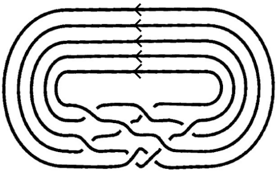
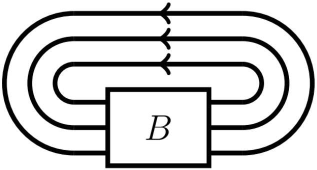
Sharpness of the Morton-Franks-Williams inequality

Ilya Alekseev

Saint Petersburg State University

Braid index: complexity measure of knots and links





Theorem (J. W. Alexander, 1923)

Any link can be presented as a closed braid.

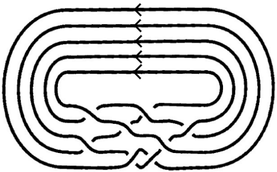
Definition

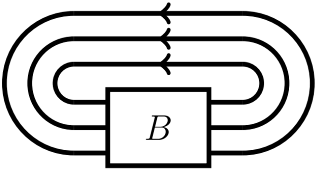
The braid index of a link $\mathcal{L}$ is the least number of strands among all braid representatives of $\mathcal{L}$.

Aim

Calculate braid index of infinite families of knots and links.

Braid index: complexity measure of knots and links





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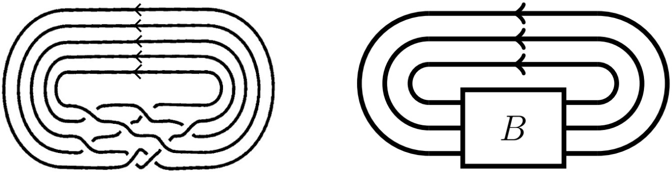

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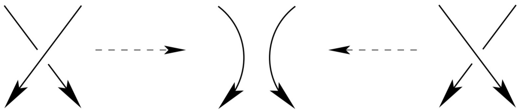
Skein polynomial (aka HOMFLY-PT polynomial)



Theorem (P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett, A. Ocneanu, 1985; J. Przytycki, P. Traczyk, 1987)

There is a unique function that maps each link diagram $\mathcal{D}$ to a two-variable Laurent polynomial $\mathcal{P}(\mathcal{D}; a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ such that:

1. isotopic links map to the same polynomial;
2. the trivial knot maps to 1;
3. relation $a\mathcal{P}(\mathcal{D}_+; a, z) - a^{-1}\mathcal{P}(\mathcal{D}_-; a, z) = z\mathcal{P}(\mathcal{D}_0, a, z)$ holds;

Morton – Franks – Williams inequality (weak form)

Theorem (H. Morton, J. Franks, R. Williams, 1986)

For any closed braid diagram $\mathcal{D}$ with n strands, one has

(MFW) $$\left(\mathrm{maxdeg}_a(\mathcal{P}_{\mathcal{D}}) - \mathrm{mindeg}_a(\mathcal{P}_{\mathcal{D}})\right)/2 + 1 \leq n.$$

Corollary

For any link $\mathcal{L}$, one has

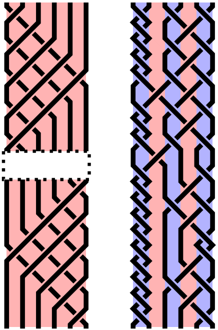
$$\left(\mathrm{maxdeg}_a(\mathcal{P}_{\mathcal{L}}) - \mathrm{mindeg}_a(\mathcal{P}_{\mathcal{L}})\right)/2 + 1 \leq \mathrm{Br}(\mathcal{L}).$$

Previous results

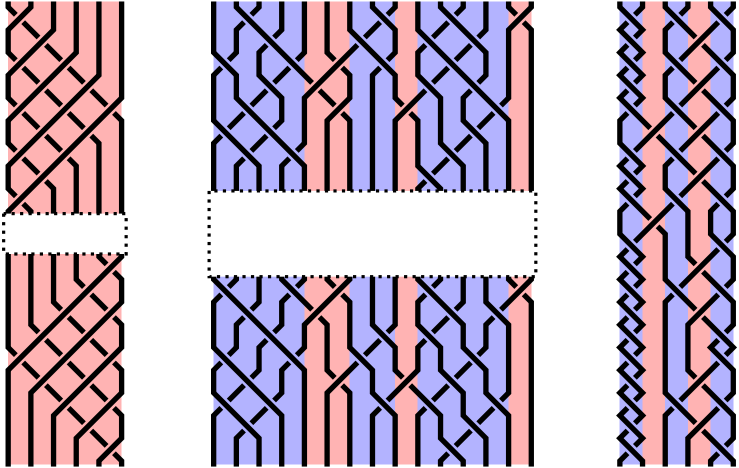
Objective

Describe infinite families of closed braid diagrams such that (MFW) is sharp.

- **J. Franks, R. F. Williams, 1987**. Closures of *positive* braids that begin and end with the half twist.
- **Y. Diao, G. Hetyei, P. Liu, 2019**. Closures of *alternating* braids that have no columns containing exactly one crossing.



Locally twisted braids



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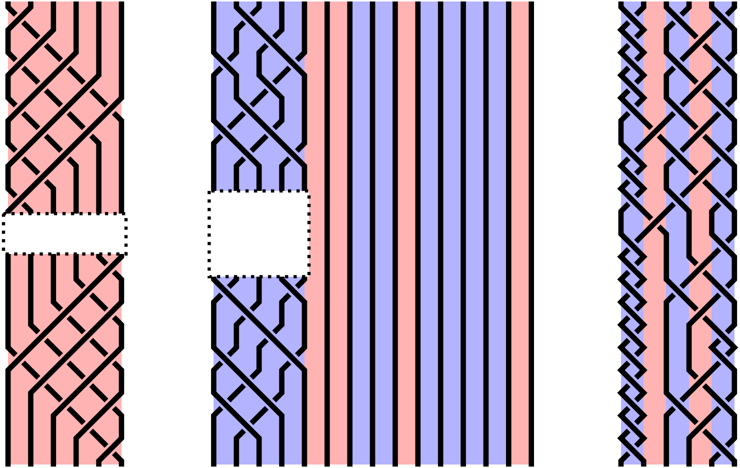

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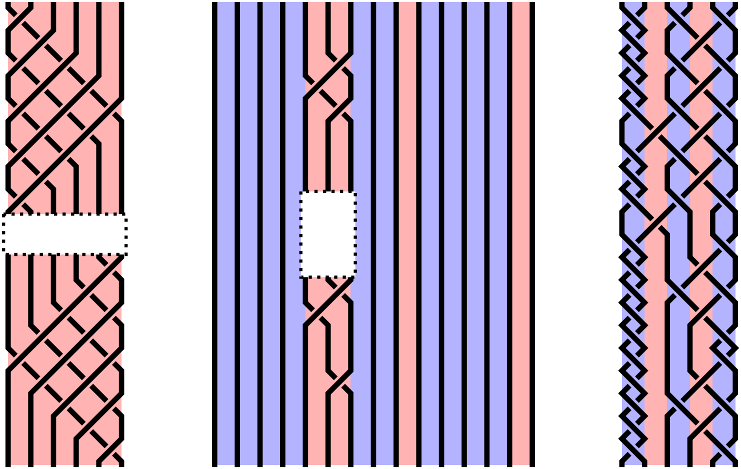

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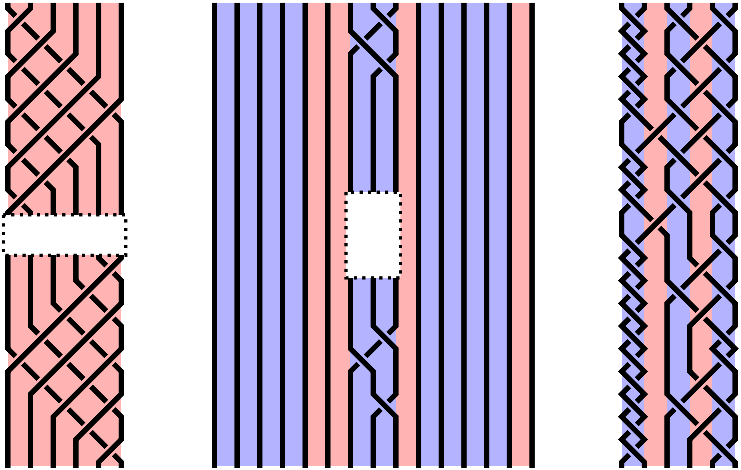

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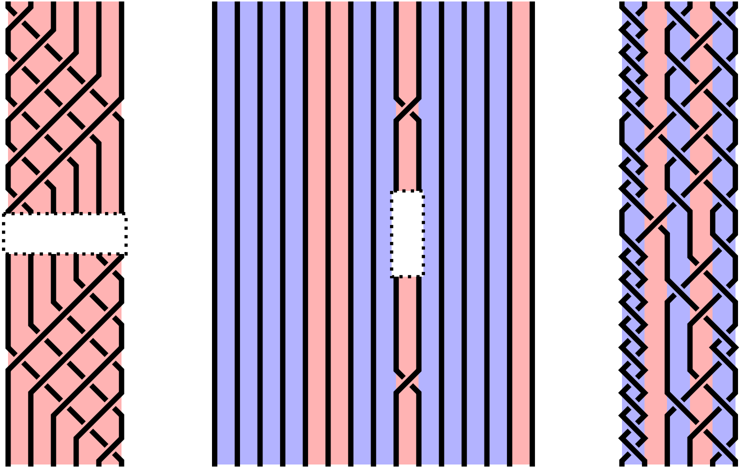

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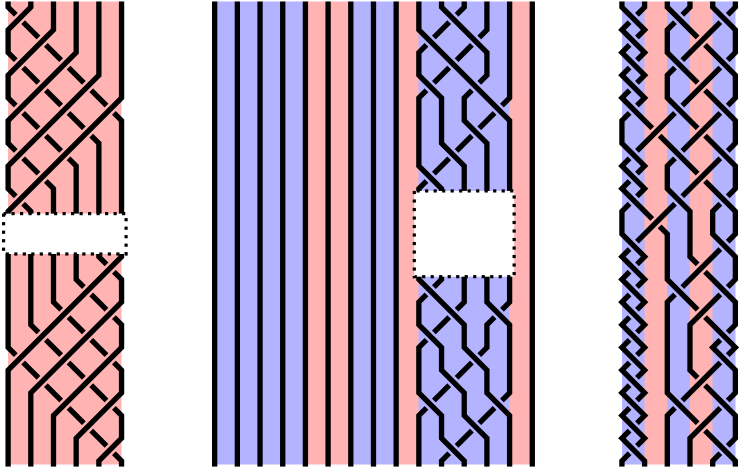

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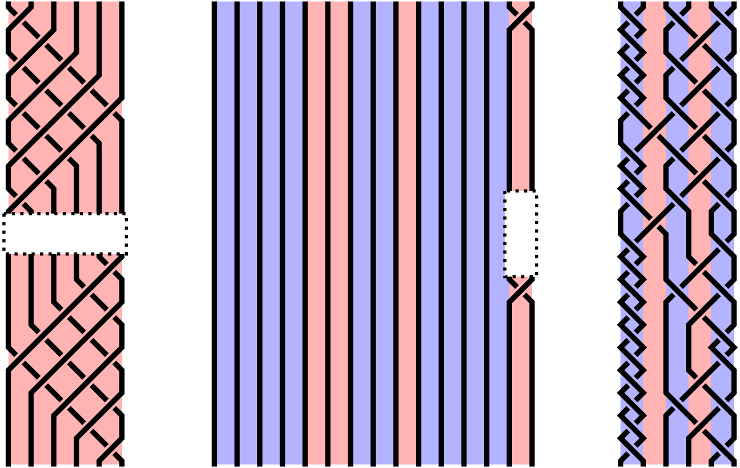

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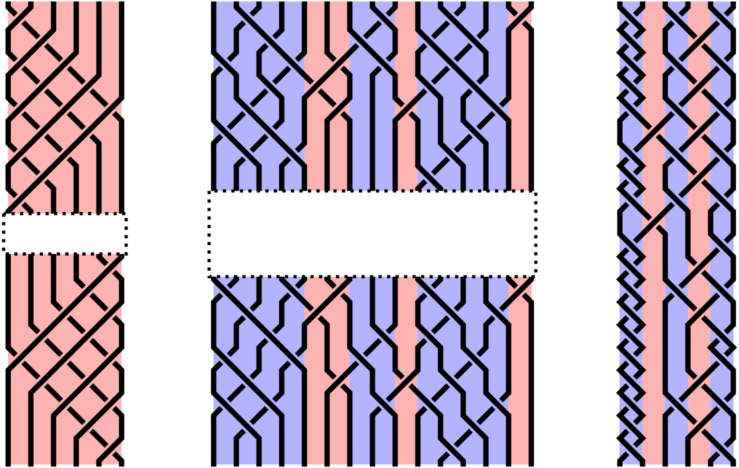

First main result and its applications

Theorem (I. A., 2019)

The Morton–Franks–Williams inequality is sharp for all closed locally twisted braid diagrams.

Folklore (weak form)

If a closed homogeneous braid diagram realizes the braid index, then it realizes the crossing number.

Corollary

Any closed locally twisted braid diagram realizes the crossing number.

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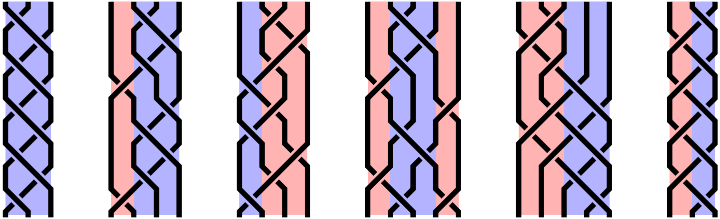
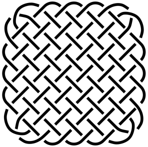
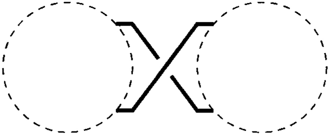
Examples of locally twisted braid diagrams



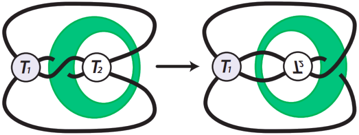
Alternating link diagrams

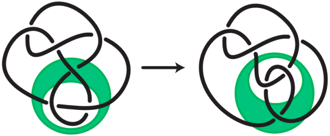




Theorem (L. Kauffman, K. Murasugi, M. Thistlethwaite, 1987)

Any *reduced* alternating diagram realizes the crossing number.

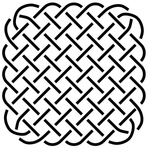


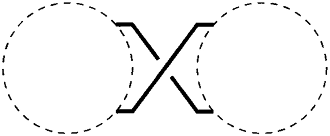


Theorem (W. Menasco, M. Thistlethwaite, 1993)

Any two reduced alternating diagrams representing the same link can be related by *flypes*.

Alternating link diagrams





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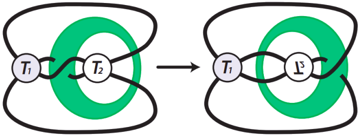

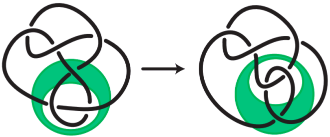

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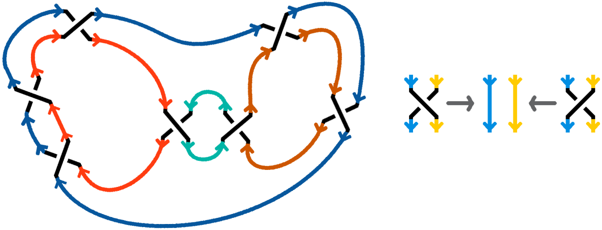
Morton–Franks–Williams inequality (strong form)

Definition

The *Seifert circles* of a link diagram $\mathcal{D}$ are simple closed curves obtained by smoothing all crossings of $\mathcal{D}$.



Theorem (S. Yamada, 1987)

Any link that admits a diagram $\mathcal{D}$ with $s(\mathcal{D})$ Seifert circles can be presented as a closed braid with $s(\mathcal{D})$ strands.

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Theorem (H. Morton, J. Franks, R. Williams, 1986)

For any diagram $\mathcal{D}$ of a link $\mathcal{L}$, one has

(MFW) $$(\mathrm{maxdeg}_a(\mathcal{P}_\mathcal{L}) - \mathrm{mindeg}_a(\mathcal{P}_\mathcal{L}))/2 + 1 \leq s(\mathcal{D}),$$

where $s(\mathcal{D})$ is the number of Seifert circles of $\mathcal{D}$.

Corollary

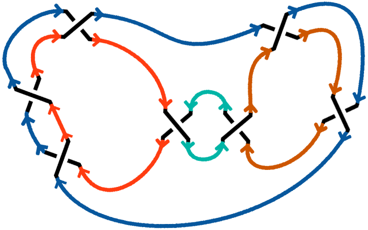
For any diagram $\mathcal{D}$ of a link $\mathcal{L}$, one has

$$(\mathrm{maxdeg}_a(\mathcal{P}_\mathcal{D}) - \mathrm{mindeg}_a(\mathcal{P}_\mathcal{D}))/2 + 1 \leq \mathrm{Br}(\mathcal{L}).$$

Sharpness of the Morton–Franks–Williams inequality for alternating diagrams

Theorem (Y. Diao, G. Hetyei, P. Liu, 2019)

The Morton–Franks–Williams inequality is sharp for all alternating diagrams that have no lonely crossings.



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Describe infinite families of link diagrams such that (MFW) is sharp.

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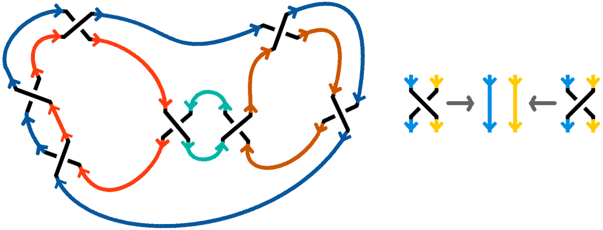

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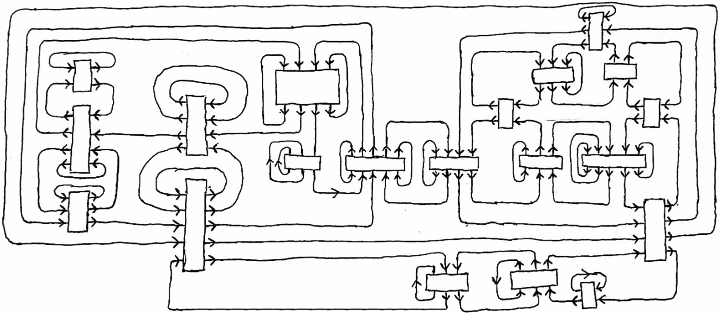
Locally twisted link diagrams



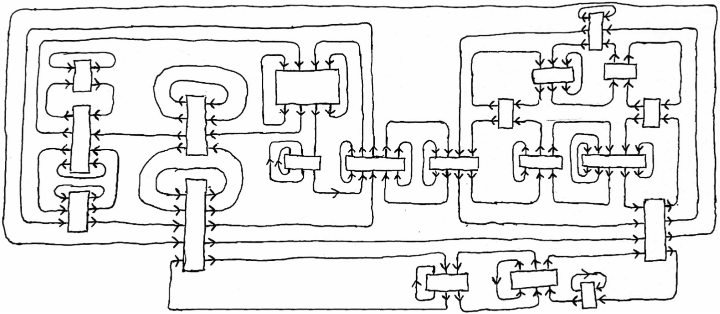
Definition

A stencil is *knitted* if any two Seifert circles share at most one braid box.

Definition

A link diagram is *locally twisted* if it can be obtained from a knitted stencil by inserting locally twisted braids.

Locally twisted link diagrams



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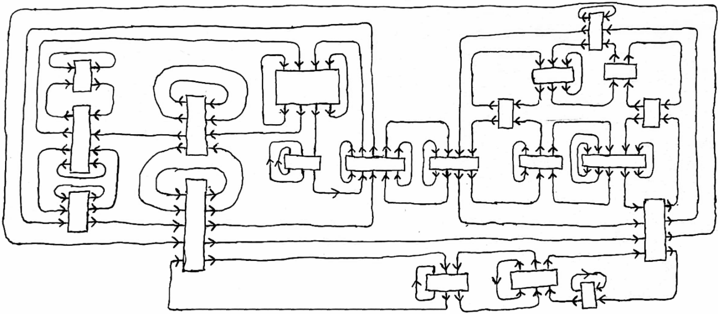

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Second main result and its applications

Theorem (I. A., 2020)

The Morton–Franks–Williams inequality is sharp for all locally twisted braid diagrams.

Folklore (moderate form)

If a *homogeneous* link diagram realizes the Seifert number, then it realizes the crossing number.

Corollary

Any homogeneous locally twisted link diagram realizes the crossing number.

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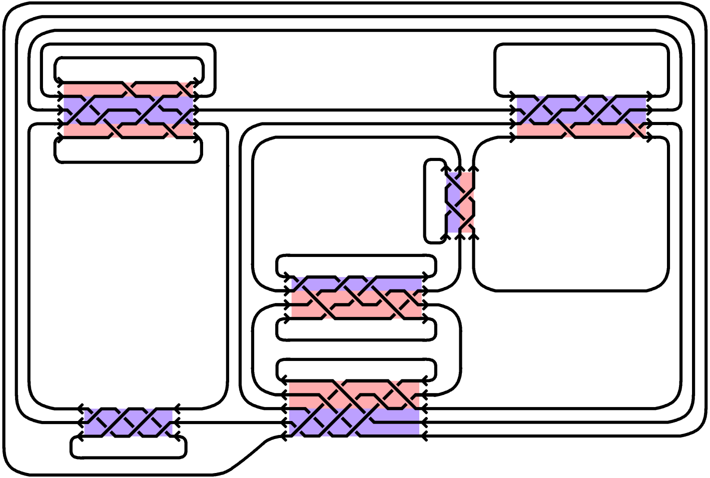
Homogeneous diagrams

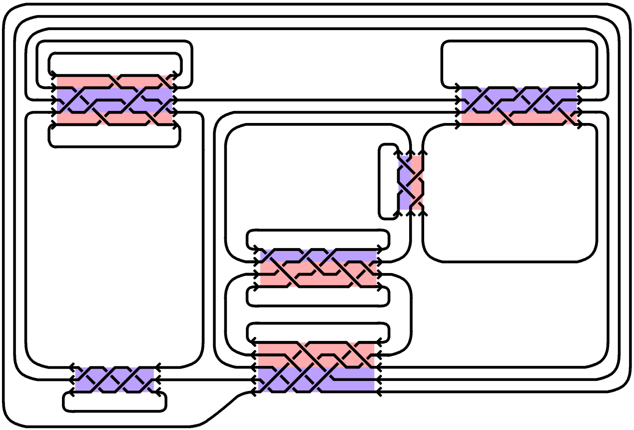
Definition

A link diagram $\mathcal{D}$ is said to be *homogeneous* if each block of the Seifert graph of $\mathcal{D}$ is homogeneous.



Homogeneous diagrams

- Any positive diagram is homogeneous.
- Any alternating diagram is homogeneous.
- The Seifert circles of $\mathcal{D}$ divide the plane into regions. If for each of these regions, crossings have the same type, then $\mathcal{D}$ is homogeneous.



Question 1

Does any locally twisted link diagram realize the crossing number?

Question 2

Without referring to the brute-force enumeration method, show that the crossing number of the knot 9_{49} is 9.

Folklore (strong form)

If for a link diagram $\mathcal{D}$,

- the canonical Seifert surface of $\mathcal{D}$ has minimal genus (among the canonical surfaces of other diagrams);
- $\mathcal{D}$ realizes the Seifert number,

then $\mathcal{D}$ realizes the crossing number.

Question 3

Describe new infinite families of link diagrams realizing the canonical genus.