4}) = 3.66386237670887 6....$

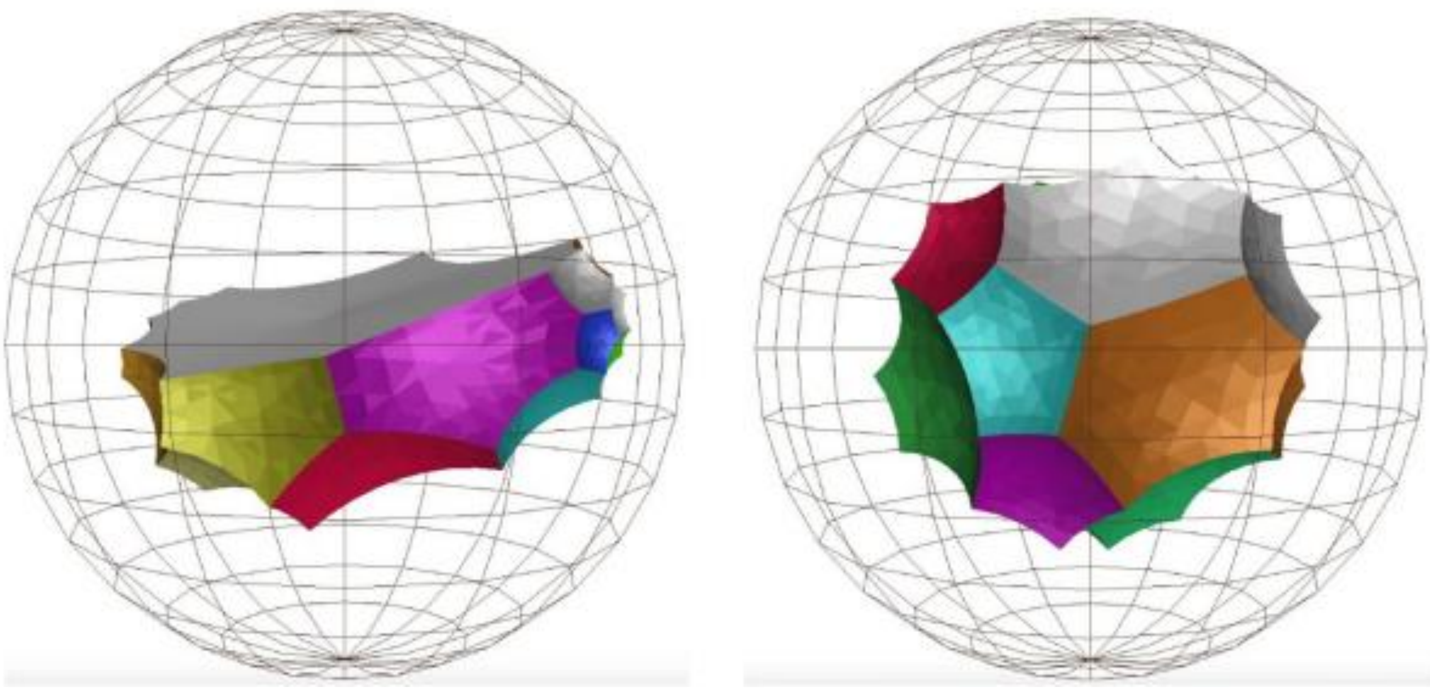
Improved upper bound

Theorem (Egorov - Vesnin)

Let P be a compact right-angled hyperbolic polyhedron with N vertices. If P is not a right-angled dodecahedron, then

$$vol(P) < (N - 14)\frac{5v_3}{8}.$$

Informally speaking, large faces reduce the volume



Left: polyhedron with minimal volume

Right: polyhedron with maximum volume

Upper bounds considering face sizes

Theorem (Egorov - Vesnin)

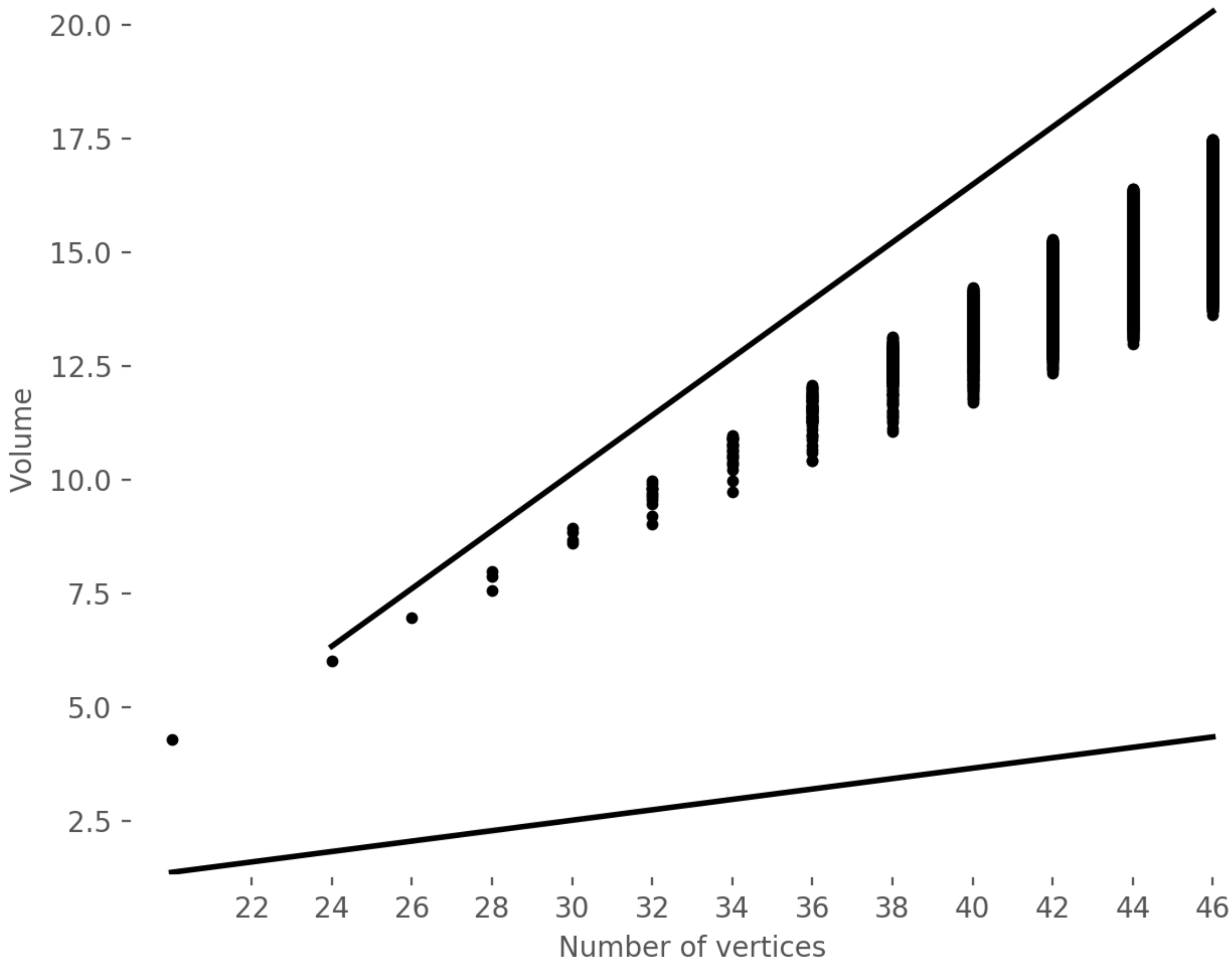
Let P be a compact right-angled hyperbolic polyhedron with N ≥ 24 vertices. Let F_1 and F_2 be two faces of P such that F_1 is n_1-gon and F_2 is n_2–gon. Then

$$vol(P) < (N - n_1 - n_2)\frac{5v_3}{8}$$

Corollary

Let P be a compact right-angled hyperbolic polyhedron with N vertices. Let F_1, F_2 and F_3 be three faces of P such that F_2 is adjanced to F_1 and F_3. Suppose that the face F_i is a n_i-gon, i = 1, 2, 3. Then

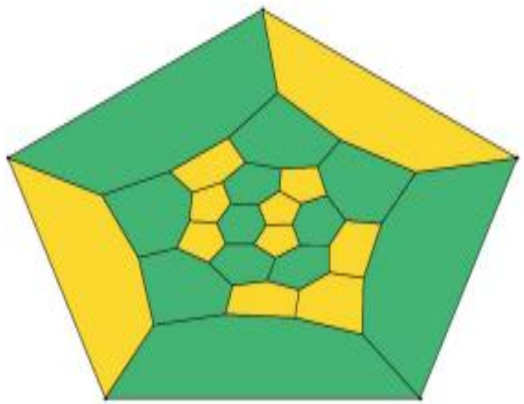
$$vol(P) < (N - n_1 - n_2 - n_3 + 4)\frac{5v_3}{8}$$

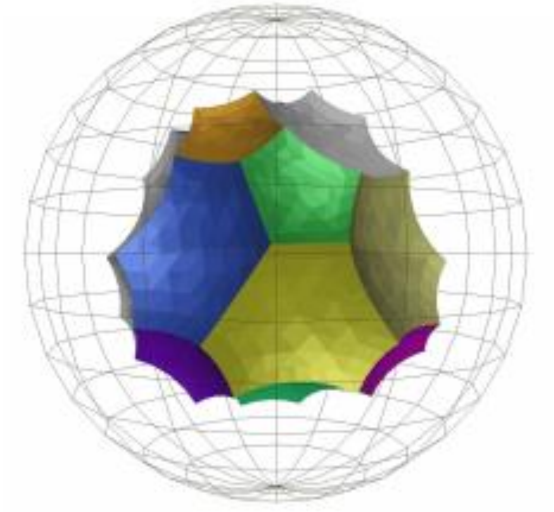


Volumes of compact right-angled polyhedra, Atkinson's lower bound, new upper bound.

Fullerenes

Fullerenes are called trivalent polyhedra, the faces of which are 5 and 6-gons.





Theorem (Došlić, 2005) Every fullerene can be realized as compact right-angled hyperbolic polyhedron.

Ideal right-angled polyhedra

Combinatorial structure

Theorem (I. Rivin, 1992)

A polyhedral graph is realized as a graph of an ideal right-angled polyhedron in H^3 if and only if it is 4-valent and cyclically 6-connected. Moreover, such an realization is unique.

Can be generated using plantri.

Euler's formula

From Euler's formula V - E + F = 2 and 4-valence, 2E = 4V, it follows that F = V + 2.

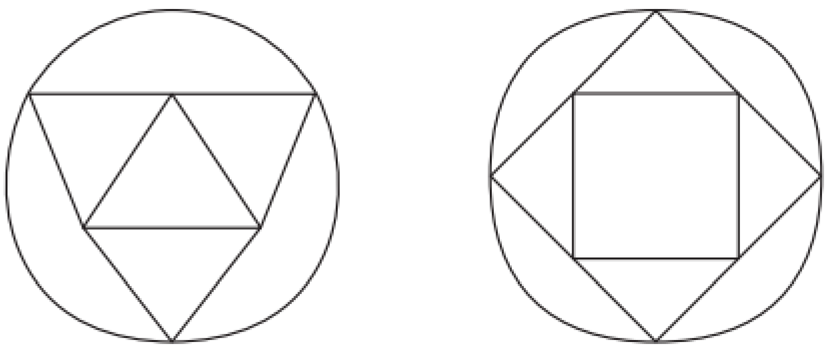
Let p_k denote the number of k-gonal faces (k ≥ 3) of P. Then

$$p_3 = 8 + \sum_{k>4} p_k(k-4)$$

Therefore, each ideal right-angled polyhedron has at least 8 triangular faces.

An octahedron is an ideal right-angled polyhedron with a minimum number of faces (8 triangles).

Aniprisms



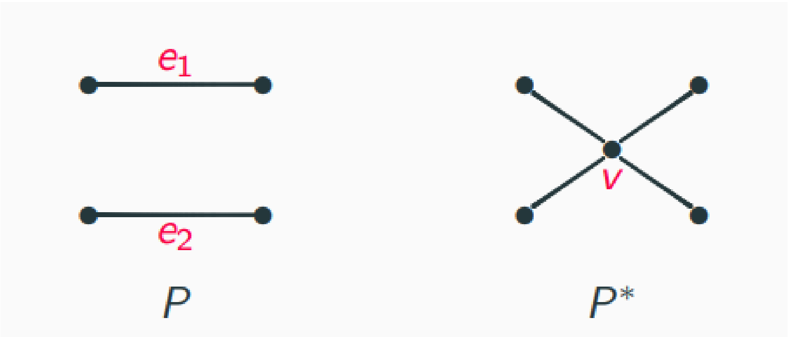
Theorem (W. Thurston, 1980)

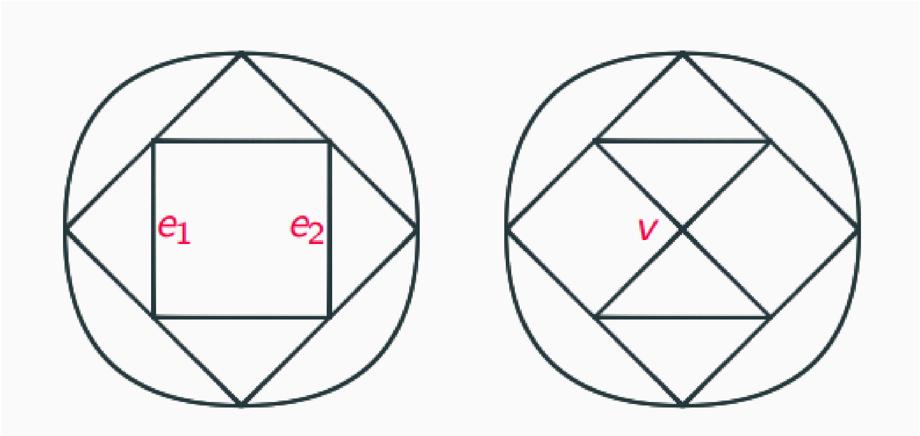
Let n≥ 3, then the volume of an ideal right-angled antiprism A(n) is expressed as follows

$$vol(A(n)) = 2n(\Lambda(\frac{\pi}{4}+\frac{\pi}{2n}) + \Lambda(\frac{\pi}{4}-\frac{\pi}{2n})).$$

Edge twisting

Let e_1 and e_2 be two non-adjacent edges that lie on the same face. Edge twisting consists in removing edges e_1 and e_2, and then, creating a new vertex v and connecting it with edges to the ends of the removed edges





A(4) A(4)*

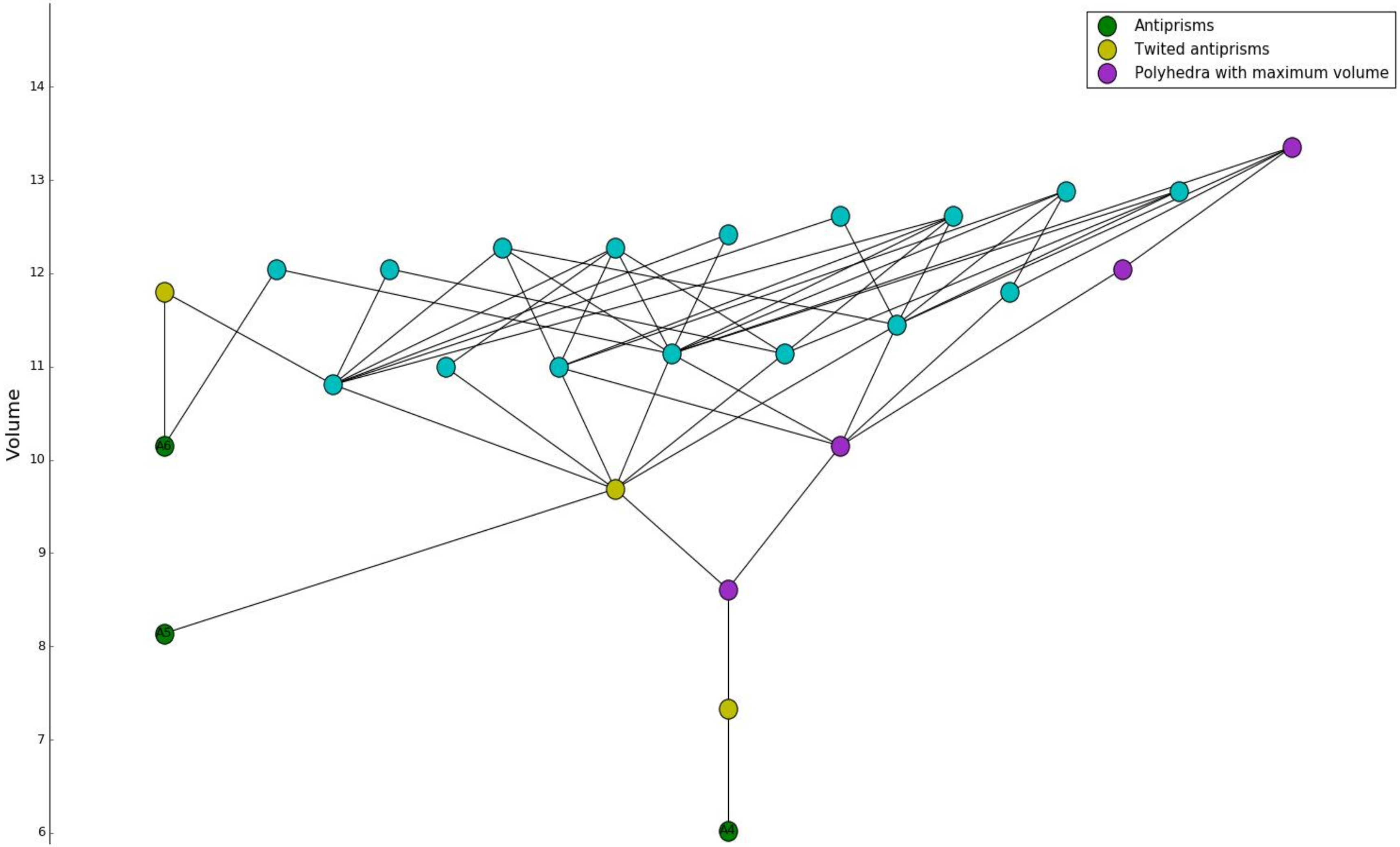
All polyhedra are obtained from A (4))

Theorem (G. Brinkmann, 2000)

Every ideal right-angled hyperbolic polyhedron is either an antiprism or can be obtained from an antiprism by edge twisting operations.

Theorem (N.Yu. Erokhovets, 2019)

Every ideal right-angled hyperbolic polyhedron either an antiprism A (n), $n \geq 3$, or is obtained from A(4) by edge twisting operations.



Antiprisms
Twited antiprisms
Polyhedra with maximum volume
Volume
14
13
12
11
10
9
8
7
6
A6
A5
A4

Atkinson’s volume bounds

Theorem (C. Atkinson, 2009)

Let P be an ideal right-angled hyperbolic polyhedron with N vertices. Then

$$(N - 2)\frac{v_8}{4} \leq vol(P) \leq (N - 4)\frac{v_8}{2}.$$

Improved upper bound

Theorem (Egorov - Vesnin)

Let P be an ideal right-angled hyperbolic polyhedron with $N \geq 10$ vertices. Then

$$vol(P) < (N - 5)\frac{v_8}{2}.$$

Upper bounds considering face sizes

Theorem (Egorov - Vesnin)

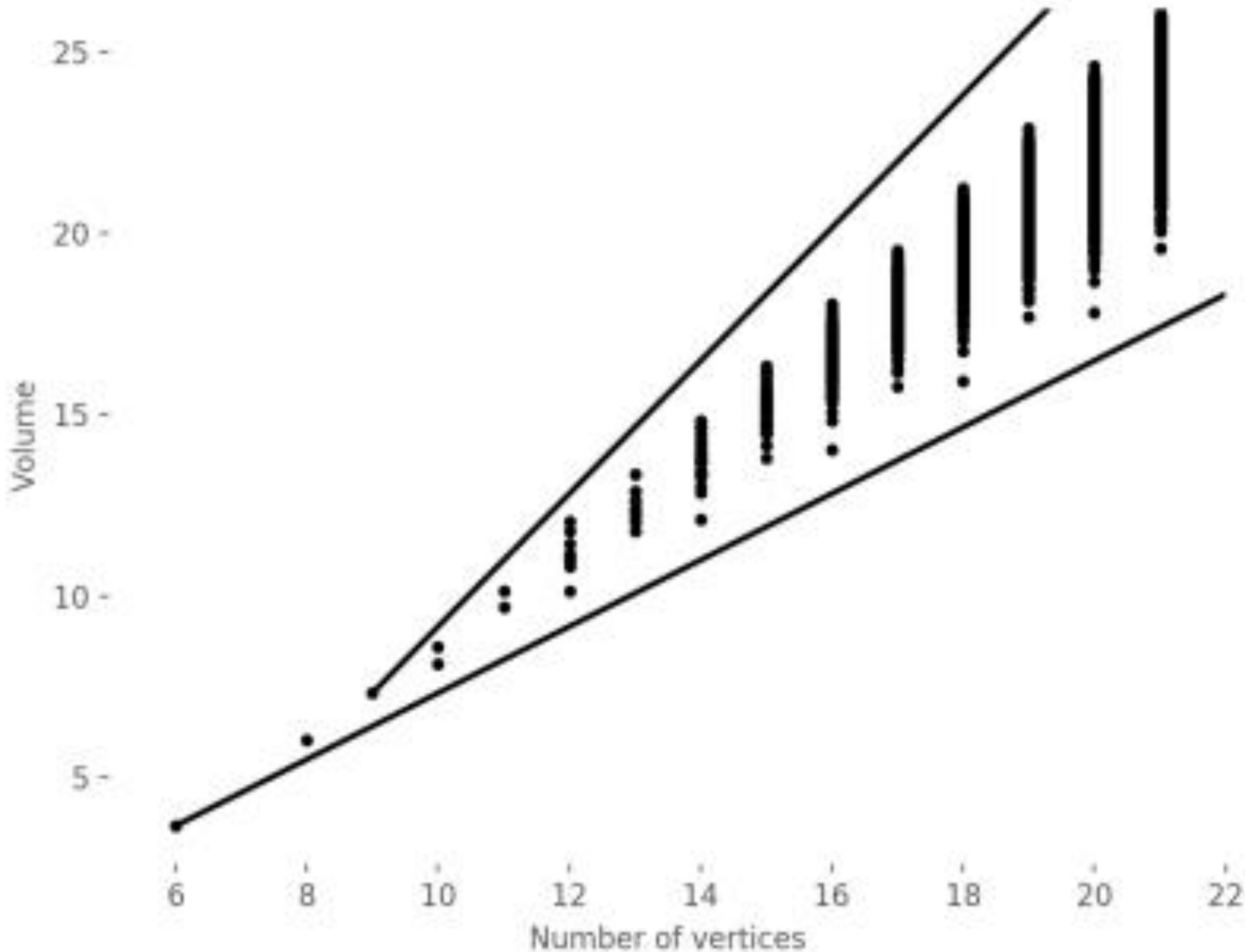
Let P be an ideal right-angled hyperbolic polyhedron with N vertices. Let F_1 *and* F_2 be two faces of P such that F_1 is n_1-gon and F_2 is n_2–gon, $n_1, n_2 \geq 4$. Then

$$vol(P) < \left(N - \frac{n_1}{2} - \frac{n_2}{2}\right)\frac{v_8}{2}$$

Corollary

Let P be an ideal right-angled hyperbolic polyhedron with N vertices. Let F_1, F_2 and F_3 be three faces of P such that F_2 is adjanced to F_1 and F_3. Suppose that the face F_i is a n_i-gon, i = 1, 2, 3. Then

$$vol(P) < (N - n_1 - n_2 - n_3 + 1)\frac{v_8}{2}$$



Volume
25
20
15
10
5
6
8
10
12
14
16
18
20
22
Number of vertices

Thanks for attention!