Algorithm for consturcting a rectangular diagram of the Seifert surface

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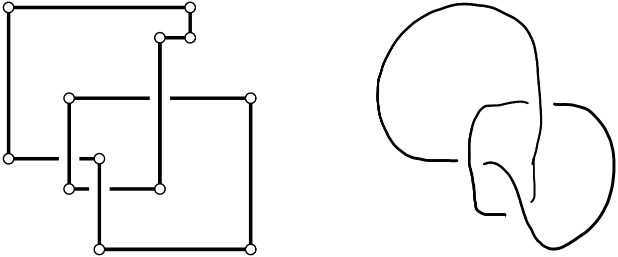
Defintions

Definition

Rectangular diagram of a link — set of vertices R in $\mathbb{T}^2$, such that any parallel $\mathbb{S}^1 \times \{\varphi\}$ and meridian $\{\theta\} \times \mathbb{S}^1$ consist only 0 or 2 vertices.

Figure: Rectangular diagram of trefoil



Definition

A rectangle in the 2-torus $\mathbb{T}^2$ is a subset of form $[\theta_1; \theta_2] \times [\varphi_1; \varphi_2]$, where $\theta_1 \neq \theta_2, \varphi_1 \neq \varphi_2$, $\theta_1, \theta_2, \varphi_1, \varphi_2 \in \mathbb{S}^1$.

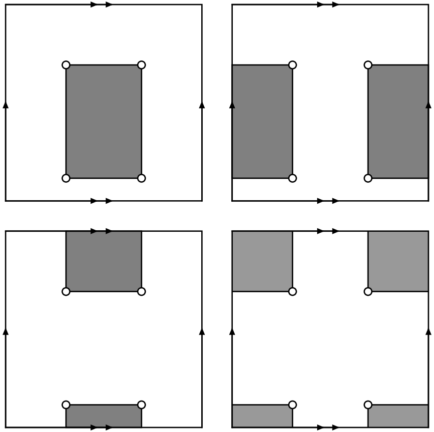


Figure: Recntangles

Definition

Two rectangles r and $\widetilde{r}$ are said to be compatible, if their intersection $r \cap \widetilde{r}$ satisfies one of the following:

1. $r_1 \cap r_2$ is empty;
2. $r_1 \cap r_2$ is a subsest of vertices of r_1;
3. $r_1 \cap r_2$ is a rectangle disjoint from the vertices of both rectangles r_1 and r_2.

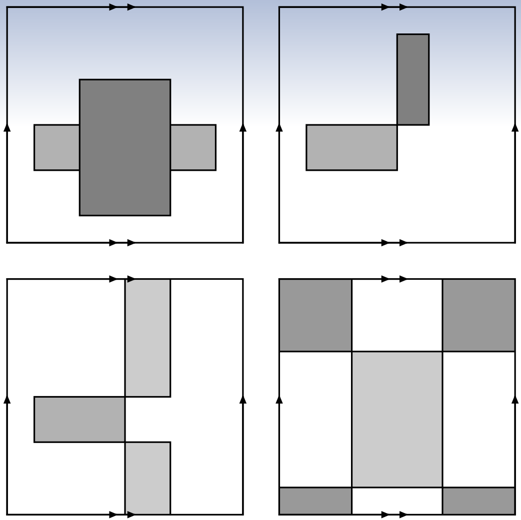


Figure: Compatible rectangles.

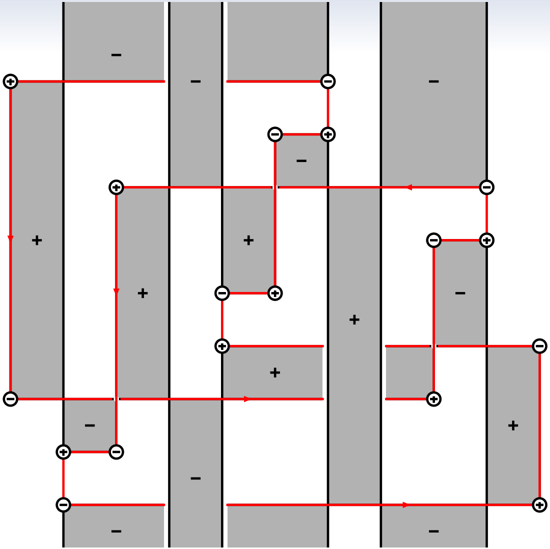
Definition (Dynnikov–Prasolov)

Rectangular diagram of surface is a collection $\Pi = \{r_1, \ldots, r_k\}$ of pairwise compatible rectagnles in $\mathbb{T}^2$, that free vertices of rectagnles is a rectangle diagram of a link.

Definition

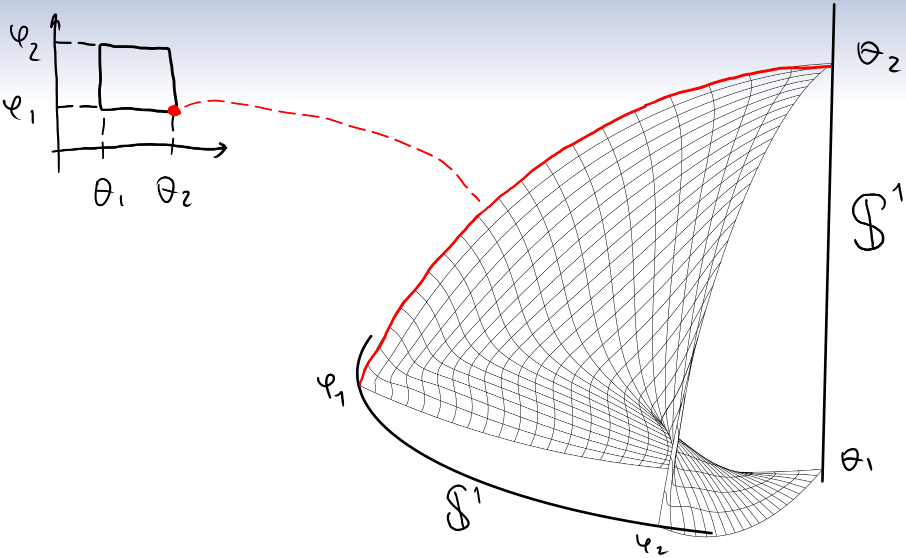
Boundary of a rectangle diagram of a surface — set of free vertices of Π and will be denoted as $\partial\Pi$.

Rectangular diagram of Seifert surface for the trefoil.

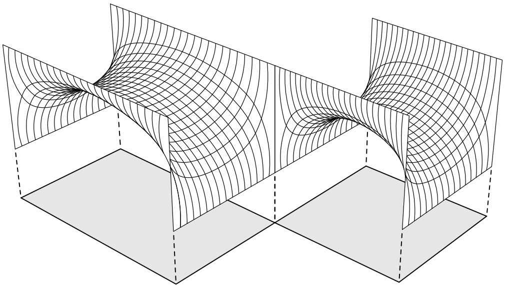


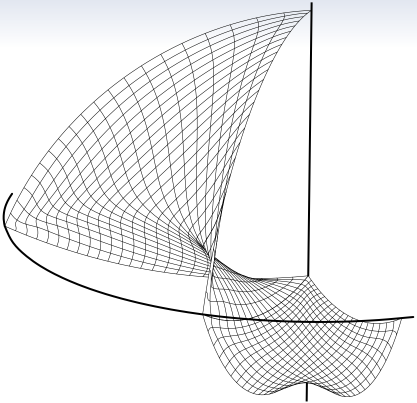
Theorem (C.2020)

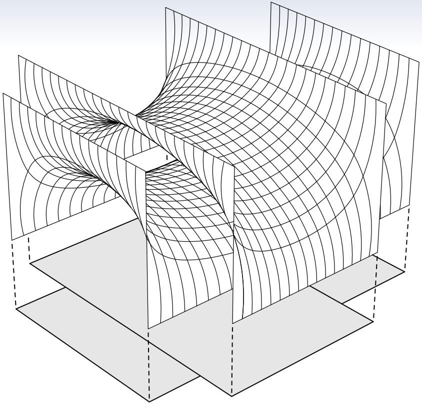
There exists an algorithm, which for any oriented diagram of a link R with complexity m, produce oriented diagram Π of Seifert surface with complexity less than $2m^4$. Moreover, $\partial\Pi$ is obtained from R by using less than $\frac{m^2}{2}$ stabilizations.

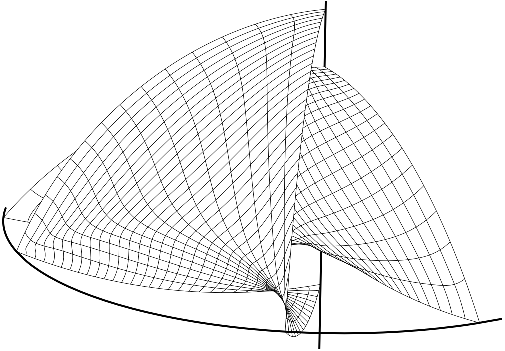


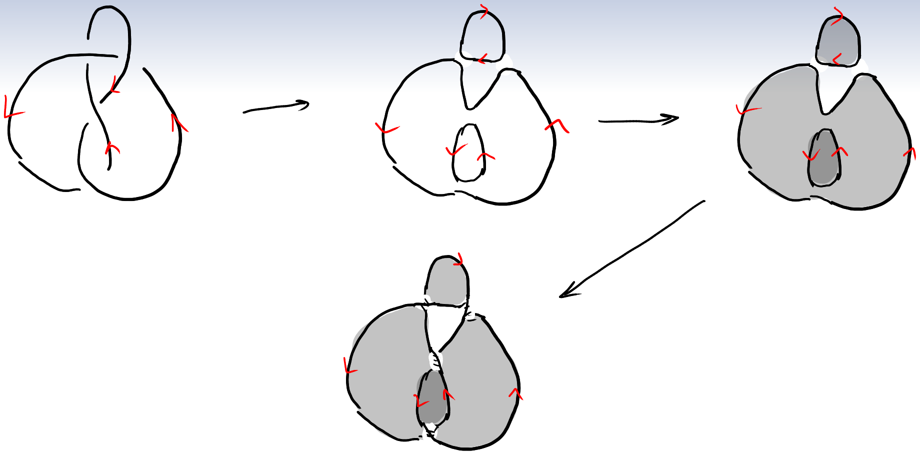
φ_2
φ_1
θ_1
θ_2
θ_2
$\mathbb{S}^1$
θ_1
φ_1
$\mathbb{S}^1$
φ_2



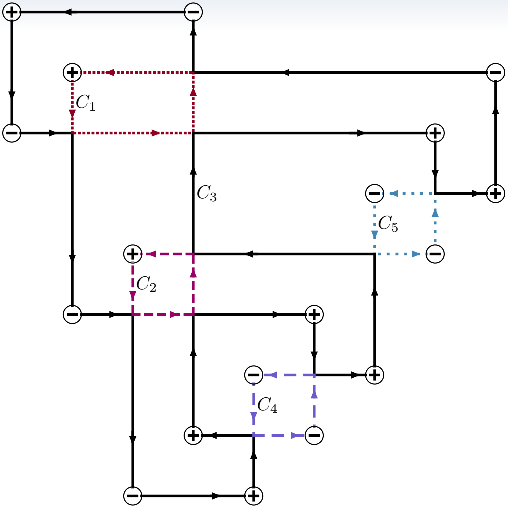








Splitting into Seifert circles.

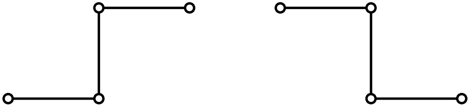


Definition

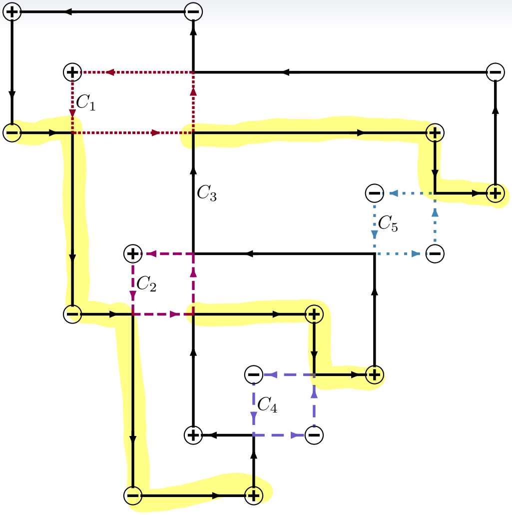
We call Seifert circle winding, if there are no

1. pieces like on the Figure;
2. non-neighbor vertical edges on any meridian.

Figure: Forbidden pieces.



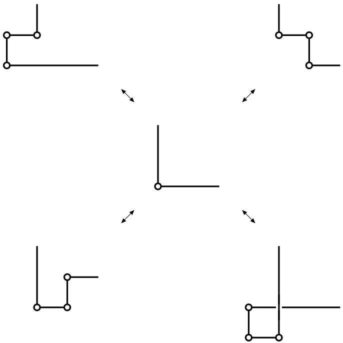
Forbidden pieces



Definition

Stabilization — replacement one rectangular diagram of a link R to another R', such $|R'| = |R| + 2$ and symmetric difference $R \triangle R'$ has the form $(\theta_i, \varphi_j), i, j = 1, 2$, and rectangle $[\theta_1; \theta_2] \times [\varphi_1; \varphi_2], \theta_1 < \theta_2, \varphi_1 < \varphi_2$ does not contain any other vertices of R, R'.

Figure: Examples of stabilizations



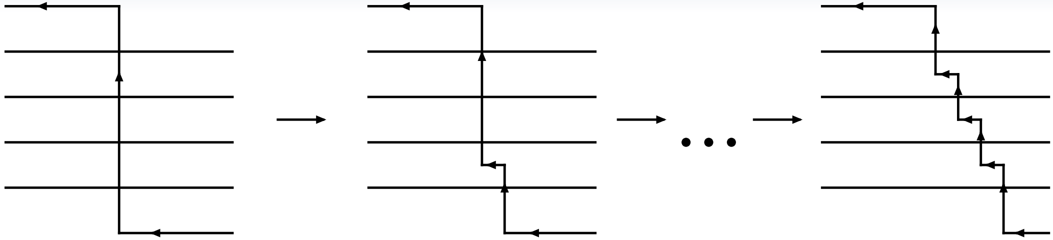
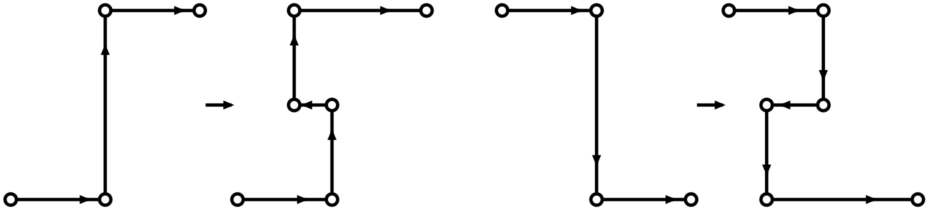


Figure: Applying stabilizations.

Figure: Removing forbidden pieces.



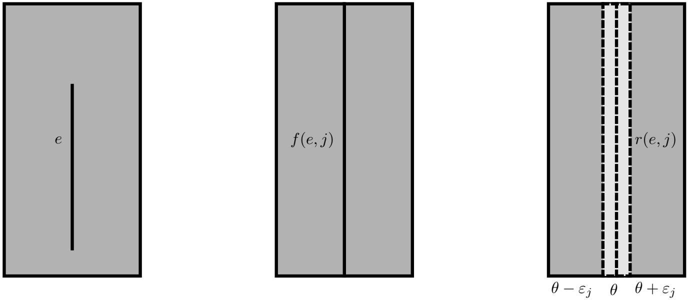
We construct sets Π_k of rectangles with the following properties:

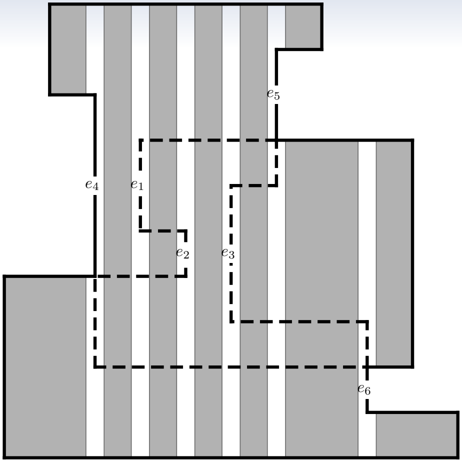
1. $\partial \Pi_i = V(C_i)$
2. For any $k \in \{1, \ldots, n\}$ all rectangles of the union $\bigcup_{i \leqslant k} \Pi_i$ are pairwise compatible.

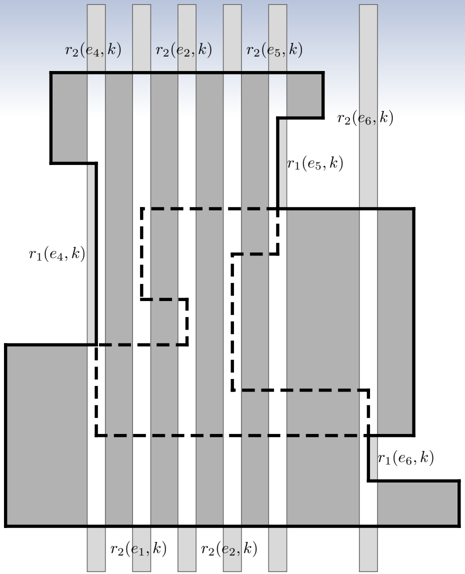
Renumber Seifert circles $C_1, \ldots C_n$ so that for any $i < k$ satisfies $D_k \not\subset D_i$. Let

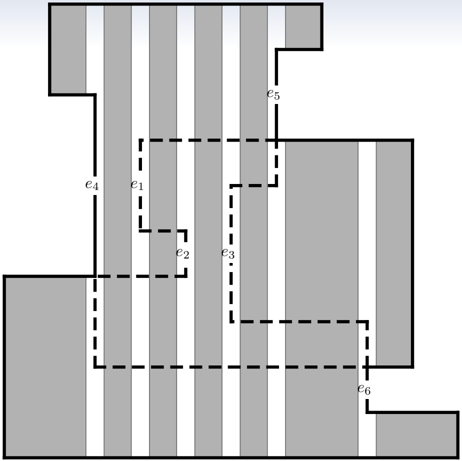
$$\varepsilon_k = \frac{k}{2(n+1)}.$$

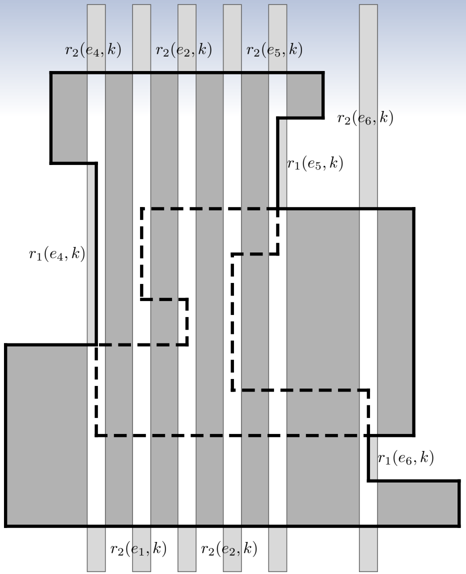
Figure: Edge e, arc $f(e,k)$, rectangle $r(e,k)$

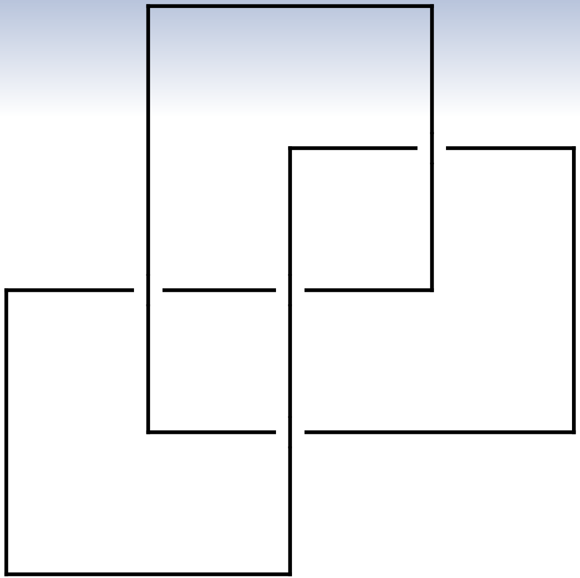


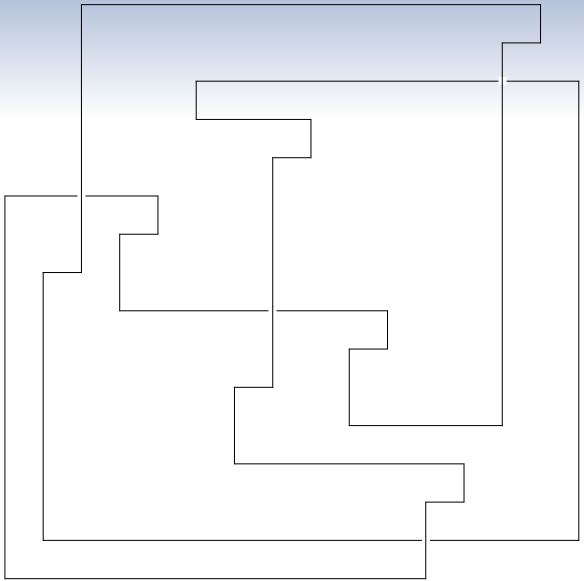

e_5
e_4
e_1
e_2
e_3
e_6

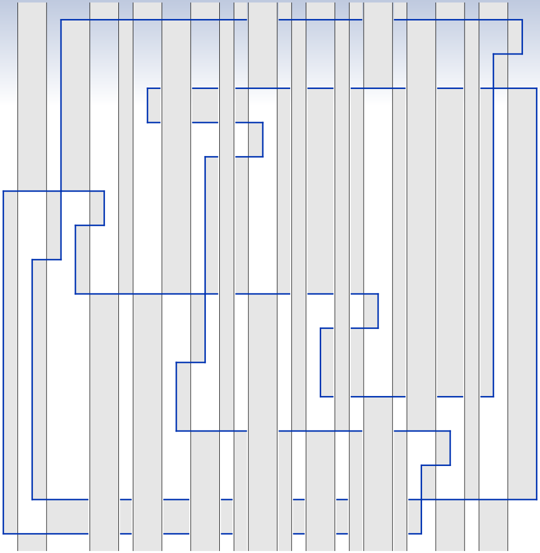

$r_2(e_4, k)$
$r_2(e_2, k)$
$r_2(e_5, k)$
$r_2(e_6, k)$
$r_1(e_5, k)$
$r_1(e_4, k)$
$r_1(e_6, k)$
$r_2(e_1, k)$
$r_2(e_2, k)$

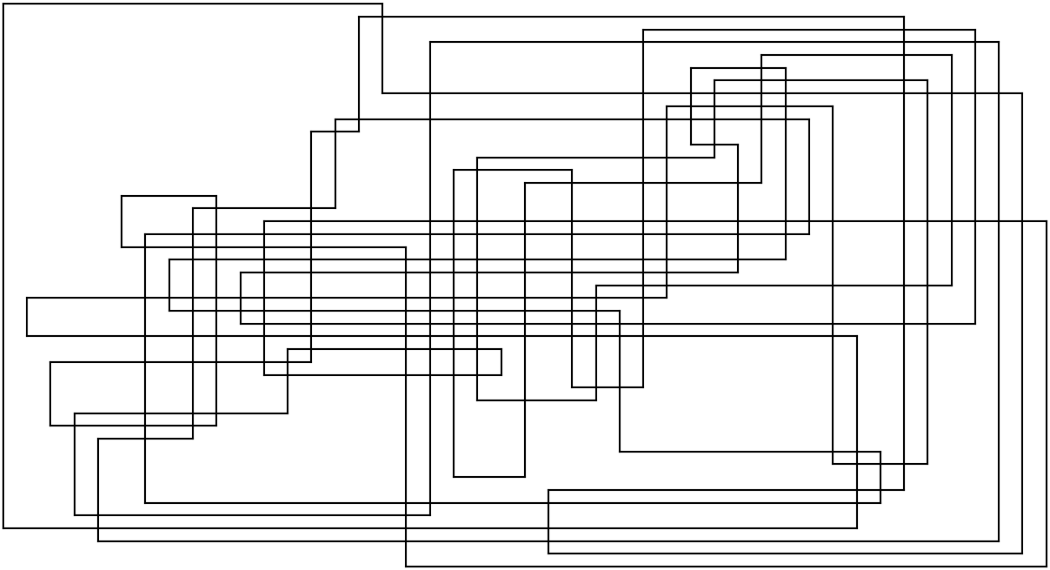

e_5
e_4
e_1
e_2
e_3
e_6


$r_2(e_4, k)$
$r_2(e_2, k)$
$r_2(e_5, k)$
$r_2(e_6, k)$
$r_1(e_5, k)$
$r_1(e_4, k)$
$r_1(e_6, k)$
$r_2(e_1, k)$
$r_2(e_2, k)$

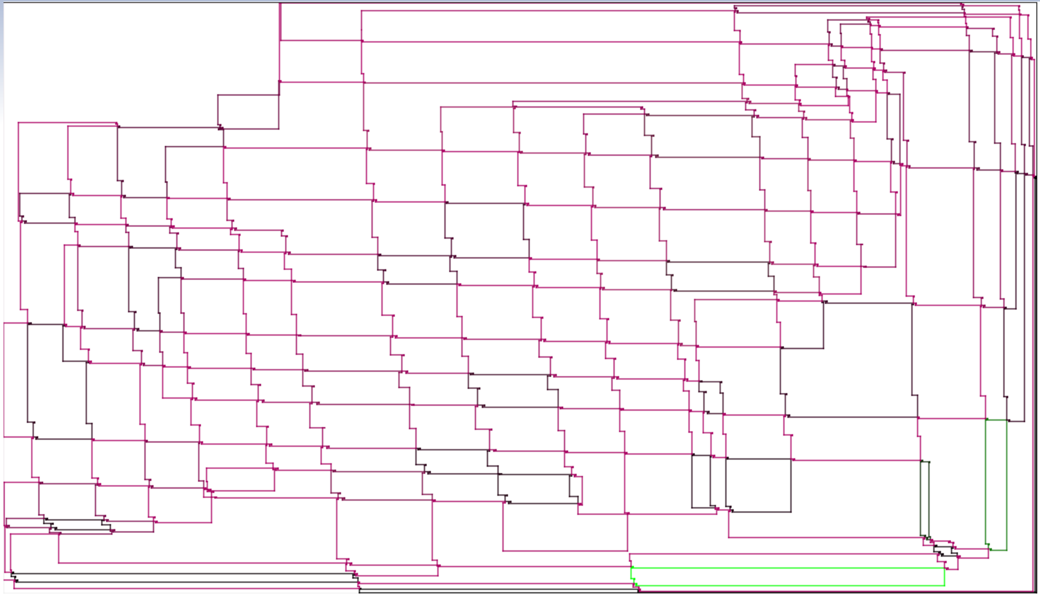








Algorithm at work.



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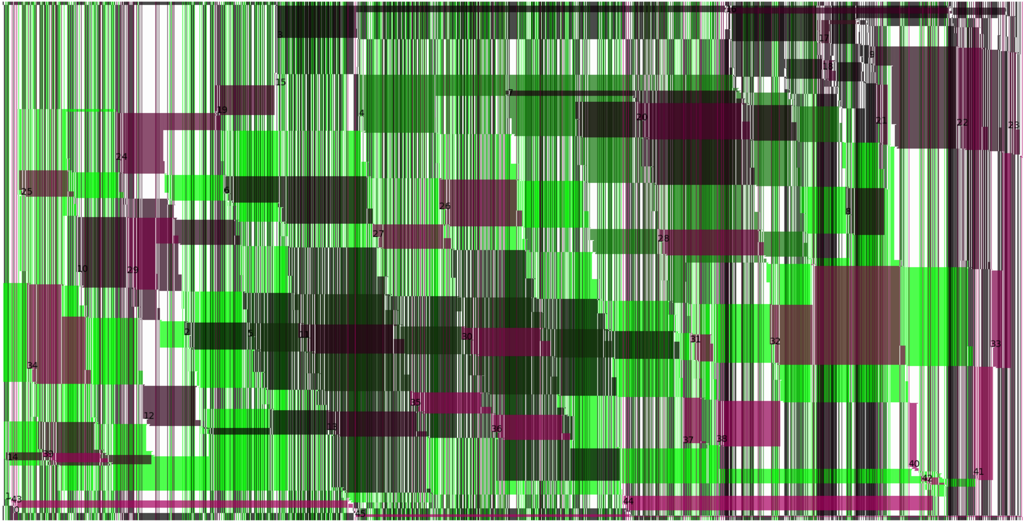

Algorithm at work.

Questions