Symplectic Khovanov homology and Jones polynomial

VIII Russian-Chinese Conference on Knot Theory and Related Topics

Zhechi Cheng

Wuhan University

December 25, 2021

Table of Contents

Background–Link Resolutions

A link is an embedding $L : \sqcup S^1 \to \mathbb{R}^3$.

A (planar) link diagram D is a generic projection of L to $\mathbb{R}^2$, where the self-intersections can be presented as isolated crossings.

A resolution at a crossing is a choice of replacing the crossing with one of the following:

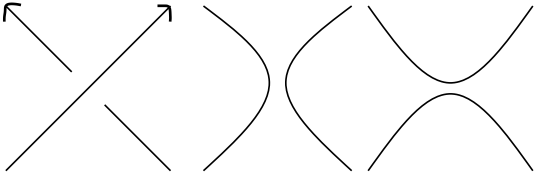

Figure: Resolution of K_+ at a crossing into K_0 and K_∞

Background–Jones Polynomial

Definition (Jones 1987, [5], Kauffman 1987, [6])

Jones polynomial for a link diagram D is $V(L) = t^{\frac{-3w(D)}{4}} \sum_d t^{\frac{ind(d)}{2}} (-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{n(d)}$ where $w(D)$ is the writhe of D, $ind(d)$ is the number of ∞-resolution minus that of 0-resolution, and $n(d)$ is the number of circles in the resolution diagram d.

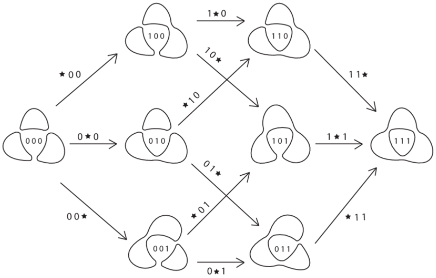

Figure: Resolutions of a trefoil, [9]

Background–Khovanov homology

Definition (Khovanov 2000, [7])

Khovanov homology $Kh^{i,j}(L)$ is a link invariant which categorifies Jones polynomial $V(L)$, i.e. $V(L) = t^j \sum_i (-1)^i dim Kh^{i,j}(L)$.

Background–Khovanov homology

Definition (Khovanov 2000, [7])

Khovanov homology $Kh^{i,j}(L)$ is a link invariant which categorifies Jones polynomial $V(L)$, i.e. $V(L) = t^j \sum_i (-1)^i dim Kh^{i,j}(L)$.

Jones polynomial

$(-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^n$

Multiplication with term t^k

Cancellation of terms of different signs

Skein relations of Jones polynomials

Khovanov homology

$\otimes^n(\mathbb{F}[\frac{1}{2}] \oplus \mathbb{F}[-\frac{1}{2}])$

Grading shift $C \to C[k]$

Differentials

Long exact sequences of homology groups

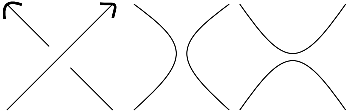

Figure: Unoriented skein relation K_+, K_0 and K_∞.

Background–Alternate approaches to $Kh^{*,*}(L)$

Symplectic Geometry

Symplectic Khovanov homology, see Seidel-Smith 2004 [8], Abouzaid-Smith 2019 [1] and etc.

Background–Alternate approaches to $Kh^{*,*}(L)$

Symplectic Geometry

Symplectic Khovanov homology, see Seidel-Smith 2004 [8], Abouzaid-Smith 2019 [1] and etc.

Complex Geometry

Derived category of coherent sheaves, see Cautis-Kamnitzer 2008 [2] and etc.

Background–Alternate approaches to $Kh^{*,*}(L)$

Symplectic Geometry

Symplectic Khovanov homology, see Seidel-Smith 2004 [8], Abouzaid-Smith 2019 [1] and etc.

Complex Geometry

Derived category of coherent sheaves, see Cautis-Kamnitzer 2008 [2] and etc.

Gauge Theory

Instanton Floer homology, see Witten 2011 [10], Xie 2021 [11] and etc.

Background–Symplectic Khovanov homology

Definition (Seidel-Smith 2004, [8])

Symplectic Khovanov homology $Kh^*_{symp}(L)$ is a singly graded link invariant.

Background–Symplectic Khovanov homology

Definition (Seidel-Smith 2004, [8])

Symplectic Khovanov homology $Kh^{*}_{symp}(L)$ is a singly graded link invariant.

Conjecture (Seidel-Smith 2004, [8])

Symplectic Khovanov homology is isomorphic to Khovanov homology $Kh^{k}_{symp}(L) \cong \bigoplus_{i-j=k} Kh^{i,j}(L)$.

Background–Symplectic Khovanov homology

Definition (Seidel-Smith 2004, [8])

Symplectic Khovanov homology $Kh^{*}_{symp}(L)$ is a singly graded link invariant.

Conjecture (Seidel-Smith 2004, [8])

Symplectic Khovanov homology is isomorphic to Khovanov homology $Kh^{k}_{symp}(L) \cong \bigoplus_{i-j=k} Kh^{i,j}(L)$.

Theorem (Abouzaid-Smith 2019, [1])

The conjecture above is true for charateristic-0 fields $\mathbb{F}$.

Main result

Theorem (C. 2020, [3])

There is a well-defined relative second grading, called **relative weight grading** *on symplectic Khovanov homology.*

Main result

Theorem (C. 2020, [3])

There is a well-defined relative second grading, called **relative weight grading** *on symplectic Khovanov homology.*

Theorem (C. 2021, [4])

Over any characteristic-0 field $\mathbb{F}$, there is an isomorphism between symplectic Khovanov homology and Khovanov homology as doubly-graded homology theories, such that

$$Kh^{i,j}(L) \cong Kh_{symp}^{i-j,-j}(L)$$

Main result

Theorem (C. 2020, [3])

There is a well-defined relative second grading, called **relative weight grading** *on symplectic Khovanov homology.*

Theorem (C. 2021, [4])

Over any characteristic-0 field $\mathbb{F}$, there is an isomorphism between symplectic Khovanov homology and Khovanov homology as doubly-graded homology theories, such that

$$Kh^{i,j}(L) \cong Kh_{symp}^{i-j,-j}(L)$$

Corollary

Symplectic Khovanov homology categorifies Jones polynomial.

Construction–Basic Assumptions

The base field $\mathbb{K}$ is characteristic-0, such as $\mathbb{Q}$ or $\mathbb{C}$.

The symplectic manifold $\mathcal{Y}_n = Hilb^n(S)\backslash D_r$, where S is a complex surface defined by equation

$$\{(u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + (z - p_1)\ldots(z - p_{2n}) = 0\} \in \mathbb{C}^3$$

and D_r consists of all the elements of length less than n.

The link $L \subset \mathbb{R}^3$ is represented by a *bridge diagram* $(\vec{\alpha}, \vec{\beta}, \vec{p})$

Construction–Bridge diagram $(\vec{\alpha}, \vec{\beta}, \vec{p})$

Let $\vec{p}$ be $2n$ points on the plane.

Let $\vec{\alpha}$ be a set of n pairwise disjoint arcs whose boundary is exactly $\vec{p}$.

Let $\vec{\beta}$ be a set of n pairwise disjoint arcs whose boundary is exactly $\vec{p}$.

Let β-arcs surpass α-arcs at the intersections.

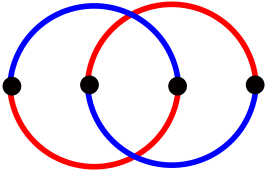

Figure: A bridge diagram for Hopf link

Construction–Symplectic Khovanov $Kh^*_{symp}(L)$

Consider a complex surface $S = \{(u, v, z) \in \mathbb{C}^3 | u^2 + v^2 + \prod(z - p_i) = 0\}$.

S is a Lefschetz fibration $\pi_z : S \to \mathbb{C}$, with singular fibers at p_i.

Each arc α_i or β_i determines uniquely a Lagrangian sphere Σ_{α_i}, Σ_{β_i} respectively, in S

n pairwise-disjoint arcs α_i or β_i determines uniquely a Lagrangian in $\mathcal{Y}_n = Hilb^n(S) \backslash D_r$

$$\mathcal{K}_\alpha = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \ldots \times \Sigma_{\alpha_n}$$

$$\mathcal{K}_\beta = \Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \ldots \times \Sigma_{\beta_n}$$

Define $Kh^*_{symp}(L) = HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$.

Construction–Symplectic Khovanov $Kh^*_{symp}(L)$

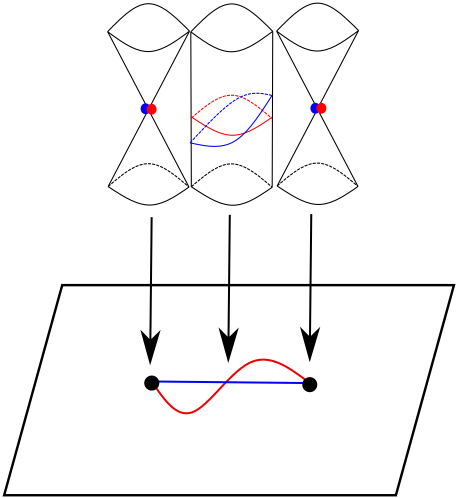

Figure: A schematic for $\mathcal{Y}_1 = Hilb^1(S) = S$.

Construction–Review Floer homology $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$

$HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ is the Lagrangian intersection Floer homology of Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$

The generators are the intersections of the two Lagrangians. If we think $Hilb^n(S)$ as the n-fold product of S, each of these intersections consists of n points from the intersection of α-arcs and β-arcs such that there will be exactly 1 point on each arc.

In the case of the unknot in the previous slide, there are two generators are from the end point intersections and two other generators from the interior intersection.

Construction–Weight grading

In order to define the weight grading, we need to work with some additional information, so we partially compactify $\mathcal{Y}_n$

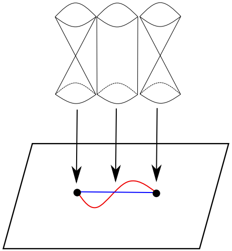

Figure: A schematic for $\mathcal{Y}_n$, in the case of $n = 1$

Construction–Weight grading

We partially compactify $\mathcal{Y}_n$ to $\bar{\mathcal{Y}}_n$

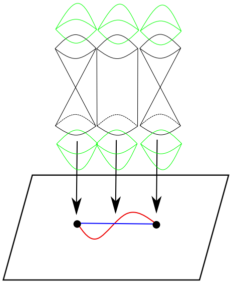

Figure: A schematic for $\mathcal{Y}_n$ compactified to $\bar{\mathcal{Y}}_n$

Construction–Weight grading

We partially compactify $\mathcal{Y}_n$ to $\bar{\mathcal{Y}}_n$

We consider Moduli space of disks with two inner marked points $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$

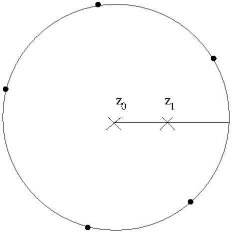

Figure: The domain of $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$

Construction–Weight grading

We partially compactify $\mathcal{Y}_n$ to $\bar{\mathcal{Y}}_n$

We consider Moduli space of disks with two inner marked points $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$, such that z_0 mapped to D_0 and z_1 mapped to D_0'. The intersection of the disk u and D_0 is 1.

Counting curves in $\mathcal{R}^2_{(0,1)}(x; y)$ with some refinement terms, we get an automorphism ϕ on $CF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$, which is a chain map of degree 0.

ϕ induces an automorphism Φ on $HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$.

The weight grading $wt(x)$ of a (generalized) eigenvector $x \in HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ is its (generalized) eigenvalue.

Construction–Weight grading

Proposition (Abouzaid-Smith 2019 [1], C. 2020 [3])

Weight grading is compatible with Floer products, i.e. if $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_2$ be compact Lagrangians given by crossingless matchings in $\mathcal{Y}_n$. For any eigenvector $x \in HF^*(\mathcal{K}_1, \mathcal{K}_2)$ and $y \in HF^*(\mathcal{K}_0, \mathcal{K}_1)$, we have

$$wt(xy) = wt(x) + wt(y)$$

.

Proof.

By counting curves in dimension-1 boundary strata, we have $\phi\mu^2(x, y) = \mu^2(\phi(x), y) + \mu^2(x, \phi(y))$ or equivalently $wt(xy)xy = (wt(x)x)y + x(wt(y)y) = (wt(x) + wt(y))xy$ □

Proof–Main Ingredients

Abouzaid-Smith singly graded isomorphism $\Phi : Kh^*_{symp} \to Kh^*$ (after gradings collapse.)

Long exact sequence that respects weight grading

$$\ldots \to Kh^{*,j_1}_{symp}(K_+) \to Kh^{*,j_2}_{symp}(K_0) \to Kh^{*,j_3}_{symp}(K_\infty) \to \ldots$$

Bigraded Khovanov homology long exact sequence

$$\ldots \to Kh^{*,j_1}(K_+) \to Kh^{*,j_2}(K_0) \to Kh^{*,j_3}(K_\infty) \to \ldots$$

Proof–Bigraded long exact sequence

Let β, γ and δ curves differ only in the shown region in the following figure. We perturb all other arcs not shown in the diagram such that they only intersect at end points.

We can deduce from Abouzaid and Smith's result in [1] that $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$, and $\mathcal{K}_\delta$ forms an exact triangle of Lagrangians.

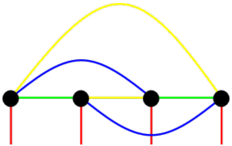

Figure: Exact triangle of Lagrangians. The red, blue, green and yellow are α, β, γ and δ curves. $\alpha + \beta = K_+$, $\alpha + \gamma = K_0$ and $\alpha + \delta = K_\infty$.

Proof–Bigraded long exact sequence

In other words, we can prove the following exact sequence

$$\ldots \xrightarrow{c_1} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \xrightarrow{c_2} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\gamma) \xrightarrow{c_3} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\delta) \xrightarrow{c_1} \ldots \quad (1)$$

or equivalently

$$\ldots \xrightarrow{c_1} Kh^*_{symp}(K_+) \xrightarrow{c_2} Kh^*_{symp}(K_0) \xrightarrow{c_3} Kh^*_{symp}(K_\infty) \xrightarrow{c_1} \ldots \quad (2)$$

In particular, there are elements $c_1 \in CF^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in CF^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $c_3 \in CF^*(\mathcal{K}_\delta, \mathcal{K}_\gamma)$ such that the maps above are Floer products with the corresponding elements.

Proof–Bigraded long exact sequence

In other words, we can prove the following exact sequence

$$\dots \xrightarrow{c_1} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \xrightarrow{c_2} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\gamma) \xrightarrow{c_3} HF^*(\mathcal{K}_\alpha, \mathcal{K}_\delta) \xrightarrow{c_1} \dots \quad (1)$$

or equivalently

$$\dots \xrightarrow{c_1} Kh^*_{symp}(K_+) \xrightarrow{c_2} Kh^*_{symp}(K_0) \xrightarrow{c_3} Kh^*_{symp}(K_\infty) \xrightarrow{c_1} \dots \quad (2)$$

In particular, there are elements $c_1 \in CF^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in CF^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $c_3 \in CF^*(\mathcal{K}_\delta, \mathcal{K}_\gamma)$ such that the maps above are Floer products with the corresponding elements.

We prove that the sum of weight grading of c_1, c_2 and c_3 is 0. Alternatively, we can show that $\mu^3(c_2, c_3, c_1) = e_{\mathcal{K}_\beta}$.

Proof–Bigraded isomorphism

Abouzaid and Smith construct an isomorphism $\Phi : Kh^*_{symp}(K) \to Kh^*(K)$ that commutes the long exact sequences:

$$\begin{array}{ccccccccc}
HF^*(\mathcal{K},\mathcal{K}_\gamma) & \longrightarrow & HF^{*+2}(\mathcal{K},\mathcal{K}_\delta) & \longrightarrow & HF^{*+1}(\mathcal{K},\mathcal{K}_\beta) & \longrightarrow & HF^{*+1}(\mathcal{K},\mathcal{K}_\gamma) & \longrightarrow & HF^{*+3}(\mathcal{K},\mathcal{K}_\delta) \\
\Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \downarrow\Phi \\
Kh^*(K_0) & \longrightarrow & Kh^{*+2}(K_\infty) & \longrightarrow & Kh^{*+1}(K_+) & \longrightarrow & Kh^{*+1}(K_0) & \longrightarrow & Kh^{*+3}(K_\infty)
\end{array}$$

Proof–Bigraded isomorphism

Abouzaid and Smith constructed an isomorphism $\Phi : Kh^*_{symp}(K) \to Kh^*(K)$ that commutes the long exact sequences:

For diagrams without any crossing, they computed Φ such that it is an isomorphism as bigraded groups. With the homological grading $k = i - j$ and weight grading $wt = -j$.

Proof–Bigraded isomorphism

Abouzaid and Smith constructed an isomorphism $\Phi : Kh^*_{symp}(K) \to Kh^*(K)$ that commutes the long exact sequences:

For diagrams without any crossing, they computed Φ such that it is an isomorphism as bigraded groups. With the homological grading $k = i - j$ and weight grading $wt = -j$.

Consider following diagram when we decompose the long exact sequences with respect to weight or Jones gradings.

$$\begin{array}{ccccccccc}
HF^{*,-j_1}(\mathcal{K},\mathcal{K}_\gamma) & \xrightarrow{c_2} & HF^{*+2,-j_2}(\mathcal{K},\mathcal{K}_\delta) & \xrightarrow{c_3} & HF^{*+1,j'_3}(\mathcal{K},\mathcal{K}_\beta) & \xrightarrow{c_1} & HF^{*+1,-j_1}(\mathcal{K},\mathcal{K}_\gamma) & \xrightarrow{c_2} & HF^{*+3,-j_2}(\mathcal{K},\mathcal{K}_\delta) \\
\Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \downarrow\Phi \\
Kh^{*,j_1}(K_0) & \longrightarrow & Kh^{*+2,j_2}(K_\infty) & \longrightarrow & Kh^{*+1,j_3}(K_+) & \longrightarrow & Kh^{*+1,j_1}(K_0,j_1) & \longrightarrow & Kh^{*+3,j_2}(K_\infty)
\end{array}$$

Proof–Bigraded isomorphism

We use induction on number of crossings.

The base case of diagrams of no crossing is true because of Abouzaid-Smith [1].

K_0 and K_∞ have one less crossing than K_+, see FIgure 9. The induction hypotheses imply that the first, second, fourth and fifth columns are isomorphism. By five lemma, the third row should also be isomorphism. This finishes the proof.

$$\begin{array}{ccccccccc}
HF^{*,-j_1}(\mathcal{K},\mathcal{K}_\gamma) & \xrightarrow{c_2} & HF^{*+2,-j_2}(\mathcal{K},\mathcal{K}_\delta) & \xrightarrow{c_3} & HF^{*+1,j_3'}(\mathcal{K},\mathcal{K}_\beta) & \xrightarrow{c_1} & HF^{*+1,-j_1}(\mathcal{K},\mathcal{K}_\gamma) & \xrightarrow{c_2} & HF^{*+3,-j_2}(\mathcal{K},\mathcal{K}_\delta) \\
\Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \Phi\downarrow & & \downarrow\Phi \\
Kh^{*,j_1}(K_0) & \longrightarrow & Kh^{*+2,j_2}(K_\infty) & \longrightarrow & Kh^{*+1,j_3}(K_+) & \longrightarrow & Kh^{*+1,j_1}(K_0,j_1) & \longrightarrow & Kh^{*+3,j_2}(K_\infty)
\end{array}$$

References I

Mohammed Abouzaid and Ivan Smith.
Khovanov homology from Floer cohomology.
J. Amer. Math. Soc., 32(1):1–79, 2019.

Sabin Cautis and Joel Kamnitzer.
Knot homology via derived categories of coherent sheaves, i: The sl(2)-case.
Duke Math. J., 142(3):511–588, 2008.

Zhechi Cheng.
Jones grading from symplectic Khovanov homology.
Ph. D. Thesis.

Zhechi Cheng.
Bigradings the symplectic Khovanov cohomology.
Preprint, arxiv: 2111.07792.

References II

Vaughan Jones.
Hecke algebra representations of braid groups and link polynomials.
Annals of Math., 126(2):335–388, 1987.

Louis H. Kauffman.
State models and the Jones polynomial.
Topology, 26(3):395–407, 1987.

Mikhail Khovanov.
A categorification of the Jones polynomial.
Duke Math. J., 101:359–426, 3 2000.

Paul Seidel and Ivan Smith.
A link invariant from the symplectic geometry of nilpotent slices.
Duke Math. J., 134(3):453–514, 2006.

References III

Aditi Vashist.
Torsion in Kovanov homology.
Master Thesis, 2008.

Edward Witten.
Khovanov homology and gauge theory.
Geometry and Topology Monographs, 18, 291-308 2011.

Yi Xie.
Instantons and annular Khovanov homology.
Adva. in Math., 388, 2021.

The End

Thank you for listening!