The relation between tunnel numbers of a cable knot and its companion

Junhua Wang

Jiangsu University of Technology

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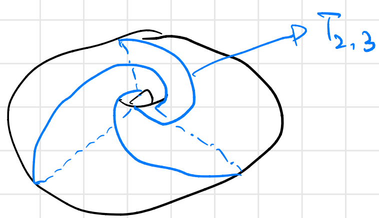
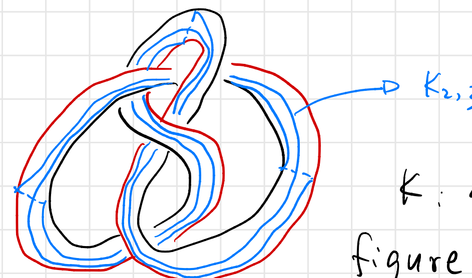
I. Definition

1. $K \subset S^3$: a knot $\Rightarrow \exists$ a homeo. $h : S^1 \times D^2 \to N(K)$

Let $T_{p,q} \subset \partial(S^1 \times D^2)$: (p,q)-torus knot, then we call $h(T_{p,q}) \subset \partial N(K)$ a <u>(p,q)-cable knot</u> of K, and denote by $K_{p,q}$. K is called a <u>companion knot</u> of $K_{p,q}$.



$\xrightarrow{h}$



K : 4_1

figure-eight knot

Remark: ① If K is unknot, then $K_{p,q}$ is a torus knot.

② The homeo. h is not unique, we can choose

(Maybe twist along meridian)

a canonical map taking longitude to a perferred longitude.

(homological trivial in knot complement)

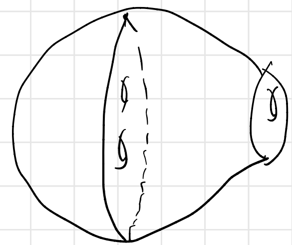
2. tunnel number

$K \subset S^3$: a non-trivial knot.

- If there are some disjoint arcs $\tau_1, \tau_2, \cdots, \tau_k$ properly embedded in knot exterior $E(K)$, such that $E(K) \setminus \bigcup_{i=1}^{k} \eta(\tau_i)$ is a handlebody, then we say $\{\tau_1, \tau_2, \cdots, \tau_k\}$ is a tunnel system of K.

- The minimal number k satisfying the above proposition is called the tunnel number of K. Denoted by $t(K)$.

· $t(K) = g(E(K)) - 1$, $g(E(K))$: Heegaard genus of $E(K)$



$E(K)$

e.g. $t(T_{p,q}) = 1 \iff g(E(T_{p,q})) = 2$

torus knot

II. Background

Question 1 : $t(K_{p,q}) \ ? \ t(K)$

① Theorem 1 (Y. Moriah , 1991)

$$t(K_{p,q}) \leq t(K)+1 .$$

② Theorem 2 (J. H. Wang and Y. Q. Zou , 2020)

$$t(K_{p,q}) \geq t(K) .$$

Idea of Thm 2

- $K_{p,q} \subset \partial N(K) \overset{\text{push}}{\rightsquigarrow} K_{p,q} \subset \mathrm{int}(N(K))$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$E(K_{p,q}) = E(K) \cup_{T^2} C_{p,q} \longrightarrow$ cable space

$E(K) \cup_{T^2}$: incompressible

$E(K_{p,q}) = E(K) \cup_{A^2} (S^1 \times D^2)$

- Given $S \subset E(K_{p,q})$: minimal Heegaard surface.

consider $S \cap T^2 \overset{\text{construct}}{\rightsquigarrow} S' \subset E(K)$: H.S. & $g(S') = g(S)$.

Remark 1. For a satellite knot, Tao Li (arXiv, 2020) proved its tunnel number is not less than that of its <u>pattern</u>.

($\Rightarrow$ $t(K_1 \# K_2) \geq \max\{t(K_1), t(K_2)\}$, Schirmer, 2016)

Thm1 + Thm2: $t(K) \leq t(K_{p,q}) \leq t(K)+1$.

Question 2: When $t(K_{p,q}) = t(K)$?

Wang-Zou (2020) gave two sufficient conditions to guarantee $t(K_{p,q}) = t(K)+1$. (Hempel distance)

S : closed connected orientable surface with $g(S) \geq 2$

curve graph $\longrightarrow$ curve complex ($\dim \leq 3g-4$)

- vertices : isotopy classes of essential simple closed curves on S
- edges : two vertices have a edge if they are disjoint

n-simplex n+1 vertices

If each edge is given with length one, then we have

an abstract metric space (S, d_S).

- The distance (Harvey) between two curves measures the complexity of their intersection.
- Hempel distance is defined for a Heegaard splitting corresponding to curves bounding disks on both sides.

Remark 2. ① For any positive integer n, there is a satellite knot K with companion k satisfying $t(K) - t(k) = n$.

(iteration of cable)

② For a satellite knot K with companion k, is it always true that $t(K) \geq t(k)$?

③ There are many results about the (super)additivity of composite knot : when $t(K_1 \# K_2) = t(K_1) + t(K_2) (+1)$?

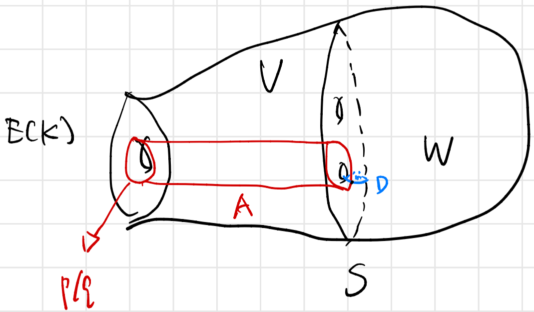
III. Result

$$t(K) \leq t(K_{p,q}) \leq t(K)+1.$$

Observation: If K is a <u>p/q - primitive</u> knot, then $t(K_{p,q}) = t(K)$.

- K is called p/q - primitive if a minimal Heegaard splitting of $E(K)$ is p/q - primitive.

P/q-primitive Heegaard Splitting $E(K) = V \cup_S W$



$\exists$ $A \subset V$: vertical annulus

$D \subset W$: essential disk

s.t. $A \cap D = \partial_+ A \cap \partial D \subset S$: single point

& $\partial_- A \cap \partial E(K)$: p/q-slope

- a minimal Heegaard surface S of $E(K)$

$\downarrow$

genus $g(S)+1$ Heegaard surface S' of $E(K_{p,q})$

$\downarrow$ K is p/q-primitive

S' is stabilized $\Rightarrow$ $g(E(K_{p,q})) = g(E(K))$

Is p/q-primitive the unique obstruction of non-degeneration of a cable knot's tunnel number?

Conjecture : $t(K_{p,q}) = t(K)$ iff K is p/q-primitive.

We consider tunnel number one,

Theorem 3 (J. J. Wang and J. H. Wang, 2021)

If $t(K_{p,q}) = t(K) = 1$, then K is p/q-primitive.

Theorem 4 (Y. Moriah, 1991)

If $t(K_{p,q}) = t(K) = 1$, then K is a torus knot T_{p_1,q_1} and

$$|p - p_1 q_1 q| = 1.$$

- $|p - p_1 q_1 q| = 1 \iff$ geometric intersection number

$$i(p_1 q_1, p/q) = 1$$

$\xRightarrow[1971]{\text{L. Moser}}$ Dehn surgery on T_{p_1,q_1} along slope p/q yields a lens space.

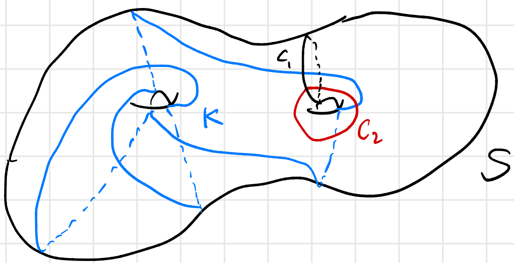
Comparison with Berge conjecture

(Berge conjecture) Let $K \subset S^3$ be a non-trivial knot which has Dehn surgery resulting in a lens space. Then K is doubly primitive.

? $t(K) = 1$

Li–Moriah–Pinsky (arXiv, 2019)

K is called doubly primitive if K can be isotoped on a genus two Heegaard surface S of S^3 intersecting essential disks in just one point on both sides of S.



$S^3 = V_1 \cup_S V_2$

$K \subset S$, $|K \cap C_i| = 1$. $C_i = \partial D_i \subset V_i$

$(i = 1, 2)$

- doubly primitive knot is tunnel number one

(push K into a handlebody $\rightsquigarrow$ genus two Heegaard surface of $E(K)$)

- torus knot is doubly primitive
- integer surgery on a doubly primitive knot yields a lens space

Maybe p/q-primitive can be regarded as some generalization of doubly primitive for a knot.

$$\begin{array}{c}(D_1, K, D_2) \\ \cap \\ S \end{array} \rightsquigarrow (A, D, E(K))$$

Q: In general, if $t(K_{p,q}) = t(K)$, is K p/q-primitive?

Thank you.