# Integrable geodesic and magnetic geodesic flows on the 2-torus.

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# Poisson bracket and Hamiltonian systems

Let $M$ be a smooth manifold, dim $M = N$.
Let $f, g \in C^\infty(M)$.

In local coordinates $y = (y^1, \ldots, y^N)$ on $M$ *the Poisson bracket* is given by:

$$h^{ij}(y) = \{y^i, y^j\}, \qquad \{f, g\} = h^{ij}(y) \frac{\partial f(y)}{\partial y^i} \frac{\partial g(y)}{\partial y^j}, \qquad i, j = 1, \ldots, N.$$

Poisson bracket allows to define a Hamiltonian system on $M$:

$$\frac{d}{dt} y^i = \{y^i, H(y)\}, \qquad i = 1, \ldots, N.$$

# Poisson bracket and Hamiltonian systems

In canonical coordinates $(y^1, \ldots, y^N) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$, $N = 2n$ we have

$$\{x^i, p_j\} = \delta^i_j, \qquad \{x^i, x^j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad i, j = 1, \ldots, n;$$

$$\{F, H\} = \sum_{j=1}^{n} \left( \frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right).$$

Canonical Hamiltonian equations:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}.$$

*The first integrals* $F = F(y)$ of this system satisfy the following condition:

$$\dot{F} = \{F, H\} = 0.$$

# Integrable geodesic flow on a 2-surface

Let

$$ds^2 = g_{ij}(x)dx^i dx^j, \qquad i, j = 1, 2$$

be a Riemannian metric on a 2-surface $\mathbb{M}^2$. The geodesic flow is called *integrable* if the Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad H = \frac{1}{2} g^{ij} p_i p_j$$

possesses an additional first integral $F : T^*\mathbb{M}^2 \to \mathbb{R}$ such that

$$\dot{F} = \{F, H\} = \sum_{j=1}^{2} \left( \frac{\partial F}{\partial x^j} \frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial H}{\partial x^j} \right) = 0$$

and $F$ is functionally independent with $H$ almost everywhere.

# Topological obstacles to the complete integrability

**Theorem** (V.V. Kozlov, 1979)
*If a genus of a surface* $\mathbb{M}^2$ *is different from 0 or 1 (that is* $\mathbb{M}^2$ *is homeomorphic neither to a sphere* $\mathbb{S}^2$ *nor to a torus* $\mathbb{T}^2$*), then the geodesic flow of any analytical Riemannian metric on this surface has no first integral which is analytical on* $T^*\mathbb{M}^2$ *and independent on the Hamiltonian.*

# Polynomial in momenta first integrals

It is known that there exist metrics of two types on the 2-torus with an integrable geodesic flow, namely:

$$ds^2 = \Lambda(x)(dx^2 + dy^2), \qquad F_1 = p_2,$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

**Conjecture about degrees of polynomial first integrals** (V.V. Kozlov). The maximal degree of any *irreducible* polynomial in momenta first integral of geodesic flow on a surface of genus $g$ seems to be not larger than $4 - 2g$.

# Cubic first integral

Choose the conformal coordinates $(x, y)$, such that $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$.

$$H = \frac{p_1^2 + p_2^2}{2\Lambda}, \quad F = a_0(x, y)p_1^3 + a_1(x, y)p_1^2 p_2 + a_2(x, y)p_1 p_2^2 + a_3(x, y)p_2^3.$$

The following relations on the metrics and coefficients of the first integral hold:

$$a_2 - a_0 = c_0, \quad a_3 - a_1 = c_1,$$

where $c_0, c_1 \in \mathbb{R}$ are Kolokoltsov constants; moreover,

$$a_1\Lambda_y + 2\Lambda a_{0x} + 3a_0\Lambda_x = 0,$$

$$3a_1\Lambda_y + 2\Lambda a_{1y} + (1 + a_0)\Lambda_x = 0.$$

$$(1 + a_0)\Lambda_y + \Lambda\left(a_{0y} + a_{1x}\right) + a_1\Lambda_x = 0,$$

It can be written in the following form:

$$\begin{pmatrix} 3a_0 & 2\Lambda & 0 \\ 1 + a_0 & 0 & 0 \\ a_1 & 0 & \Lambda \end{pmatrix} \begin{pmatrix} \Lambda \\ a_0 \\ a_1 \end{pmatrix}_x + \begin{pmatrix} a_1 & 0 & 0 \\ 3a_1 & 0 & 2\Lambda \\ 1 + a_0 & \Lambda & 0 \end{pmatrix} \begin{pmatrix} \Lambda \\ a_0 \\ a_1 \end{pmatrix}_y = 0.$$

# Integrable geodesic flow on the 2-torus

**Theorem** (N.V. Denisova, V.V. Kozlov)
*Suppose that the geodesic flow on the 2-torus admits a homogeneous in momenta first integral $F_n$ which is independent on the Hamiltonian. Suppose that*
*1) either $F_n$ is even on $p_1$, $p_2$*
*2) or $F_n$ is even on $p_1(p_2)$ and odd on $p_2(p_1)$,*
*then there exists an additional polynomial in momenta first integral of degree $\leq 2$.*

**Theorem** (N.V. Denisova, V.V. Kozlov)
*Suppose that the geodesic flow on the 2-torus admits a homogeneous in momenta first integral $F_n$ which is independent on the Hamiltonian. The metric $\Lambda(x, y)$ is assumed to be a trigonometric polynomial. Then there exists an additional polynomial in momenta first integral of degree $\leq 2$.*

# Integrable geodesic flow on the 2-torus

**Theorem** (M. Bialy, A.E. Mironov)
*If the Hamiltonian system has an integral F which is a homogeneous polynomial of degree* $n$*, then on the covering plane* $\mathbb{R}^2$ *there exist the global semi-geodesic coordinates* $(t, x)$ *such that*

$$ds^2 = g^2(t,x)dt^2 + dx^2, \qquad H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$$

*and F can be written in the form:*

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k.$$

*Here the last two coefficients can be normalized by the following way:*

$$a_{n-1} = g, \ a_n = 1.$$

# Integrable geodesic flow on the 2-torus

*The condition* $\{F, H\} = 0$ *is equivalent to the quasi-linear PDEs*

$$U_t + A(U)U_x = 0, \qquad (1)$$

*where* $U^T = (a_0, \ldots, a_{n-1}),\ a_{n-1} = g,$

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \ldots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \ldots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \ldots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

# Quasi-linear system of PDEs

Quasi-linear systems of the form

$$A(U)U_x + B(U)U_y = 0,$$

$$U_t = A(U)U_x, \qquad U = (u_1, \dots, u_n)^T$$

appears in such areas like

- gas-dynamics
- non-linear elasticity
- integrable geodesic flows on 2-torus

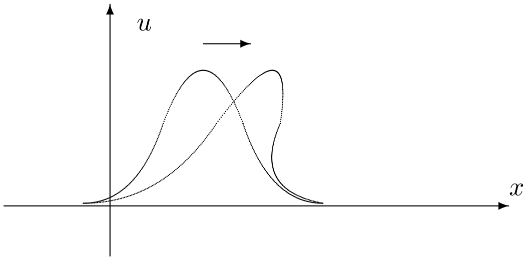
and many others.

# Hopf equation (inviscid Burgers' equation)

Consider the following equation $u_t + uu_x = 0$. The solution of the Cauchy problem $u|_{t=0} = g(x)$ is given by the implicit formula

$$u(x,t) = g(x - ut).$$

It follows from this formula that the higher any point is placed, the faster it is.



# Semi-Hamiltonian systems

**Theorem** (M. Bialy, A.E. Mironov)
*(1) is semi-Hamiltonian system. Namely, there is a regular change of variables*

$$U \mapsto (G_1(U), \ldots, G_n(U))$$

*such that for some* $F_1(U), \ldots, F_n(U)$ *the following conservation laws hold:*

$$(G_i(U))_x + (F_i(U))_y = 0, \qquad i = 1, \ldots, n.$$

*Moreover, in the hyperbolic domain, where eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of A(U) are real and pairwise distinct, there exists a change of variables*

$$U \mapsto (r_1(U), \ldots, r_n(U))$$

*such that the system can be written in Riemannian invariants:*

$$(r_i)_x + \lambda_i(r)(r_i)_y = 0, \qquad i = 1, \ldots, n.$$

**S.P. Tsarev:** the generalized hodograph method.

## Polynomial integrals of the geodesic flow on a 2-surface

**Theorem** (G. Abdikalikova, A.E. Mironov)
*On a 2-surface introduce the coordinates* $ds^2 = g^2(t,x)dt^2 + dx^2$. *The Hamiltonian takes the form* $H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$. *The corresponding geodesic flow has a local polynomial in momenta first integral of the fourth degree:*

$$F_4 = \frac{a_0}{g^4}p_1^4 + \frac{a_1}{g^3}p_1^3 p_2 + \frac{a_2}{g^2}p_1^2 p_2^2 + p_1 p_2^3 + p_2^4.$$

*Here*

$$a_0(t,x) = \frac{3(c_2 + t + 3c_3^2)}{5c_3^2}, \quad a_2(t,x) = -\frac{6(2c_2 + 2t + c_3^2)}{5c_3^2},$$

$$a_1(t,x) = -\frac{3\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2},$$

$$g(t,x) = \frac{2\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2},$$

*where* $c_1, c_2, c_3$ *are arbitrary constants.*

# Magnetic geodesic flow (systems with gyroscopic forces)

$$\frac{d}{dt}y^i = \{y^i, H(y)\}_{mg}, \qquad i = 1, \dots, N.$$

In coordinates $(y^1, \dots, y^N) = (x^1, \dots, x^n, p_1, \dots, p_n)$, $N = 2n$ magnetic Poisson bracket is given by

$$\{x^i, p_j\}_{mg} = \delta^i_j, \qquad \{x^i, x^j\}_{mg} = 0, \qquad \{p_i, p_j\}_{mg} = \Omega_{ij}(x),$$

Consider a Hamiltonian system

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2$$

on the 2-torus in presence of a magnetic field with $H = \frac{1}{2}g^{ij}p_ip_j$ and the Poisson bracket:

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left( \frac{\partial F}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x^i} \right) + \Omega(x^1, x^2) \left( \frac{\partial F}{\partial p_1}\frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2}\frac{\partial H}{\partial p_1} \right).$$

# The only known examples of integrable geodesic flows on the 2-torus on all energy levels

**Integrable geodesic flow**

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad F_1 = p_1;$$

$$ds^2 = (\Lambda_1(x) + \Lambda_2(y))(dx^2 + dy^2), \qquad F_2 = \frac{\Lambda_2 p_1^2 - \Lambda_1 p_2^2}{\Lambda_1 + \Lambda_2}.$$

**Integrable magnetic geodesic flow**

$$ds^2 = dx^2 + dy^2, \quad \omega = B dx \wedge dy, \quad B = const \neq 0, \quad F_1 = \cos\left(\frac{p_1}{B} - y\right);$$

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y) dx \wedge dy, \qquad F_1 = p_1 + u(y).$$

# Magnetic geodesic flow and its integrability

**Theorem** (S.V. Bolotin, V.V. Ten)
*Let* $H = \frac{p_1^2+p_2^2}{2}$ *and the magnetic form* $\omega = \lambda(x,y)dx \wedge dy$*. The magnetic geodesic flow possesses an additional polynomial first integral iff the Fourier spectrum of* $\lambda(x,y)$ *lies on a straight line going through the origin and the average of* $\lambda(x,y)$ *over the whole torus is equal to 0.*

**Consequence** (S.V. Bolotin, V.V. Ten)
*The degree of any irreducible polynomial first integral of such magnetic geodesic flow is equal to 1.*

# Quadratic first integrals on several energy levels

$$H = \frac{p_1^2 + p_2^2}{2\Lambda(x^1, x^2)}, \quad \dot{x}^j = \{x^j, H\}_{mg}, \quad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2.$$

**Theorem** (A., Bialy, Mironov, 2017)
*Consider the magnetic flow of the Riemannian metric* $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$ *with the non-zero magnetic form* $\omega$*. Suppose the magnetic flow admits a first integral* $F_2$ *on all energy levels such that* $F_2$ *is quadratic in momenta. Then in some coordinates we have*

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \omega = -u'(y) dx \wedge dy$$

*so there exists another integral* $F_1$ *which is linear in momenta:* $F_1 = p_1 + u(y)$*, and* $F_2$ *can be written as a combination of* $H$ *and* $F_1$*.*

**I.A. Taimanov:** There is no additional irreducible quadratic first integral with analytic periodic coefficients even on 2 different energy levels!

# Integrals of higher degrees on several energy levels

**Lemma** (A., Valyuzhenich, 2019) *Suppose that the geodesic flow on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral $F$ of an arbitrary degree $N$ on $N+2$ different energy levels $\{H = E_1\}, \{H = E_2\} \dots$. Then $F$ is the first integral of the same flow on all energy levels.*

**Theorem** (A., Valyuzhenich, 2019) *Suppose that the geodesic flow on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral $F$ of an arbitrary degree $N$ with analytic periodic coefficients on $N+2$ different energy levels $\{H = E_1\}, \{H = E_2\} \dots$. Then the magnetic field and the metric are functions of one variable and there exists a linear in momenta first integral $F_1$ on all energy levels.*


S. Agapov, A. Valyuzhenich, "*Polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels*", Disc. Cont. Dyn. Syst. - A, **39**:11 (2019), 6565-6583.

S. Agapov, "*On first integrals of two-dimensional geodesic flows*", Sib. Math. Journ., **61**:4 (2020), 563–574.

# Certain generalizations

**Lemma** *Suppose that the geodesic flow of the metric* $ds^2 = \Lambda(x, y)(dx^2 + dy^2)$ *on a 2-surface admits a rational in momenta first integral*

$$F = \frac{\sum_{k=0}^{M} a_k(x, y) p_1^{M-k} p_2^k}{\sum_{s=0}^{N} b_s(x, y) p_1^{N-s} p_2^s}$$

*with analytic coefficients* $a_k(x, y), b_s(x, y)$. *Denote*

$$f_1 = a_0 - a_2 + a_4 - \dots, \quad g_1 = a_1 - a_3 + a_5 - \dots,$$

$$f_2 = b_0 - b_2 + b_4 - \dots, \quad g_2 = b_1 - b_3 + b_5 - \dots,$$

*here* $a_k = 0$ *while* $k > M$, $b_s = 0$ *while* $s > N$. *Then*

$$\left(\frac{f_1 - ig_1}{f_2 - ig_2}\right)_x - i\left(\frac{f_1 - ig_1}{f_2 - ig_2}\right)_y = 0.$$

S. Agapov, "*On first integrals of two-dimensional geodesic flows*", Sib. Math. Journ., **61**:4 (2020), 563–574.

# Certain generalizations (toward rational integrals)

**Theorem** *Suppose that the geodesic flow of the metric* $ds^2 = \Lambda(x,y)(dx^2 + dy^2)$ *on the 2-torus admits a rational in momenta first integral*

$$F = \frac{a_0(x,y)p_1 + a_1(x,y)p_2}{b_0(x,y)p_1 + b_1(x,y)p_2}.$$

*All the coefficients* $a_k(x,y), b_s(x,y)$ *are assumed to be analytic periodic in both variables functions. Also assume that at least one of them does not vanish anywhere. Then* $\Lambda(x,y)$ *is a constant.*

S. Agapov, "*On first integrals of two-dimensional geodesic flows*", Sib. Math. Journ., **61**:4 (2020), 563–574.

# Certain generalizations (toward rational integrals)

**Lemma** *Suppose that the magnetic geodesic flow on the 2-torus admits a rational in momenta first integral* $F = \frac{Q_N}{R_M}$ *on* $\frac{N+M+1}{2}$ *or* $\frac{N+M+2}{2}$ *(depending on the parity* $N+M$*) pairwise distinct energy levels* $\{H = C_1\}, \{H = C_2\}, \dots$*. Here* $Q_N, R_M$ *are inhomogeneous polynomials in momenta of degrees* $N$ *and* $M$ *accordingly,* $C_j > 0$ *for any* $j$*. Then* $F$ *is the first integral of the same flow on all energy levels simultaneously.*

S. Agapov, "*On first integrals of two-dimensional geodesic flows*", Sib. Math. Journ., **61**:4 (2020), 563–574.

# Certain generalizations (toward rational integrals)

**Theorem** *Suppose that the magnetic geodesic flow of the metric* $ds^2 = \Lambda(x,y)(dx^2 + dy^2)$ *on the 2-torus in a non-zero magnetic field admits a rational in momenta first integral*

$$F = \frac{a_0(x,y)p_1 + a_1(x,y)p_2 + g(x,y)}{b_0(x,y)p_1 + b_1(x,y)p_2 + h(x,y)}$$

*on at least two distinct energy levels. All the coefficients* $a_k(x,y), b_s(x,y)$ *are assumed to be analytic periodic in both variables functions. Also assume that at least one of them does not vanish anywhere. Then there exist a linear in momenta first integral on all energy levels simultaneously.*

S. Agapov, "*On first integrals of two-dimensional geodesic flows*", Sib. Math. Journ., **61**:4 (2020), 563–574.

# Quadratic first integrals on a fixed energy levels

For a Riemannian metric $ds^2 = \Lambda(x,y)(dx^2+dy^2)$ and quadratic in momenta first integral on the 2-torus on a fixed energy level we obtain the following system

$$A(U)U_x + B(U)U_y = 0,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \Lambda \\ u_0 \\ f \\ g \end{pmatrix}.$$

Magnetic field has the form: $\Omega = \frac{1}{4}(g_x - f_y).$

**M. Bialy, A.E. Mironov:** This system is proved to be semi-Hamiltonian.

# The only known explicit non-trivial solution

**Dorizzi B., Grammaticos B., Ramani A. and Winternitz P.:**

$$A(U)U_x + B(U)U_y = 0, \qquad U = (\Lambda, u_0, f, g)^T, \qquad \Omega = \frac{1}{4}(g_x - f_y).$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix}.$$

Explicit solution:

$$U_0(x,y) = \begin{pmatrix} \Lambda(x,y) \\ u_0(x,y) \\ f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 2E - 2h(x,y) \\ -8q(x,y) - 4(E - h(x,y)) \\ -4R'(y) \\ 4S'(x) \end{pmatrix}, \quad \Omega = S''(x) + R''(y),$$

$$h(x,y) = \frac{1}{2}(S')^2 + \frac{1}{2}(R')^2 + SR'' + RS'' + \mu_1 - \mu_2, \quad q(x,y) = \frac{1}{2}(S')^2 + SR'' + \mu_2,$$

here $\mu_1(x,y) = (S')^2 + \frac{1}{2}\beta_2 S^2 - \beta_3 S, \quad \mu_2(x,y) = -(R')^2 - \frac{1}{2}\beta_1 R^2 - \beta_3 R$ and

$$S'' = \alpha S^2 + \beta_1 S + \gamma_1, \quad R'' = -\alpha R^2 + \beta_2 R + \gamma_2.$$

# Quadratic first integrals on a fixed energy level

**Theorem** (A., Bialy, Mironov, 2017)
*There exist real analytic Riemannian metrics on the 2-torus which are arbitrary close to the Liouville metrics (and different from them) and a non-zero analytic magnetic fields such that magnetic geodesic flows on the energy level* $\{H = \frac{1}{2}\}$ *have polynomial in momenta first integral of degree two.*

# Toward the search for first integrals of higher degrees on a fixed energy level


H.M. Yehia, "*On certain two-dimensional conservative mechanical systems with a cubic second integral*", Journ. of Phys. A, **35** (2002), 9469–9487.

A.A. Elmandouh, H.M. Yehia, "*New integrable systems with a quartic integral and new generalizations of Kovalevskaya's and Goriatchev's cases*", Reg. Chaot. Dyn, **13**:1 (2008), 57–69.

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Thank you for your attention!