Sequence of virtual link invariants arising from flat links

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Virtual knots are virtual diagrams modulo generalized Reidemeister moves.

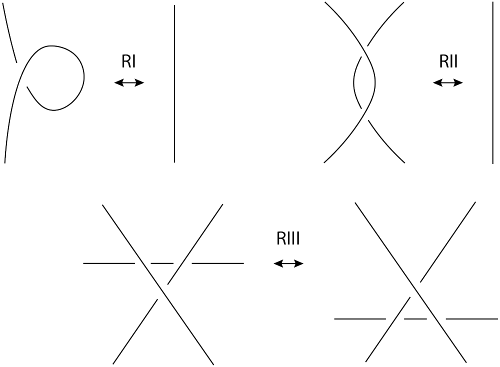

Figure: Classical Reidemeister moves

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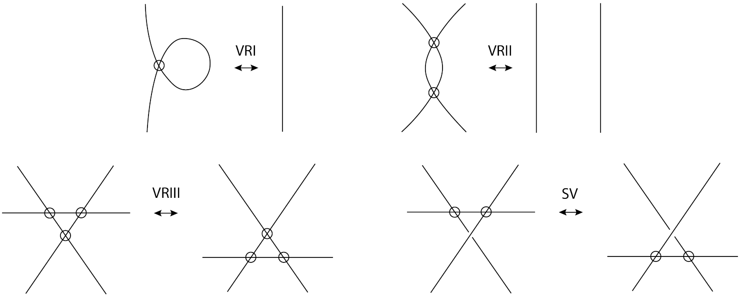

Figure: Virtual Reidemeister moves

Flat knots are flat diagrams modulo virtual Reidemeister moves and flat Reidemeister moves.

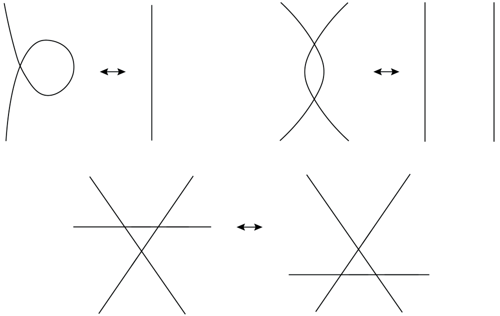

Figure: Flat Reidemeister moves

Flat knots may also be considered as virtual knots modulo crossing change operation.

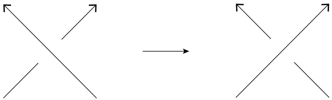

Figure: Crossing change operation

To define F–polynomials we assign to each classical crossing the following weights: $\mathrm{sgn}(c), \mathrm{Ind}(c)$ and $\nabla J_n(D_c)$.

Sign of a crossing $\text{sgn}(c)$ is defined by

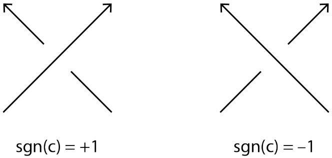

Figure: Sign of a classical crossing

To define an index of a crossing $\text{Ind}(c)$ assign an integer value to each arc in a way satisfying the rule below

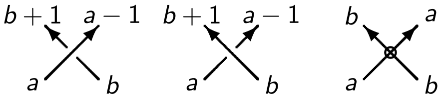

Figure: Cheng coloring

Such integer labeling, called a Cheng coloring, always exists for an oriented virtual knot diagram.

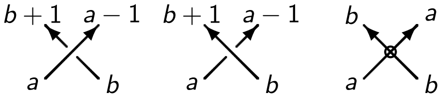

Figure: Cheng coloring

Using this coloring Z. Cheng and H. Gao assigned an integer value Ind(c) to each classical crossing c of a virtual knot diagram

$$\mathrm{Ind}(c) = \mathrm{sgn}(c)(a - b - 1)$$

S. Satoh and K. Taniguchi introduced another invariant of virtual knots - the n-th writhe $J_n(D)$. For each $n \in \mathbb{Z} \setminus \{0\}$ the n-th writhe of an oriented virtual link diagram D is defined as

$$J_n(D) = \sum_{Ind(c) = n} \text{sgn}(c)$$

K. Kaur, M. Prabhakar and A. Vesnin defined n-th dwrithe of D, denoted by $J_n(D)$

$$\nabla J_n(D) = J_n(D) - J_{-n}(D)$$

$\nabla J_n(D)$ is a flat virtual knot invariant.

For every classical crossing c of D we consider a diagram D_c, obtained by smoothing crossing c against orientation.

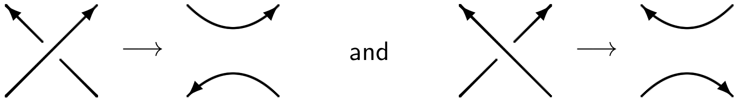

Figure: Smoothing against orientation

Flat knots $[D_c]$ corresponding to D_c satisfy following properties:

- For a crossing c involved in RI $[D_c] = [D^{\pm}]$.
- Two crossings involved in RII have the same $[D_c]$.
- For crossings involved in RIII and SV $[D_c]$ are preserved.
- $[D_c]$ of the crossing is preserved under Reidemeister moves and crossing change operation if it isn't involved in it.

These properties allowed K. Kaur, M. Prabhakar and A. Vesnin to construct a family of polynomial invariants, called F–polynomials using $\nabla J_n(D_c)$ as a weight. These invariants are defined by

$$F_K^n(t,\ell) = \sum_{c\in C(D)} \operatorname{sgn}(c) t^{\operatorname{Ind}(c)} \ell^{\nabla J_n(D_c)}$$

$$- \sum_{c\in T_n(D)} \operatorname{sgn}(c) \ell^{\nabla J_n(D_c)} - \sum_{c\notin T_n(D)} \operatorname{sgn}(c) \ell^{\nabla J_n(D)},$$

where $T_n(D) = \{c \in C(D) : \ |\nabla J_n(D_c)| = |\nabla J_n(D)|\}$

Definition

Let G be an abelian group and $w : C(\mathcal{D}) \to G$ be a function which assigns a *value* $w(c) \in G$ to a classical crossing $c \in C(D)$ for all diagrams $D \in \mathcal{D}$. Function w is said to be a *weight function*, write $w \in W_G$, if it satisfies *weight function conditions* (C1)–(C3)

(C1) w is *local*, i.e. if D' is obtained from D by a generalized Reidemeister move such that a crossing $c \in D$ is not involved in this move and $c' \in D'$ is the corresponding crossing, then $w(c') = w(c)$;

(C2) if diagram D' is obtained from D by RIII-move and involved classical crossings $a, b, c \in D$ have weights $w(a)$, $w(b)$ and $w(c)$, as well as involved crossings of $a', b', c' \in D'$ have weights $w(a')$, $w(b')$ and $w(c')$, then $w(a') = w(a)$, $w(b') = w(b)$ and $w(c') = w(c)$.

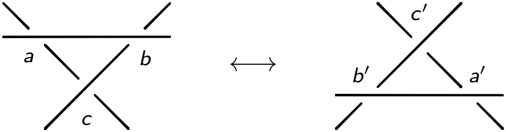

Figure: RIII move.

(C3) if diagram D' is obtained from D by SV-move and involved classical crossing $c \in D$ has weight $w(c)$, as well as involved classical crossing $c' \in D'$ has weight $w'(c')$, then $w'(c') = w(c)$.

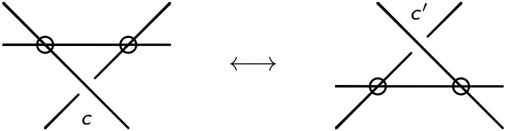

Figure: SV move.

Definition

Let w be a weight function, a digram D' is obtained from D by RII-move and α, β are crossings involved. If $w(\beta) = -w(\alpha)$, then we say that w is an odd weight function and write $w \in W_G^{odd}$. If $w(\beta) = w(\alpha)$, then we say that w is an even weight function and write $w \in W_G^{even}$.

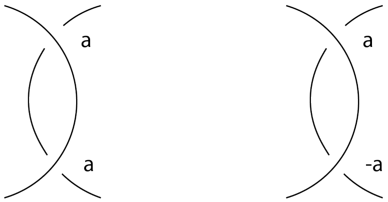

Figure: Even and odd weight functions on RII crossings.

An even weight function such that for every crossing obtained by RI–move its value is fixed is a Chord Index introduced by Cheng.

Definition

A subset $C'(\mathcal{D}) \subset C(\mathcal{D})$ is said to be *consistent* if the characteristic function $1_{C'(\mathcal{D})} : C(\mathcal{D}) \to \{0,1\} \subset \mathbb{Z}$ of the set $C'(\mathcal{D})$ is an even weight function.

Definition

Let $C'(\mathcal{D}) \subset C(\mathcal{D})$ be consistent. Then $w' : C'(\mathcal{D}) \to G$ is said to be *a weight function defined for* $C'(\mathcal{D})$ if w' satisfies weight function conditions (C1) - (C3) for all crossings in $C'(\mathcal{D})$.

If $C'(\mathcal{D}) \subset C(\mathcal{D})$ is consistent, and $w' : C'(\mathcal{D}) \to G$ is a weight function, then w' can be extended to $w : C(\mathcal{D}) \to G$ by defining

$$w(c) = \begin{cases} w'(c), & c \in C'(\mathcal{D}), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}(L)$ be the set of all diagrams of an ordered oriented virtual link L, For a diagram $D \in \mathcal{D}(L)$ denote by $R(w, D)$ the set of values $w(c)$, where c is a classical crossing in D that may be reduced by a single RI-move. Then take a union over all diagrams of L:

$$R(w, L) = \bigcup_{D \in \mathcal{D}(L)} R(w, D).$$

Suppose there are two weights $v \in W_{G_1}^{odd}$ and $w \in W_{G_2}^{even}$. Take $g \in G_2$ such that either $g \notin R(w, L)$ or $R(v, L) = \{0\}$. Then I–function is defined by

$$I(D; v, w, g) = \sum_{w(c)=g} v(c).$$

Assume that our weight functions are local with respect to crossing change operation. Then to every weight function $w : C'(\mathcal{D}) \to G$ we can associate a weight function $w^* : C'(\mathcal{D}) \to G$ induced by taking a mirror image, i.e. $w^*(c) = w(c^*)$.
We define a flat I–function by

$$I_f(D; v, w, g) = \sum_{w(c)=g} v(c) + \sum_{w^*(c)=g} v^*(c).$$

Theorem

$I(D; v, w, g)$ is an ordered oriented virtual link invariant,

$I_f(D; v, w, g)$ is an ordered oriented flat virtual link invariant.

Applying the type-1 smoothing to a classical crossing $c \in D$, which belongs to a single component, we obtain a link diagram with one less classical crossing and the same number of components.

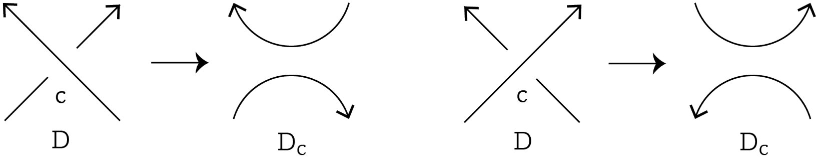

Figure: Type-1 smoothing.

Consider a diagram of an $(n-1)$-component ordered oriented virtual link. Applying the type-2 smoothing to a classical crossing c such that two meeting arcs belong to the i-th component we obtain a diagram of an ordered n-component link.

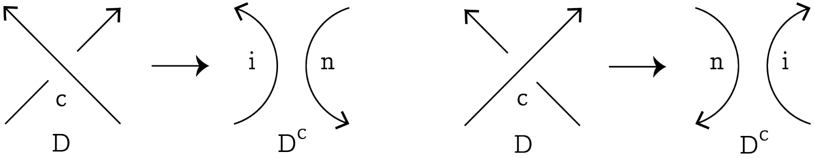

Figure: Type-2 smoothing.

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Consider a diagram of an n-component ordered oriented virtual link, and assume that crossing c belong to components n and $(n-1)$. Then after type-3 smoothing we will get a diagram of an $(n-1)$-component ordered link.

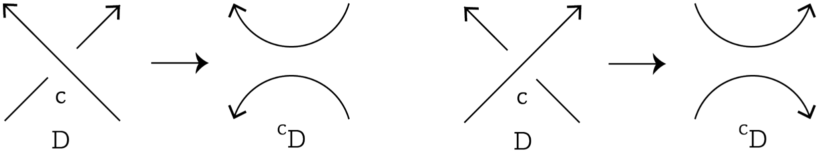

Figure: Type-3 smoothing.

Let us denote by $\mathcal{VL}_{\text{flat}}$ a free $\mathbb{Z}$-module generated by ordered oriented flat virtual links. For a virtual link diagram D denote by $[D]$ a flat virtual link whose diagram is obtained from D by replacing all classical crossings by flat crossings, then $[D] \in \mathcal{VL}_{\text{flat}}$.

Theorem

Functions S_i defined as

$$S_1(c) = [D_c], \qquad S_2(c) = [D^c], \qquad \textit{and} \quad S_3(c) = [{}^c D],$$

are even weight functions taking values in $\mathcal{VL}_{flat}$.

Corollary

Functions B^i and B^i_{flat} defined by

$$B^i = \sum_{c\in C_i(D),\ [D^c]\neq[D\sqcup O]} \mathrm{sgn}(c)[K^c],$$

$$B^i_{flat} = \sum_{c\in C_i(D),\ [D^c]\neq[D\sqcup O]} \mathrm{sgn}(c)([K^c]-[K^{c^*}]),$$

are virtual link and flat virtual link invariants. $C_i(D)$ are crossings c s.t. both arcs belong to the i-th component of a link.

These new invariants appear to be useful studying connected sums of virtual knots. As an example we give a new proof of Kishino knot being nontrivial.

Let K be a Kishino knot. We will show that $B^1(K) \neq 0$, hence K is distinguished from the unknot by B^1.

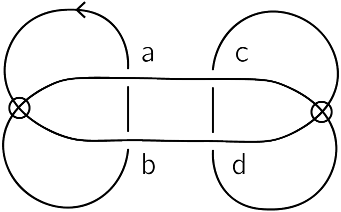

Figure: Kishino knot.

To calculate $B^1(K)$ we find signs of all crossings

$$\mathrm{sgn}(a) = \mathrm{sgn}(d) = -1 \quad \textit{and} \quad \mathrm{sgn}(b) = \mathrm{sgn}(c) = 1.$$

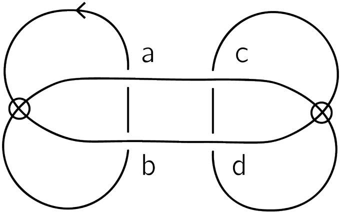

Figure: Kishino knot.

$$B^1(K) = -[K^a] + [K^b] + [K^c] - [K^d].$$

Each smoothing provide an ordered oriented 2-component virtual link. One can check, that

$$[K^a] = [K^d] \quad and \quad [K^b] = [K^c],$$

so we only need to prove that $[K^a]$ and $[K^b]$ are distinct.

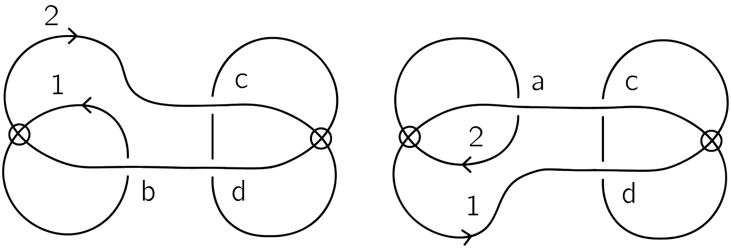

Figure: Diagrams of $K_1 = K^a$ and $K_2 = K^b$.

To simplify the notation we denote K^a as K_1 and K^b as K_2. The only crossings corresponding to 2-nd component of K_1 are c and d and there are no such crossings in K_2. Hence

$$B^2_{\text{flat}}(K_1) = [K_1^c] - [K_1^{c^*}] - [K_1^d] + [K_1^{d^*}] \quad \textit{and} \quad B^2_{\text{flat}}(K_2) = 0,$$

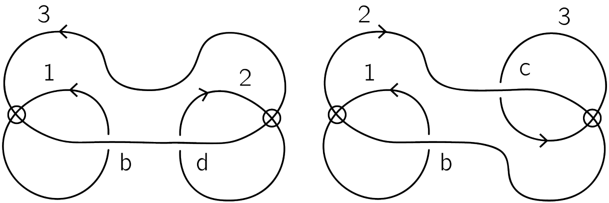

Figure: Diagrams of K_1^c and K_1^d.

Note that in K_1^c and in $K_1^{d^*}$ the 3-rd component is nontrivially linked with two other components, but for $K_1^{c^*}$ and K_1^d it is not true.

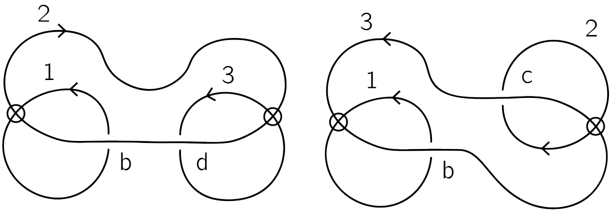

Figure: Diagrams of $K_1^{c^*}$ and $K_1^{d^*}$.

$$B^2_{\text{flat}}(K_1) = [K_1^c] - [K_1^{c^*}] - [K_1^d] + [K_1^{d^*}]$$

Since

$$[K_1^c] \neq [K_1^{c^*}], [K_1^d], \quad \text{and} \quad [K_1^{d^*}] \neq [K_1^{c^*}], [K_1^d]$$

there are no cancelations and $B^2_{\text{flat}}(K_1) \neq 0$. Therefore Kishino knot is not equivalent to the unknot.

Now we define a recursive procedure to define virtual link invariants.

- Take an invariant of ordered oriented flat virtual links A_1 taking values in a group G.
- Take an even weight function $F_1 \in W^{even}_{\mathcal{VL}_{\text{flat}}}$
- Take a couple of weight functions $w_1 \in W^{even}_{H_1}$ and $u_1 \in W^{odd}_{\mathbb{Z}}$

The invariant may be extended to a homomorphism $A_1 : \mathcal{VL}_{\text{flat}} \to G$, hence, its composition with F_1 defines a weight function $A_1 \circ F_1 \in W^{even}_G$.

$$v_1 = u_1 * (A_1 \circ F_1)$$

$$I(D; v_1, w_1, h_1) = \sum_{w_1(c)=h_1} v_1(c)$$

$$I_f(D; v_1, w_1, h_1) = \sum_{w_1(c)=h_1} v_1(c) + \sum_{w_1(c^*)=h_1} v_1(c^*),$$

We continue the procedure by taking on each step some weight functions

$$u_i \in W_G^{odd},\ F_i \in W_{\mathcal{VL}_{\mathrm{flat}}}^{even},\ w_i \in W_{H_i}^{even}.$$

We get a family of ordered oriented flat virtual link invariants A_i defined by

$$A_i(D) = I_f(D; v_{i-1}, w_{i-1}, h_{i-1}), \qquad \text{for} \quad i > 1,$$

where

$$v_{i-1} = u_{i-1} * (A_{i-1} \circ F_{i-1}).$$

To make this procedure correct we require $R(u_i, D) = \{0\}$ for all D.

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To make this procedure correct we require $R(u_i, D) = \{0\}$ for all D.

Corollary

Given three weight functions $u \in W_H^{odd}$, $w \in W_G^{even}$ and $F \in W_{\mathcal{VL}_{flat}}^{even}$ such that $R(u, L) = \{0\}$ for all links L and a sequence $\{g_i \in G\}_{i \in \mathbb{N}}$ there is an infinite sequence of weight functions $\{v_i\}_{i \in \mathbb{N}}$ and corresponding invariants generated by them.

Theorem

Let $S = \{s_1, \ldots, s_k\}$ *be a finite set of weight functions where* $s_i = A_{m_i} \circ F_{m_i}$ *for some* $m_i \in \mathbb{N}$, $w \in W_{\mathbb{Z}}^{odd}$ *and* $v \in W_{\mathbb{Z}}^{even}$ *such that* $R(v, D) = \{0\}$ *for all* D. *Then*

$$F(t, \ell_1, \ldots, \ell_k) = \sum_{c \in C(D)} w(c) t^{v(c)} \ell_1^{s_1(c)} \cdots \ell_k^{s_k(c)} - \sum_{c \in T(D)} w(c) \ell_1^{s_1(c)} \cdots \ell_k^{s_k(c)} - \sum_{c \notin T(D)} w(c) \ell_1^{A_{m_1}(D)} \cdots \ell_k^{A_{m_k}(D)}$$

is a link invariant, where $T(D) = \{c \in C(D) \mid s_i(c) \in R(s_i, L) \text{ for all } i\}$.

Example

Let $v = \mathrm{Ind}$, $w = \mathrm{sgn}$, $A_1 = \nabla J_n$ and $s_1 = \nabla J_n(D_c) = A_1 \circ F_1$, where F_1 is a type-1 smoothing. Then $F(t, \ell_1)$ coincide with n-th F-polynomial.

Take F_1 to be a type-1 smoothing, and let $F_i = F_1$ for $i \geq 2$. Define $A_1 = I_f(D; \mathrm{sgn}, \mathrm{Ind}, n) = \nabla J_n$. By taking $u_i = \mathrm{sgn} * \mathrm{Ind}$ for $i \geq 1$ we define $\nabla J_{n,m}(D)$

$$\nabla J_{n,m}(D) = A_2(D) = I_f(D; u_2 * (A_1 \circ F_1), w_1, m) =$$
$$= \sum_{\mathrm{Ind}(c)=m} \mathrm{sgn}(c)\mathrm{Ind}(c)\nabla J_n(D_c) - \sum_{\mathrm{Ind}(c)=-m} \mathrm{sgn}(c)\mathrm{Ind}(c)\nabla J_n(D_c)$$
$$= \sum_{\mathrm{Ind}(c)=|m|} \mathrm{Ind}(c)\,\mathrm{sgn}(c)\nabla J_n(D_c).$$

Proposition

A family of $F^{n,m,k}$-polynomials is an oriented virtual knot invariant.

$$F_D^{n,m,k}(t,\ell_1,\ell_2) = \sum_{c\in C(D)} \mathrm{sgn}(c) t^{\mathrm{Ind}(c)} \ell_1^{\nabla J_n(D_c)} \ell_2^{\nabla J_{m,k}(D_c)}$$

$$- \sum_{c\in T(D)} \mathrm{sgn}(c) \ell_1^{\nabla J_n(D_c)} \ell_2^{\nabla J_{m,k}(D_c)} - \sum_{c\notin T(D)} \mathrm{sgn}(c) \ell_1^{\nabla J_n(D)} \ell_2^{\nabla J_{m,k}(D)}$$

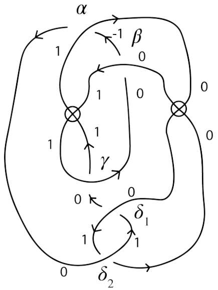

Figure: A knot, distinguished by $F^{n,m,k}$ but not by F_n from the unknot.

Let $L = K_1 \cup K_2$ be an ordered oriented virtual 2-component link, where K_1 is the first component, and K_2 is the second component. Suppose that L is presented by its diagram. Denote by $O(L)$ the set of crossings where K_1 passes over K_2. Define the over linking number $O_{\ell k}$ for L as follows:

$$O_{\ell k}(L) = \sum_{c \in O(L)} \operatorname{sgn}(c).$$

Analogously, denote by $U(L)$ the set of crossings where K_1 passes under K_2 and define the under linking number $U_{\ell k}$ for L as follows:

$$U_{\ell k}(L) = \sum_{c \in U(L)} \operatorname{sgn}(c).$$

In virtual links the two linking numbers $O_{\ell k}(L)$ and $U_{\ell k}(L)$ may differ, as can be seen for the ordered oriented virtual Hopf link $\mathcal{H}$. It is easy to see that $O_{\ell k}(\mathcal{H}) = -1$ and $U_{\ell k}(\mathcal{H}) = 0$. Also note that with reversing order of components in $\mathcal{H}$, the two linking numbers will exchange.

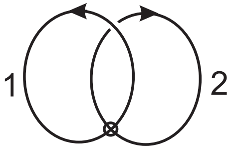

Figure: Virtual Hopf link $\mathcal{H}$.

Definition

For an ordered oriented virtual 2-component link L define its *span* by

$$\mathrm{span}(L) = O_{\ell k}(L) - U_{\ell k}(L).$$

Let D be a diagram of an ordered oriented virtual 2-component link $L = K_1 \cup K_2$, and $C_{12}(D)$ be the set of all classical crossings in D in which K_1 and K_2 meet. For $c \in C_{12}(D)$ let ${}^cD$ be a knot diagram obtained by type-3 smoothing at $c \in D$. For $n, k \in \mathbb{Z}$ consider a set

$$I_{n,k} = \{c \in D : \nabla J_n({}^cD) = k\}.$$

Definition

An *(n,k)-span* for a 2-component link L is defined by:

$$\mathrm{span}_{n,k}(L) = \sum_{c \in O(L) \cap I_{n,k}} \mathrm{sgn}(c) - \sum_{c \in U(L) \cap I_{n,k}} \mathrm{sgn}(c).$$

Definition

For a 2-component link L its (n,k)-fspan is defined as follows:

$$\mathrm{fspan}_{n,k}(L) = \mathrm{span}_{n,k}(L) + \mathrm{span}_{n,-k}(L).$$

Proposition

$$\widetilde{F}_K^{n,k,m}(t,\ell,v) = \sum_{c\in C(D)} \mathrm{sgn}(c) t^{\mathrm{Ind}(c)} \ell^{\nabla J_n(D_c)} v^{\mathrm{fspan}_{k,m}(D^c)}$$

$$- \sum_{c\in T(D)} \mathrm{sgn}(c) \ell^{\nabla J_n(D_c)} v^{\mathrm{fspan}_{k,m}(D^c)} - \sum_{c\notin T(D)} \mathrm{sgn}(c) \ell^{\nabla J_n(D)} v^{\mathrm{fspan}_{k,m}(D^c)},$$

where

$$T(D) = \{c \in C(D) \mid \nabla J_n(D_c) = \pm \nabla J_n(D) \textit{ and } \mathrm{fspan}_{k,m}(D^c) = 0\}$$

is an oriented virtual knot invariant.

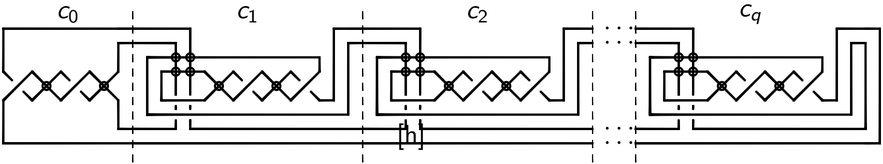

Figure: Virtual knot VK_q.

Virtual knots VK_3 and VK_4 can be distinguished by polynomial $\widetilde{F}^{2,2,2}(t, \ell, v)$, but not by F-polynomials.

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Thank you!