On checkerboard colorability and arrow polynomial

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VIII Chinese-Russian Conference on Knot Theory and Related Topics
Dec 26, 2021

Outline

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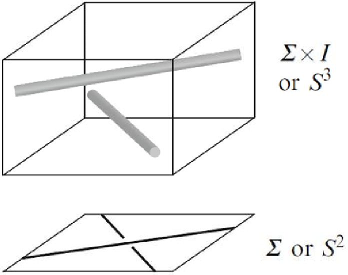
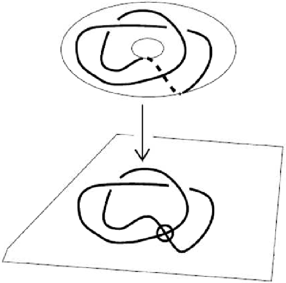
- A *knot* is meant a smooth embedding of the circle S^1 in $\mathbb{R}^3$ (or in the sphere S^3) as well as the image of this embedding. A *link* is a smooth embedding (image) of several disjoint circles in $\mathbb{R}^3$.
- A *virtual knot* is a knot in thickened orientable surface $\Sigma * I$ (where Σ is orientable surface and $I = [0, 1]$).
- A *twisted knot* is a knot in thickened surface $\Sigma * I$ (where Σ is orientable or non-orientable surface and $I = [0, 1]$).

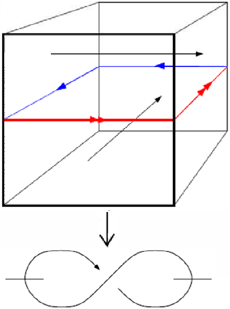
[1] L. H. Kauffman, Virtual knot theory, European J. Comb. 20 (1999) 663-690.

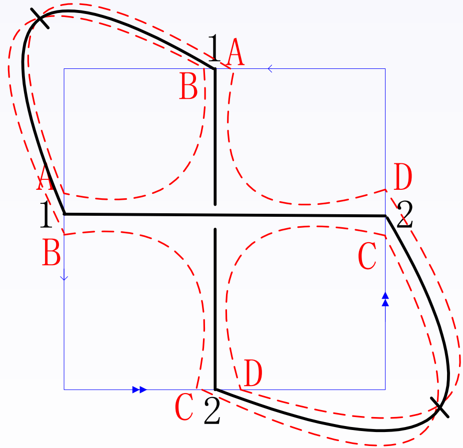
[2] M. O. Bourgoin, Twisted link theory, Algebr. Geom. Topol. 8(3) (2008) 1249-1279.



Moving between links and link diagrams






1
A
B
A
D
1
2
B
C
D
C
2

Abstract link diagram

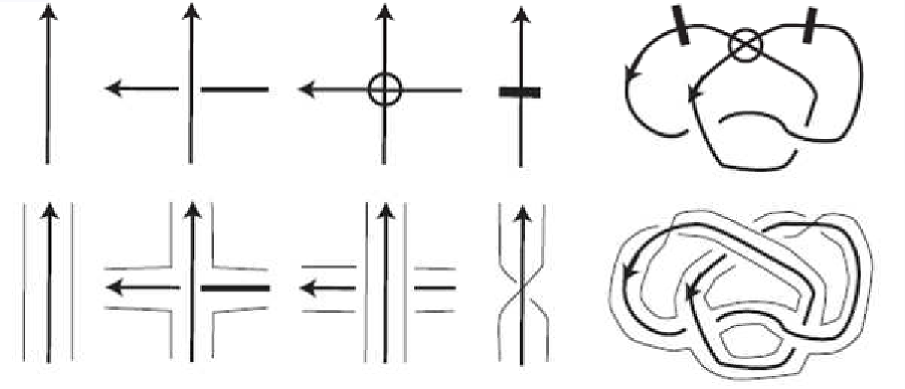
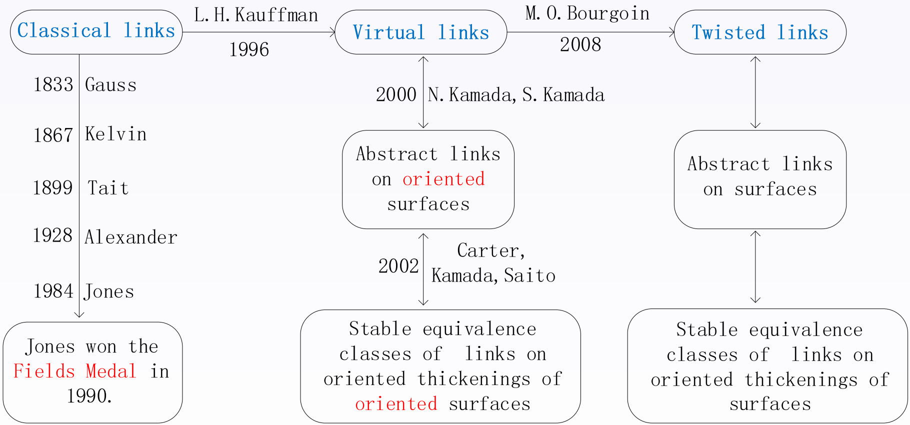


Figure: A twisted link diagram and its associated abstract link diagram.


Classical links
L.H.Kauffman
1996
Virtual links
M.O.Bourgoin
2008
Twisted links
1833 Gauss
1867 Kelvin
1899 Tait
1928 Alexander
1984 Jones
Jones won the Fields Medal in 1990.
2000 N.Kamada, S.Kamada
Abstract links on oriented surfaces
2002 Carter, Kamada, Saito
Stable equivalence classes of links on oriented thickenings of oriented surfaces
Abstract links on surfaces
Stable equivalence classes of links on oriented thickenings of surfaces

- L. Kelvin, On vortex atoms, Phil. Mag. 34 (1867) 15-24.
- J. W. Alexander, Topological invariants of knots and links, Trans. Amer Math. Soc 30(2) (1928) 275-306.
- V.F.R. Jones, A polynomial invariant for knots via Von Neumann algebras, Bull. Amer. Math. Soc. 12(1) (1985) 103-111.
- N. Kamada, S. Kamada, Abstract link diagrams and virtual knots, J. Knot Theory Ramifications 9(1) (2000) 93-106.
- J. S. Carter, S. Kamada and M. Saito, Stable equivalences of knots on surfaces and virtual knot cobordisms, J. Knot Theory Ramifications 11(3) (2002) 311-322.

- A *twisted link diagram* is a 4-regular plane graph whose edges may contain bars and vertices are endowed with the following structure: each vertex either has an over-crossing and under-crossing or is marked by a virtual crossing.

- A *virtual link diagram* is a twisted link diagram without bars.
- A *link diagram* is a virtual link diagram without virtual crossings.

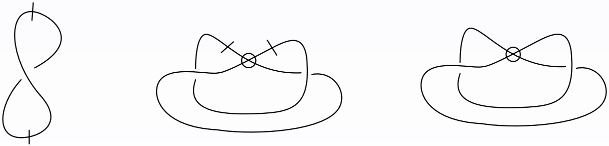
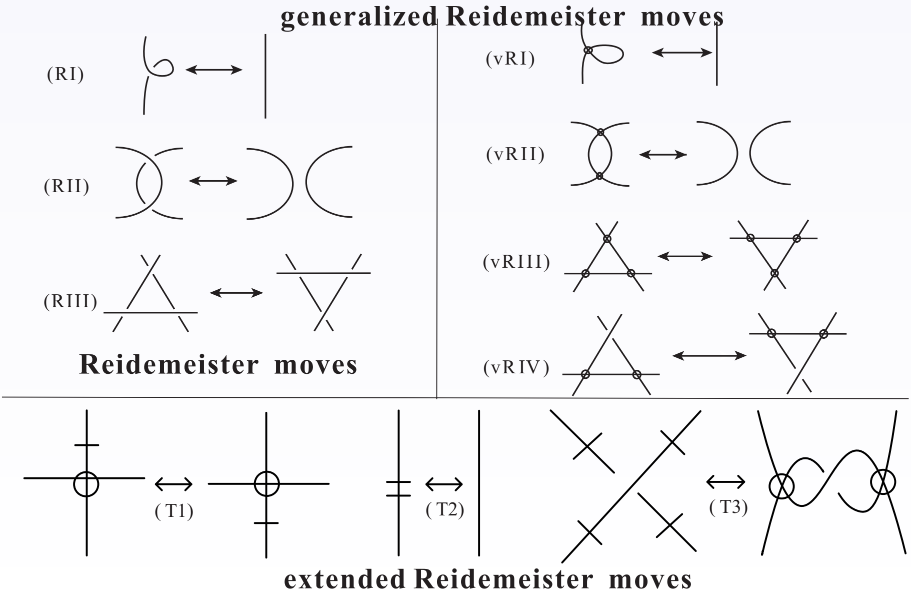


Figure: A onefoil twisted knot (left), a nonorientable twisted knot (middle), a virtual knot (right).

Reidemeister moves



- Two twisted link diagrams are said to be *equivalent* if they are related by a finite sequence of extended Reidemeister moves.
- A *twisted link* is an equivalence class of twisted link diagrams modulo extended Reidemeister moves.

- Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves.
- A *virtual link* is an equivalence class of virtual link diagrams modulo generalized Reidemeister moves.

Outline

- The *faces* of a twisted link diagram are closed curves that run along the immersed curve and have the relationship with the crossings, virtual crossings, and bars.

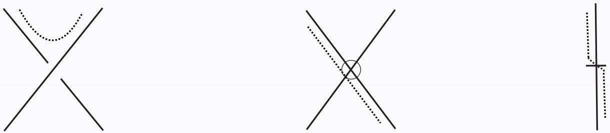


Fig.: Faces and crossings, virtual crossings, and bars.

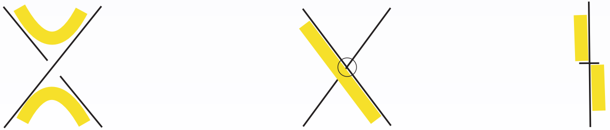


Fig.: Color on crossings, virtual crossings, and bars.

- A twisted link diagram is *checkerboard colorable* if its faces can be assigned one of two colors such that the arcs of the link diagram between two crossings always separate faces of one color from those of the other.
- An twisted link is said to be *checkerboard colorable* if it has a checkerboard colorable twisted link diagram.

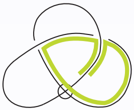
[1] M. O. Bourgoin, Twisted link theory, Algebr. Geom. Topol. 8(3)(2008) 1249-1279.

N. Kamada and V.O. Manturov independently introduced the notion of*checkerboard colorablilty* of a virtual link diagram.

[1] N. Kamada, On the Jones polynomials of checkerboard colorable virtual links, Osaka J. Math. 39(2) (2002) 325-333.

[2] V.O. Manturov, Knot theory, Chapman Hall/CRC CRC Press LLC, 2004.

- All classical link diagrams and classical links are checkerboard colorable.
- The checkerboard colorability of a virtual link diagram is not necessarily preserved by generalized Reidemeister moves.



Outline

Let L be an oriented virtual link with n components. If L is checkerboard colorable, then

- Jones polynomial: $EXP(V_L(t))$ is integer if n is odd, otherwise is half-integer if n is even.
- Sawollek (Alexander-Conway) polynomial: $Z_L(x, 1) = 0$ and $Z_L(x, x) = 0$

When $n = 1$,

- index polynomial: $Q(L) \in Z[t^2]$.
- index polynomial of the flat virtual link $\overline{L}$ of L: $FQ(L) \in Z[t^2]$.

- N. Kamada, On the Jones polynomials of checkerboard colorable virtual links, Osaka J. Math. 39(2) (2002) 325-333.
- T. Imabeppu, On Sawollek polynomials of checkerboard colorable virtual links, J. Knot Theory Ramifications, 25(2)(2016)ArticleID: 1650010, 19pp.
- Y. H. Im, K. Lee and S. Y. Lee, Index polynomial invariant of virtual links, J. Knot Theory Ramifications 19(5) (2010) 709-725.
- Y. H. Im, K. Lee and H. Son, An index polynomial invariant for flat virtual knots, European J. Combin. 31(8) (2010) 2130-2140.

- In 2016, Imabeppu explained that the above first five criteria are independent from the others.
- Imabeppu determined checkerboard colorability of virtual knots up to four classical crossings, with seven exceptions (that is, virtual knots 4.55, 4.56, 4.59, 4.72, 4.76, 4.77, 4.96).

Outline

Arrow polynomial of an oriented virtual link diagram was defined by Dye and Kauffman in 2009.

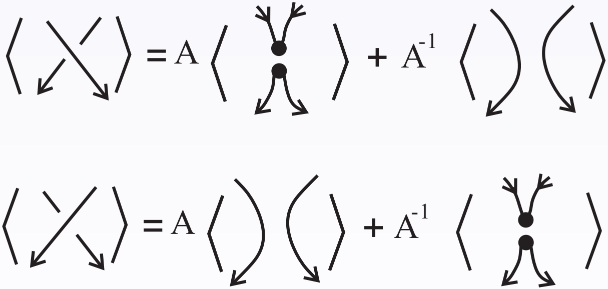


Figure: Oriented state expansion.

[1] H. A. Dye, L. H. Kauffman, Virtual crossing number and the arrow polynomial, J. Knot Theory Ramifications 18(10) (2009) 1335-1357.

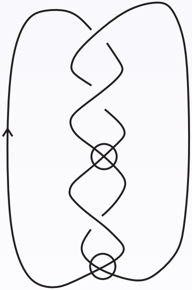


Figure: Virtualized trefoil 3.7^*.

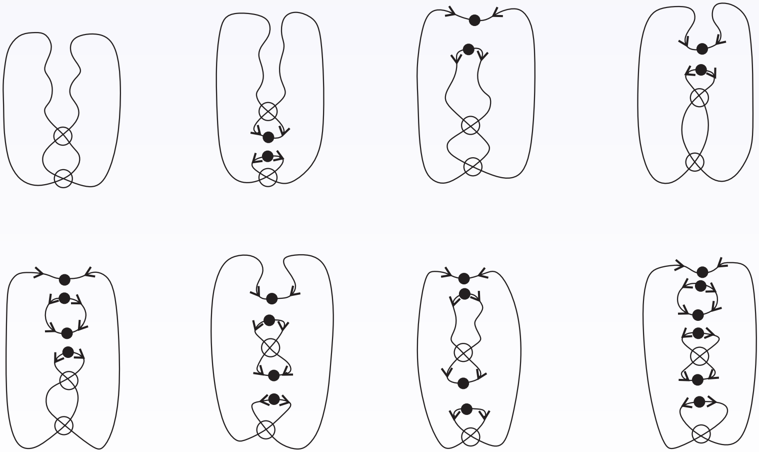


Figure: States of the virtualized trefoil 3.7^*.

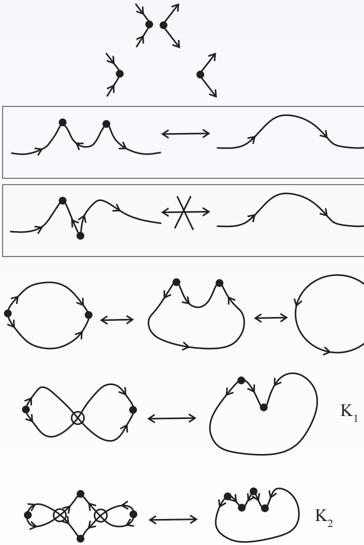


Figure: Reduction rule for the arrow polynomial.

Definition

The Arrow polynomial $\langle D\rangle_A$ of an oriented virtual link diagram D is defined by

$$\langle D\rangle_A = \sum_S A^{\alpha-\beta} d^{|S|-1}\langle S\rangle \tag{1}$$

where S runs over the oriented bracket states of the diagram, α denotes the number of smoothings with coefficient A in the state S and β denotes the number with coefficient A^{-1}, $d = -A^2 - A^{-2}$, $|S|$ is the number of circle graphs in the state, and $\langle S\rangle$ is a product of extra variables $K_1, K_2, \ldots$ associated with the non-trivial circle graphs in the state S.

Theorem (Dye and Kauffman, 2009)

Let D be an oriented virtual link diagram. $\langle D\rangle_A$ is an invariant under the generalized Reidemeister moves except RI.

Let D be an oriented virtual link diagram with writhe $\omega(D)$. The normalized version is defined by

$$\langle D\rangle_{NA} \;=\; (-A^3)^{-\omega(D)}\langle D\rangle_A. \tag{2}$$

For an oriented virtual link L represented by an oriented link diagram D, we denote $\langle D\rangle_{NA}$ by $\langle L\rangle_{NA}$, and call it the *normalized arrow polynomial* of L.

Theorem (Dye and Kauffman, 2009)

Let L be an oriented virtual link. Then $\langle L\rangle_{NA}$ is an invariant under the generalized Reidemeister moves.

$$\langle 3.7^* \rangle_{NA} \;=\; -A^{-3}(-A^{-5} + K_1^2 A^{-5} - K_1^2 A^3). \tag{3}$$

Note that if a reduced state has the form:

$$A^m d^l (K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}). \tag{4}$$

Then the *k-degree of the state* is:

$$i_1 \times j_1 + i_2 \times j_2 + \cdots + i_v \times j_v. \tag{5}$$

- A *surviving state* is a summand of $\langle D \rangle_A$.

- Let $AS(D)$ denote the set of k-degrees obtained from the set of surviving states of a diagram D.
- If the summand has no K_C variables, then the k-degree is zero.

Example

(1) If $A^3K_1K_4$ is a summand of $\langle D\rangle_A$, then 5 is an element of $AS(D)$.
(2) If a virtual link L has a total of 4 summands with subscripts summing to: 2, 2, 1, 0 then $AS(L) = \{2, 1, 0\}$.

Lemma (Dye and Kauffman, 2009)

For an oriented virtual link diagram D, $AS(D)$ is invariant under the generalized Reidemeister moves.

Dye and Kauffman showed that the phenomenon of cusped states and extra variables K_n only occurs for virtual links.

Theorem (Dye and Kauffman, 2009)

If D be a classical link diagram, then $AS(D) = \{0\}$.

Theorem (Deng, Jin and Kauffman, 2021)

Let D be an oriented checkerboard colorable virtual link diagram. Then $\langle D\rangle_{NA}$ has

(1) $AS(D)$ only contains even integer;

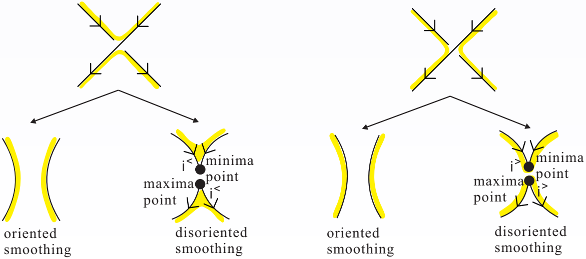
(2) for any summand $A^s K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}$ with $1 \leq i_1 < i_2 < \cdots < i_v$, $j_t \geq 1$ for $t = 1, 2, \cdots, v$, and $v \geq 1$ of $\langle D\rangle_{NA}$, we have $2i_v \leq \sum_{t=1}^{v} i_t \cdot j_t$.

[1] Q. Deng, X. Jin, L. H. Kauffman, On arrow polynomials of checkerboard colorable virtual links, J. Knot Theory Ramifications, 30(7)(2021)ArticleID:2150053, 17pp.

Proof

(1) Braiding diagram and its closure are both checkerboard colorable, introduced by L. H. Kauffman and S. Lambropoulou in 2004.

(2) Claim 1. For any circle graph of σ contains maxima point and minima point alternatively. If the circle graph is reduced, it still holds.
Claim 2. If $i^{<}Xi^{<}Y$ or $i^{>}Xi^{>}Y$ is a word of a circle graph of σ, then X and Y are both of even length.
Claim 3. For any reduced circle graph of σ, it is trivial or the labels of its cusps are all different.
Claim 4. For the state σ, its k-degree is an even integer.



$$
\begin{aligned}
\langle 4.55\rangle_{NA} &= A^4 + A^{-4} + 1 - (A^4 + A^{-4} + 2)K_1^2 + 2K_2 \\
\langle 4.56\rangle_{NA} &= A^4\left(-\left({K_1}^2 - 1\right)\right) + \frac{1 - {K_1}^2}{A^4} - 2{K_1}^2 + 2K_2 + 1 \\
\langle 4.59\rangle_{NA} &= \frac{A^8\left(K_2 - {K_1}^2\right) + A^4\left(3 - 2{K_1}^2\right) - {K_1}^2 + K_2}{A^4} \\
\langle 4.72\rangle_{NA} &= 1 \\
\langle 4.76\rangle_{NA} &= \frac{A^8\left(K_2 - {K_1}^2\right) + A^4\left(3 - 2{K_1}^2\right) - {K_1}^2 + K_2}{A^4} \\
\langle 4.77\rangle_{NA} &= \frac{A^8\left(K_2 - {K_1}^2\right) + A^4\left(3 - 2{K_1}^2\right) - {K_1}^2 + K_2}{A^4} \\
\langle 4.96\rangle_{NA} &= \frac{{K_1}^2}{A^6} + A^4(K_3 - K_1K_2) - A^2\left({K_1}^2 - 1\right) - K_1K_2 + K_1.
\end{aligned}
$$

4.55, 4.56, 4.59, 4.76, 4.77: non-checkerboard colorable.
Micah W. Chrisman: 4.72 is checkerboard colorable by (Boden-Gaudreau-Harper-Nicas-White, https://arxiv.org/pdf/1506.01726.pdf, Theorem 8.3).

Outline

N. Kamada had defined a polynomial according to the pole diagram of a twisted link diagram in 2012, here we apply the notion of Kauffman and Dye to reformulate the arrow polynomial.

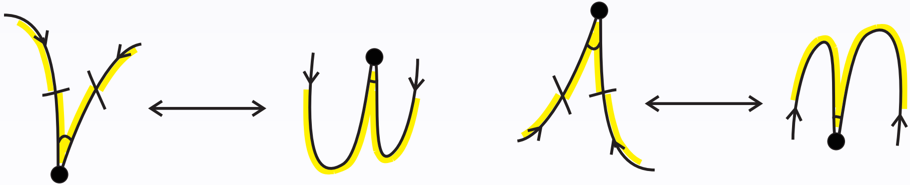
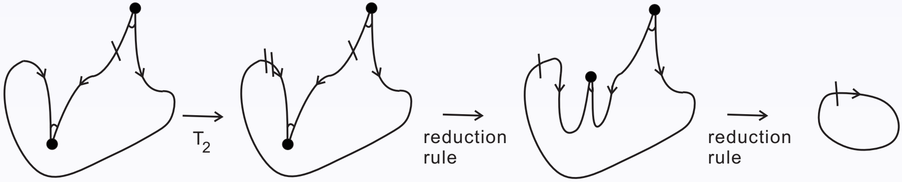
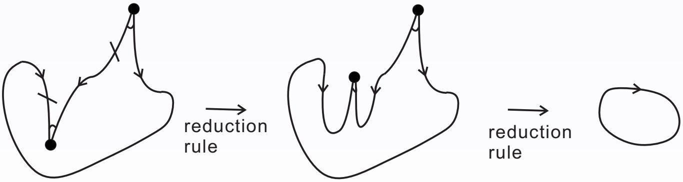


Figure: Reduction rule of a cusp with two bars for the arrow polynomial.

[1] N. Kamada, Polynomial invariants and quandles of twisted links, Topology Appl., 159(2012), 999-1006.

Reduce the state of a diagram





(1) K_i with odd bars (every edge contains at most one bar) becomes a trivial loop with one bar by using T2 and the new reduction rule.
(2) K_i with even bars becomes a trivial loop, K_1, $K_2, \cdots$ by using T2 and the new reduction rule.

Definition (Kamada, 2012)

The arrow polynomial $\langle D\rangle_A(A, M)$ of an oriented twisted link diagram D is defined by

$$\langle D\rangle_A(A, M) = \sum_S A^{\alpha-\beta} d^{|S|-1} \langle S\rangle \tag{6}$$

$\langle S\rangle$ is a product of extra variables $K_1, K_2, \cdots$ and $M^{o(S)}$, where $o(S)$ is the number of trivial loops with a bar in the state S.

[1] N. Kamada, Polynomial invariants and quandles of twisted links, Topology Appl., 159(2012), 999-1006.

Let D be an oriented twisted link diagram with writhe $\omega(D)$. The normalized version is defined by

$$\langle D\rangle_{NA}(A, M) = (-A^3)^{-\omega(D)}\langle D\rangle_A(A, M). \tag{7}$$

For an oriented twisted link L represented by an oriented link diagram D, we denote $\langle D\rangle_{NA}(A, M)$ by $\langle L\rangle_{NA}(A, M)$, and call it the *normalized arrow polynomial* of L.

Theorem (N. Kamada, 2012)

Let L be an oriented twisted link. Then $\langle L \rangle_{NA}(A, M)$ is an invariant under the extended Reidemeister moves.

Proof

It is sufficient to check the invariance under move T3.

[1] N. Kamada, Polynomial invariants and quandles of twisted links, Topology Appl., 159(2012), 999-1006.

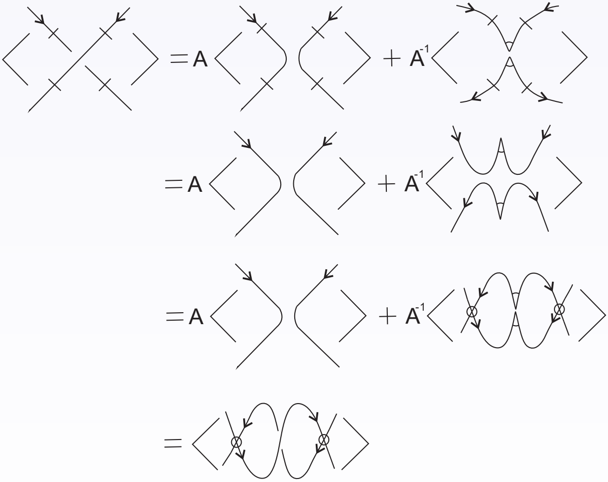


Figure: The invariance under move T3 for the arrow polynomial.

Theorem (Deng, 2022^{+})

For an oriented twisted link diagram D, $AS(D)$ is invariant under the extended Reidemeister moves.

[2] Q. Deng, One conjecture on cut points of virtual links and the arrow polynomial of twisted links, arXiv:2103.12283.

- By substituting $\frac{t^i+t^{-i}}{2}$ for K_i and $(-A^2-A^{-2})^{-1}M$ for M, $\langle L\rangle_{NA}(A,M)$ turns into the polynomial $X_D(A,M,t)$ (or $Y_D(A,M,t)$) (Miyazawa polynomial) defined by Kamada in 2010.
- By substituting 1 for K_i and $(-A^2-A^{-2})^{-1}M$ for M, $\langle L\rangle_{NA}(A,M)$ turns into the product of $(-A^2-A^{-2})^{-1}$ and the twisted Jones polynomial defined by Bourgoin in 2008.

Theorem (Deng, 2022^{+})

Let D be an oriented checkerboard colorable twisted link diagram. Then $\langle D\rangle_{NA}(A, M)$ has

(1) $\langle D\rangle_{NA}(A, M)$ does not contain M;

(2) $AS(D)$ only contains even integer;

(3) For any summand $A^s K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}$ with $1 \le i_1 < i_2 < \cdots < i_v$, $j_t \ge 1$ for $t = 1, 2, \cdots, v$, and $v \ge 1$ of $\langle D\rangle_{NA}$, we have $2i_v \le \sum_{t=1}^{v} i_t \cdot j_t$.

Corollary (Deng, 2022^{+})

Let L be a checkerboard colorable twisted link and D its diagram. Then D must contain even bars.

We can get the same statement as shown by M. O. Bourgoin in 2008.

Corollary (Bourgoin, 2008)

If a twisted link has a checkerboard colorable diagram, then its twisted Jones polynomial is the Jones polynomial.

Example

For 4 inequivalent twisted knots with 2 classical crossings, the latter three twisted knots are non-checkerboard colorable by above Theorem.

$$\begin{aligned}
\langle 2.1\rangle_{NA} &= A^{-6}(A^2 - A^{-4} + 1).\\
\langle 2.2\rangle_{NA} &= A^{-6}(A^2 - A^{-4} + 1)M.\\
\langle 2.3\rangle_{NA} &= A^{-6}\left(M\left(-\frac{K_1}{A^4} + A^2 - K_1 + 2\right)\right).\\
\langle 2.4\rangle_{NA} &= A^{-6}\left(-\frac{M^2}{A^4} + A^2 + K_1 - M^2 + 1\right).
\end{aligned}$$

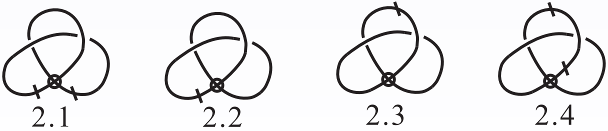


Figure: Four inequivalent twisted knots 2.1, 2.2, 2.3, 2.4.

Thanks for your attention!