Intersection formulas for parities on virtual knots

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Contents

Parity (V. Manturov, 2009).

Parity is a rule to assign numbers 0 and 1 to the (classical) crossings of diagrams of a knot in a way compatible with Reidemeister moves.

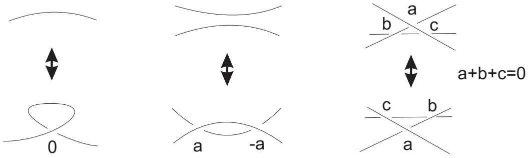
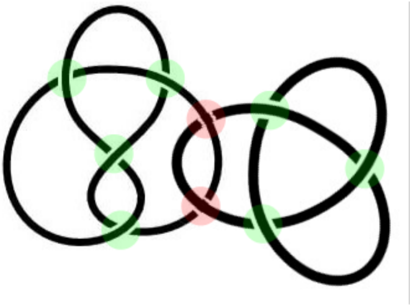


Figure: Parity axioms.

Applications of parity:

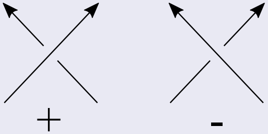
- strengthening of knot invariants
- extention of knot invariants (via parity projection between knot theories)
- minimality in a strong sense (parity bracket)

Link parity



Generalized linking number

$lk = \frac{1}{2} \sum_{c \text{ odd}} sgn(c)$

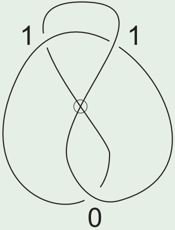


Gaussian parity

Definition

Gaussian parity of a crossing is the parity of the number of (classical) crossings that lie on a half of the knot corresponding to the crossing.

Example



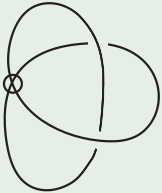
Odd writhe number $J = \sum_{c \text{ odd}} sgn(c) = -2$.

Virtual diagram

Definition

A *virtual diagram* is a generic immersion of a framed 4-graph into $\mathbb{R}^2$ (and also the image of this immersion), the image of each vertex is endowed with a classical crossing structure (with a choice for underpass and overpass specified) and intersection points of different edges are called *virtual crossings* and marked by a circle.

Example



Virtual knot

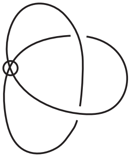
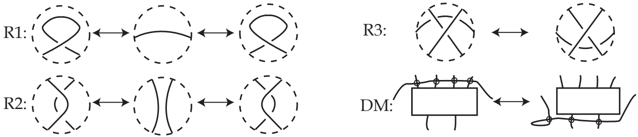


Figure: Diagram of the virtual trefoil

Definition

A *virtual link* is an equivalence class of virtual diagrams modulo Reidemeister moves and detour moves.



Knots in thickened surfaces

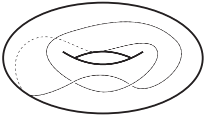


Figure: Virtual trefoil

Definition

A *virtual knot* is a knot in a thickened oriented surface $S^2_g \times I$ considered up to isotopies and stabilizations/destabilizations.

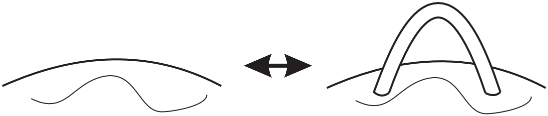


Figure: Stabilization move.

Gauss diagrams

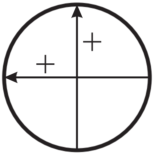
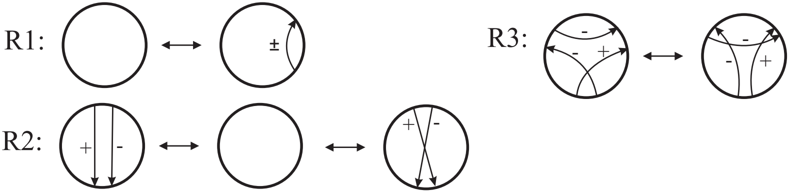


Figure: Gauss diagram of the virtual trefoil

Definition

A *virtual knot* is an equivalence class of Gauss diagrams modulo Reidemeister moves.



Flat and free knots

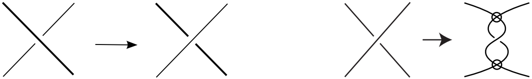


Figure: Crossing change and virtualization

Definition

Flat knots are virtual knots modulo crossing changes.
Free knots are virtual knot modulo crossing changes and virtualizations.

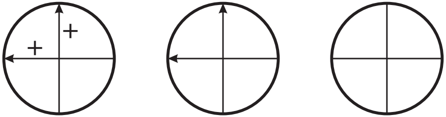


Figure: A virtual, a flat and a free knots

Knot theories

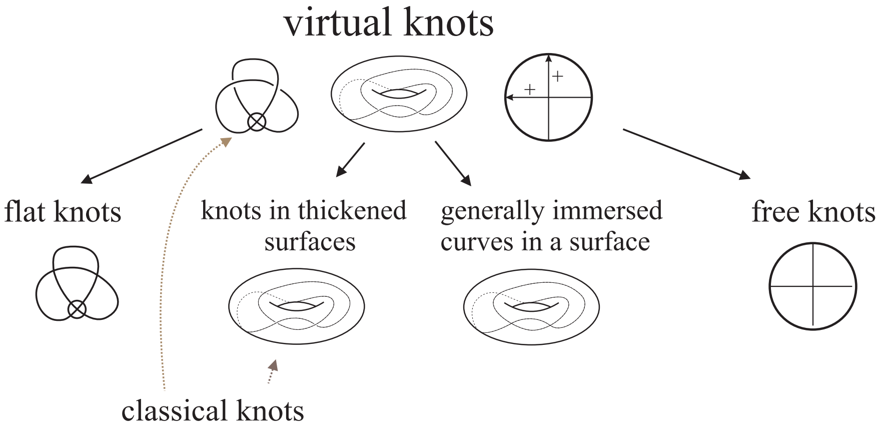


Diagram categories

A *diagram category* is a category $\mathfrak{K}$ whose

- Objects are diagrams of some knot $\mathcal{K}$;
- Morphisms are compositions of *elementary morphisms*. Elementary morphisms are:
    - isotopies
    - Reidemeister moves
    - symmetries of diagrams.

For a diagram $D \in \mathfrak{K}$, let $\mathcal{V}(D)$ be the set of crossings of D.

Remark

The map $D \mapsto \mathcal{V}(D)$ is a functor from $\mathfrak{K}$ to the category of finite sets and partial bijections between them.

Parity with coefficients in a group A

Definition

A parity p with coefficients in an abelian group A on a diagram category $\mathfrak{K}$ is a family of maps $p_D\colon \mathcal{V}(D) \to A$ defined for any diagram D such that for any elementary morphism $f\colon D \to D'$:

1. $p_{D'}(v') = p_D(v)$ *for any correspondent crossings* $v \in \mathcal{V}(D)$ *and* $v' \in \mathcal{V}(D')$ *;*
2. $p_D(v) = 0$ *if f is a decreasing first Reidemeister move and v is the disappearing crossing;*
3. $p_D(v_1) + p_D(v_2) = 0$ *if f is a decreasing second Reidemeister move and v_1 and v_2 are the disappearing crossings;*
4. $p_D(v_1) + p_D(v_2) + p_D(v_3) = 0$ *if f is a third Reidemeister move and* v_1, v_2, v_3 *take part in the move.*

Relation $2p = 0$ for a parity

Proposition

For any parity p and any crossing a one has $2p(a) = 0$.

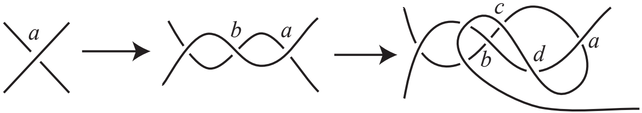


Figure: Proof of the equality $2p(a) = 0$

Oriented parity

Definition

A parity p with coefficients in an abelian group A on a diagram category $\mathfrak{K}$ is a family of maps $p_D\colon \mathcal{V}(D) \to A$ defined for any diagram D such that for any elementary morphism $f\colon D \to D'$:

1. $p_{D'}(v') = p_D(v)$ *for any correspondent crossings* $v \in \mathcal{V}(D)$ *and* $v' \in \mathcal{V}(D')$;
2. $p_D(v) = 0$ *if* f *is a decreasing first Reidemeister move and* v *is the dissapearing crossing;*
3. $p_D(v_1) + p_D(v_2) = 0$ *if* f *is a decreasing second Reidemeister move and* v_1 *and* v_2 *are the dissapearing crossings;*
4. $\epsilon(v_1)p_D(v_1) + \epsilon(v_2)p_D(v_2) + \epsilon(v_3)p_D(v_3) = 0$ *if* f *is a third Reidemeister move and* v_1, v_2, v_3 *take part in the move, and* $\epsilon(v_i)$ *is the incidence index of the crossing* v_i *to the triangle of the move.*

Oriented parity

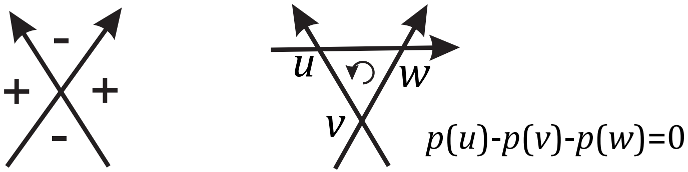


Figure: Incidence indices and an oriented parity relation

Remark

Oriented parity does not depend on orientation of the knot.

Example

The Gaussian index $ip(v) = D \cdot D_v^l$ is an oriented parity with coefficient in $\mathbb{Z}$.

Parities on virtual knots

For a long time, the Gaussian parity *ip* was the only one knows parity on virtual knots.
In [N., Parity on based matrices, arxiv:2110.04915] the *reduced stable parity* was constructed.
The construction below generalizes the construction of the reduced stable parity.

Potentials

Definition

Let $\mathcal{K}$ be a virtual knot and p be an oriented parity on diagrams of the knot $\mathcal{K}$ with coefficients in an abelian group A. For any diagram D of the knot and any two arc a and b in D, define the *potential* as shown in the figure. Analogously, one defines the potentials $\delta_{b,a}, \delta_{a,\bar{b}}, \delta_{\bar{b},a}, \delta_{\bar{a},\bar{b}}, \delta_{\bar{b},\bar{a}}$

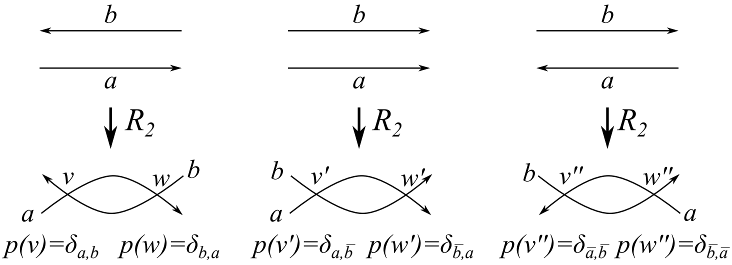


Figure: Potentials between arcs

Parity cycle

Definition

Let $\mathcal{K}$ be a virtual knot and p be an oriented parity on diagrams of the knot $\mathcal{K}$ with coefficients in an abelian group A. For any diagram D of the knot, define the *parity cycle* by the formula

$$\delta_D^p = \sum_{a\in\mathcal{A}(D)} \delta_a \cdot a \in C_1(D,A).$$

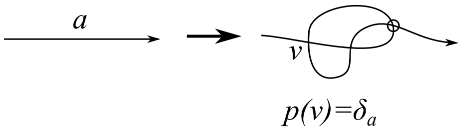


Figure: The potential $\delta_a = \delta_{a,\bar{a}}$ of an arc a

Intersection formula

Theorem

Let p be an oriented parity with coefficients in an abelian group A on the diagrams of an oriented virtual knot $\mathcal{K}$, D be a diagram of $\mathcal{K}$, and δ_D^p be the parity cycle. Then

1. δ_D^p *is a genuine 1-cycle in the cell complex* $C_*(\tilde{D}, A)$, *i.e.* $d\delta_D^p = 0$;
2. δ_D^p *is an invariant cycle;*
3. δ_D^p *is normalized:* $D \cdot \delta_D^p = 0$;
4. *(the intersection formula) for any crossing* $v \in \mathcal{V}(D)$ *we have*

$$p_D(v) = D_v^l \cdot \delta_D^p.$$

Theorem

The intersection formula defines an isomorphism $\mathcal{P}(\mathcal{K}, A) \simeq \mathcal{NIC}(\mathcal{K}, A)$ *between the set* $\mathcal{P}(\mathcal{K}, A)$ *of oriented parities with coefficients in* A *and the set* $\mathcal{NIC}(\mathcal{K}, A)$ *of normalized invariant cycles with coefficients in* A.

Biquandle

Definition

Let B be a set with binary operations $\circ, *$ such that:

- $x \circ x = x * x$
- the maps $*x, \circ x : B \to B$ and $S : B \times B \to B \times B$ where $S(x, y) = (y * x, x \circ y)$ are bijections.
- $(x \circ y) \circ (z \circ y) = (x \circ z) \circ (y * z)$
- $(x \circ y) * (z \circ y) = (x * z) \circ (y * z)$
- $(x * y) * (z * y) = (x * z) * (y \circ z)$

Then $(B, \circ, *)$ is called a *biquandle*.
For a knot diagram D, the map $c\colon \mathcal{A}(D) \to X$ which satisfies the conditions in Fig. is called a *colouring*. The set of the colouring of D is denoted by $Col_B(D)$.

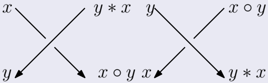


Figure: Colouring rule

Biquandle cocycle and parity cycle

Definition

A map $\theta\colon B \to A$ is called a *biquandle 1-cocycle* of B with coefficients in the group A if for any $x, y \in B$

$$\theta(x) - \theta(x \circ y) = \theta(y) - \theta(y * x).$$

Proposition

Let θ be a biquandle 1-cocycle with coefficients in A. For a diagram D and a colouring $c \in Col_B(D)$ consider

$$\delta^\theta_{D,c} = \sum_{a \in \mathcal{A}(D)} (\theta \circ c)(a) \cdot a \in C_1(D, A)$$

and $\delta^\theta_D = \sum_{c' \in Col_B(D)} \delta^\theta_{D,c'}$. Then δ^θ_D is an invariant 1-cycle with coefficients in A.

Colouring monodromy

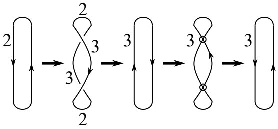
Definition

The *colour monodromy group* $Mon_B(D)$ of the diagram D as the subgroup in the permutation group of the colouring set $Col_B(D)$, formed by permutations $f_*\colon Col_B(D) \to Col_B(D)$ where $f\colon D \to D$ is an arbitrary morphism (a composition of isotopies and Reidemeister moves).

Example

Consider the biquandle $B = \{1, 2, 3\}$ with the operations given by the matrices

$$\circ = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \qquad * = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$



Parity quasi-index

Definition

Let p be an oriented parity with coefficients in a group A on diagrams of an oriented virtual knot $\mathcal{K}$ and δ be its parity cycle. Let D be a knot diagram. For any crossing v denote

$$\pi_D(v) = \delta_c - \delta_a = \delta_b - \delta_d.$$

The map $\pi_D\colon \mathcal{V}(D) \to A$ is called the *parity quasi-index* of p.

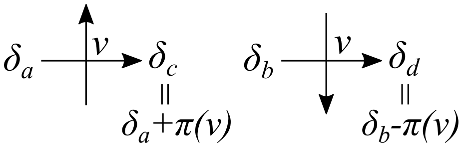


Figure: The quasi-index of a crossing

Quasi-index

Definition

Let $\mathcal{K}$ be a virtual knot. A family of maps $\pi_D \colon \mathcal{V}(D) \to A, D \in \mathfrak{K}$, is called a *quasi-index* on the diagrams of the knot $\mathcal{K}$ if the following conditions hold

(Q0) for any Reidemeister move $f \colon D \to D'$ and any crossing $v \in \mathcal{V}(D)$ which does not take part in the move, one has $\pi_D(v) = \pi_{D'}(f_*(v))$;

(Q2) $\pi_D(v_1) = \pi_D(v_2)$ for any crossings $v_1,\ v_2 \in \mathcal{V}(D)$ to which a decreasing second Reidemeister move can be applied;

(Q3) if $v_1, v_2, v_3 \in \mathcal{V}(D)$ are the crossings which take part in a third Reidemeister move $f \colon D \to D'$ then there exists an element $\lambda(f) \in A$ such that

$$\pi_{D'}(f_*(v_i)) = \pi_D(v_i) + \epsilon_\Delta(v_i) \cdot \lambda(f),\ \ i = 1, 2, 3,$$

where $\epsilon_\Delta(v_i)$ is the incidence index of the crossing v_i to the disappearing triangle Δ.

A quasi-index π is called an *index* on the diagrams of the knot $\mathcal{K}$ if the terms $\lambda(f)$ in condition (Q3) are equal to 0 for all third Reidemeister moves.

Note on index

In 2004 V. Turaev assigned a (signed) index n to crossings of flat knot diagrams and defined a polynomial invariant (index polynomial). This polynomial was numerously rediscovered in various forms afterwards by A. Henrich, Z. Cheng, Y.H. Im, K. Lee, S.Y. Lee, L. Kauffman and others. The notion of index was axiomatized by Z. Cheng in 2017 (under the name *chord index*). M. Xu gave the most general formulation of index (*weak chord index*) which allows to define index polynomial.

Remark

A parity p is not an index, but a *signed index*, i.e. $sgn \cdot p$ is an index.

Intersection formula for the quasi-index

Theorem

Let $\mathcal{K}$ be a virtual knot and p be an oriented parity with coefficients in a group A on the diagrams of the knot $\mathcal{K}$. Let δ be the parity cycle of p and $\pi_D \colon \mathcal{V}(D) \to A$, $D \in \mathfrak{K}$, be the parity quasi-index. Then

1. *the family of maps π is a quasi-index on diagrams of the knot $\mathcal{K}$;*
2. *for any diagram $D \in \mathfrak{K}$ there is a unique element $\rho(D) \in A$ such that*

$$\delta_D = \sum_{v \in \mathcal{V}(D)} \pi_D(v) \cdot D_v^r + \rho(D) \cdot D \in H_1(D, A). \tag{1}$$

Corollary (Intersection formula for the quasi-index)

$$p_D(v) = \sum_{v' \in \mathcal{V}(D)} \pi_D(v') \cdot (D_v^l \cdot D_{v'}^r) - \rho(D) \cdot ip_D(v)$$

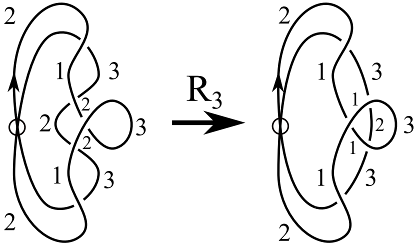
Example of quasi-index which is not an index

Example

Consider the biquandle $B = \{1, 2, 3\}$ with the operations given by the matrices

$$\circ = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \qquad * = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

and the 1-cocycle $\theta \in H^1(B, \mathbb{Z}_3)$ such that $\theta(1) = 0$, $\theta(2) = 1$ and $\theta(3) = -1$.



Quasi-index and biquandle 1-cocycle

Remark

Let $\mathcal{K}$ be a virtual knot, B be a biquandle and $\theta \in H^1(B, A)$ be a cocycle which defines an oriented parity p with coefficients in A on the diagrams of the knot $\mathcal{K}$. Then the following conditions ensure that the quasi-index π of the parity p is an index on the diagrams of $\mathcal{K}$: for all $x, y, z \in B$

$$\theta(x) - \theta(x \circ y) - \theta(x \circ z) + \theta\left((x \circ y) \circ (z \circ y)\right) = 0,$$
$$\theta(x) - \theta(x \circ y) - \theta(x * z) + \theta\left((x \circ y) * (z \circ y)\right) = 0,$$
$$\theta(x) - \theta(x * y) - \theta(x * z) + \theta\left((x * y) * (z * y)\right) = 0.$$

The conditions above come from the equations $\pi_{D'}(v_i') - \pi_D(v_i) = 0$ for the crossings v_i participating in third Reidemeister moves.

The signature of a quasi-index

Definition

Let π be an quasi-index on the diagrams of a virtual knot $\mathcal{K}$ with coefficients in an abelian group A. Let D be a diagram of $\mathcal{K}$. Then the *signature* $\sigma(\pi)$ of the quasi-index π is defined by the formula

$$\sigma(\pi) = \sum_{v \in \mathcal{V}(D)} \pi_D(v) \cdot (D \cdot D^r_v) = - \sum_{v \in \mathcal{V}(D)} \pi_D(v) \cdot ip_D(v) \in A.$$

Note that $\sigma(\pi) = D \cdot \delta_D$ where δ is the correspondent invariant 1-cycle.

Proposition

Let π be an quasi-index on the diagrams of a virtual knot $\mathcal{K}$ with coefficients in an abelian group A. Then the signature $\sigma(\pi)$ is invariant under Reidemeister moves.

Quasi-index monodromy

Example

Consider the constant index $\pi \equiv 1$ with coefficients in $\mathbb{Z}$ on the unknot diagrams. Take the diagram D below and set $\rho(D) = 0$. Consider the morphism $f\colon D \to D$ which consists of two first, one second Reidemeister move and detour moves. Then the reminder term ρ changes by 1. Hence, we have a monodromy group $Mon(\pi)$ which coincides with the coefficient group $\mathbb{Z}$.

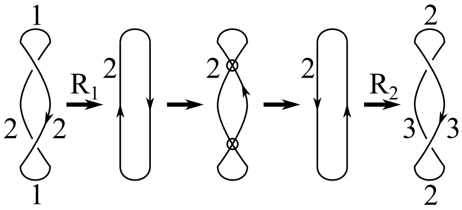


Figure: Quasi-index monodromy

Loop index values

Definition

Let π be an index on diagrams of $\mathcal{K}$ with coefficients in A. Denote the index values of the loops of type $l-$ and $r+$ by $\pi^\bullet \in A$ and the values of the loops of type $l+$ and $r-$ by $\pi^\circ \in A$.
The index π is *l_+-reduced* if $\pi^\circ = 0$. Analogously, the index π is *r_+-reduced* if $\pi^\bullet = 0$. And π is *R1-reduced* if $\pi^\circ = \pi^\bullet = 0$.

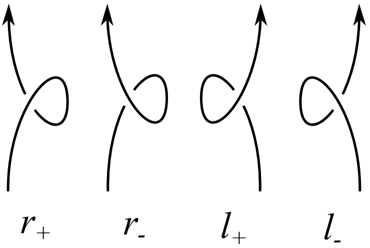


Figure: Types of loops

Intersection formula for l_+-reductions of indices

Theorem

Let π be an index on diagrams of a virtual knot $\mathcal{K}$ with coefficients in an abelian group A and let $\pi^\circ \in A$ be the index value of the crossings of types l_+ and r_-. Then the formula

$$\bar{\delta}^\pi_D = \sum_{v\in\mathcal{V}(D)\colon\, \pi_D(v)\neq\pi^\circ} sgn(v)\pi_D(v)\cdot D^-_v$$

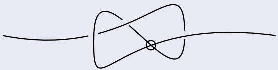
defines an invariant 1-cycle $\bar{\delta}^\pi_D$ and the formula

$$\bar{p}^\pi_D(v) = \sum_{v'\in\mathcal{V}(D)\colon\, \pi_D(v')\neq\pi^\circ} sgn(v')\pi_D(v')\cdot(D^l_v\cdot D^-_{v'})$$

defines an oriented parity $\bar{p}^\pi$ on diagrams of the knot $\mathcal{K}$ with coefficients in the group $\bar{A} = A/\langle\bar{\sigma}(\pi)\rangle$, where $\bar{\sigma}(\pi) = \sigma(\bar{\pi})$ is the signature of the l_+-reduction $\bar{\pi}$.

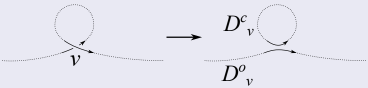
Definition

A *long virtual knot* is an equivalence class of long virtual knot diagrams modulo isotopies, Reidemester moves and detour moves.



Definition

Given a crossing $v \in \mathcal{V}(D)$ of a long knot diagram D, the oriented smoothing at v splits the diagram D into the *open half* D^o_v of D at the crossing v and the *closed half* D^c_v of D at v.



For a crossing $v \in \mathcal{V}(D)$ of the diagram D define its *order* $o(v)$ as follows: $o(v) = 1$ if $D^c_v = D^r_v$, and $o(v) = -1$ if $D^c_v = D^l_v$.

Intersection formula for long knots

Remark

The case of long knots has two features.

1. Since a long knot is not closed, there is no normalization condition on the parity 1-cycle.
2. Using closed halves, one can localize the increments which the parity quasi-index produces under Reidemeister moves, and resolve the problem of constructing the parity cycle from the parity quasi-index.

Theorem

Let π be a quasi-index on diagrams of a long virtual knot $\mathcal{K}$ with coefficients in an abelian group A, and $\rho \in A$ be an arbitrary element. Then the formula

$$p^{\pi}_D(v) = -o(v) \sum_{v' \in \mathcal{V}(D)} o(v')\pi_D(v') \cdot (D^c_v \cdot D^c_{v'}) + o(v) \cdot ip_D(v) \cdot \rho$$

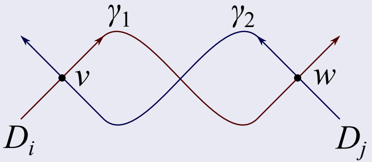
defines an oriented parity p^{π} on diagrams of $\mathcal{K}$ with coefficients in A.

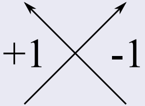
Theorem

Let $D = D_1 \cup \cdots \cup D_d$ be a diagram of an oriented virtual link $\mathcal{L}$, and p be an oriented parity with coefficients in A. Let δ^p be the parity cycle of p. For any crossings v and w of components D_i and D_j denote the path from v to w in D_i (along the orientation of the component) by γ_1, and the path from w to v in D_j by γ_2. Let $\gamma = \gamma_1\gamma_2$. Then

$$\eta_\gamma(v)p_D(v) + \eta_\gamma(w)p_D(w) = \gamma \cdot \delta^p_D$$

where $\eta_\gamma(v)$ and $\eta_\gamma(w)$ are the incidence indices of the crossings v and w to the cycle γ.





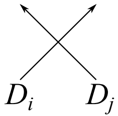
Link parity

Example (Link parity)

Let $\mathcal{L} = K_1 \cup \cdots \cup K_d$ be an oriented virtual link with d components, and A be an abelian group. Choose an arbitrary $(d-1)$-vector $l = (l_1, \ldots, l_{d-1})$ in A^{d-1}. Denote also $l_d = 0$.
Let $D = D_1 \cup \cdots \cup D_d$ be a diagram of the link $\mathcal{L}$. For a crossing $v \in \mathcal{V}(D)$ of components D_i and D_j, its *link parity* $lp^l_D(v)$ is defined by the formula

$$lp^l_D(v) = l_i - l_j.$$

Then by definition $lp^l(v) = 0$ for any self-crossing if D. Hence, $\delta^{lp^l} = 0$. Thus, a parity on links can not be restored from the parity cycle in general.



Remark

Let $\mathcal{P}(\mathcal{L}, A)$ be the set of oriented parities with coefficients in A on the diagrams of the link $\mathcal{L}$ and $\mathcal{NIC}(\mathcal{L}, A)$ be the set of normalized invariant cycles with coefficients in A on the diagrams of $\mathcal{L}$. The intersection formula defines a homomorphism

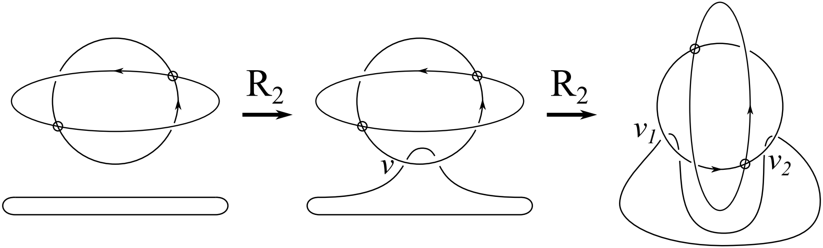
$$\Delta\colon \mathcal{P}(\mathcal{L}, A) \to \mathcal{NIC}(\mathcal{L}, A).$$

Then $\ker \Delta = \mathcal{LP}(\mathcal{L}, A)$ is the subgroup consisting of link parities. Note that $\mathcal{LP}(\mathcal{L}, A) \simeq A^{d-1}$.
On the other hand, in general the map Δ is not an epimorphism.

Intersection formula for links

Example

Let D be the diagram of a link $\mathcal{L}$ with three components. Consider a constant 1-cycle δ on $\mathcal{L}$ with coefficients in $\mathbb{Z}_2$: $\delta_D(a) = 1$ for any arc $a \in \mathcal{A}(D)$. Then δ is a normalized invariant cycle.
There is no parity whose parity cycle is δ.



Derived parities

Theorem

Let p be an oriented parity with coefficients in a group A on the diagrams of the knot $\mathcal{K}$. Then the formula

$$p'_D(v) = \sum_{v' \in \mathcal{V}(D)} p_D(v') \cdot (D^l_v \cdot D^-_{v'})$$

defines an oriented parity p' with coefficients in the group $\bar{A} = A/\langle \sigma(p) \rangle$ where $\sigma(p) = \sum_{v \in \mathcal{V}(D)} sgn(v) p(v) \cdot ip(v) \in A$ is the signature of the parity p.

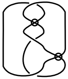


Figure: The knot 3.1. The signatures are $\sigma(ip) = 4$, $\sigma(ip') = 0$, $\sigma(ip'') = 1$. The parity vectors are $ip = (-1, -1, 2)$, $ip' = (-1, 1, -1)$, $ip'' = (-1, 0, 1)$, $ip''' = 0$.

Derived parities. Example

Example

Consider the knots 2.1 and 4.4. Then $LK_0(K_1) = LK_0(K_2) = -t^{-1} - t$ but $LK_1(K_1) = -2t^{-1}$ and $LK_1(K_2) = 0$ are different. Thus, the derived index polynomials can be more sensitive than the conventional one. Here

$$LK_n(D) = LK(ip^{(n)})(D) = \sum_{v \in \mathcal{V}(D):\ ip^{(n)}(v) \neq 0} sgn(v) t^{sgn(v) \cdot ip^{(n)}(v)}$$

is the linking invariant (odd index polynomial) of the derived parity $ip^{(n)}$.

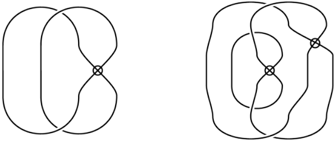


Figure: The knots 2.1 and 4.4

Open questions

1. Cohomology of biquandles generates parity cycles and parities. Is it true that any parity can be obtained by this construction? In particular, for a knot K describe the cohomology group $H^1(B(K), A)$ of the fundamental biquandle $B(K)$ of K.
2. Describe the colouring monodromy groups $Mon_B(\mathcal{K})$. Which knots have trivial monodromy? Describe the (quasi)index monodromy groups $Mon(\pi)$. Which indices π have trivial monodromy?
3. Find nontrivial examples of quasi-indices which are not indices. What is the meaning of the index conditions on biquandle 1-cocycles?

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