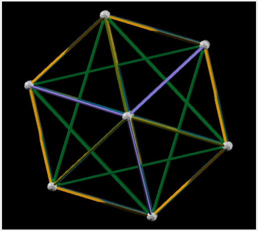
Algebraic Fibrations of Certain Hyperbolic 4-Manifolds

Fangting ZHENG[1] (joint work with Jiming MA)[2]

[1] Xi'an-Jiaotong Liverpool University, Suzhou, China
[2] Fudan University, Shanghai, China



Russian-Chinese Conference on Knot Theory
and Related Topics
24th-26th December 2021

http://labtd.nsu.ru/8RCCKT/index.html

Presentation Outline

1 Background

2 Manifolds Constructed via Colored Right-angled Polytopes

3 Algebraic fibration

4 our work

Definition

A group G **virtually algebraically fibers** if there is a finite index subgroup G' admitting an epimorphism $G' \to \mathbb{Z}$ with finitely generated kernel.

(Virtually) Algebric Fiberation

Definition

A group G **virtually algebraically fibers** if there is a finite index subgroup G' admitting an epimorphism $G' \to \mathbb{Z}$ with finitely generated kernel.

- **(1962) Stallings Fibration Theorem**
  If G is the fundamental group of a compact, irreducible, orientable 3-manifold M and G virtually algebraic fibers then the kernel is the fundamental group of a surface S, and the corresponding finite cover of M is an S-bundle over a circle.

(Virtually) Algebric Fiberation

Definition

A group G **virtually algebraically fibers** if there is a finite index subgroup G' admitting an epimorphism $G' \to \mathbb{Z}$ with finitely generated kernel.

- **(1962) Stallings Fibration Theorem**
  If G is the fundamental group of a compact, irreducible, orientable 3-manifold M and G virtually algebraic fibers then the kernel is the fundamental group of a surface S, and the corresponding finite cover of M is an S-bundle over a circle.
- **(2008) Agol- (2010) Wise-(2012)Przytycki and Wise-(2013) Agol-(2013) Liu**
  With the exception of a limited class of closed graph manifolds, every compact irreducible 3-manifold M with $\chi(M) = 0$ does virtually fiber.

- **(2018) Kielak**
  A finitely generated virtually RFRS group G admits algebraic fibration if and only if its first L^2-Betti number $\beta_1^{(2)}$ vanishes.

(Virtually) Algebric Fiberation

- **(2018) Kielak**
  A finitely generated virtually RFRS group G admits algebraic fibration if and only if its first L^2-Betti number $\beta_1^{(2)}$ vanishes.

- **(2019) Agol-Stover**
  Congruence RFRS towers

(Virtually) Algebric Fiberation

- **(2018) Kielak**
  A finitely generated virtually RFRS group G admits algebraic fibration if and only if its first L^2-Betti number $\beta_1^{(2)}$ vanishes.

- **(2019) Agol-Stover**
  Congruence RFRS towers

- **(2017) Friedl-Viduss and (2018) Stover**
  Virtually algebraic fibrations of complex hyperbolic manifolds.

In higer (especial for even) dimensions, this is not right in general.

- **(2017) Jankiewicz-Norin-Wise**
  Virtually fibering right-angled Coxeter groups.

(Virtually) Algebric Fiberation

In higer (especial for even) dimensions, this is not right in general.

- **(2017) Jankiewicz-Norin-Wise**
  Virtually fibering right-angled Coxeter groups.

- **(2021) Ma-Zheng**
  Algebraic fibrations of certain hyperbolic 4-manifolds with fiber-kernels finitely generated but not finitely presented fiber kernel. (i.e. no hope to be a 3-manifold bundle over S^1)

(Virtually) Algebric Fiberation

In higer (especial for even) dimensions, this is not right in general.

- **(2017) Jankiewicz-Norin-Wise**
  Virtually fibering right-angled Coxeter groups.

- **(2021) Ma-Zheng**
  Algebraic fibrations of certain hyperbolic 4-manifolds with fiber-kernels finitely generated but not finitely presented fiber kernel. (i.e. no hope to be a 3-manifold bundle over S^1)

- **(2021) Italiano-Martelli-Migliorini**, **(2021) Battista-Martelli**
  Some cusped hyperbolic n-manifolds, $5 \leq n \leq 8$, that fiber algebraically. And the algebraic fibrations may be promoted to a perfect circle-valued Morse functions.

- **(1973) Scott's theorem**
  Finitely generated 3-manifold groups are all finitely presented
  Ahlfors' Finiteness Theorem, *Sullivan's Cusp Finiteness Theorem* ect.

- **(1973) Scott's theorem**
  Finitely generated 3-manifold groups are all finitely presented
  Ahlfors' Finiteness Theorem, *Sullivan's Cusp Finiteness Theorem* ect.

- **(1998) Kapovich**
  a complex hyperbolic manifold M, if $\pi_1(M)$ algebraic fibers, then the fiber-kernel is not finitely presented.

- **(1973) Scott's theorem**
  Finitely generated 3-manifold groups are all finitely presented
  Ahlfors' Finiteness Theorem, *Sullivan's Cusp Finiteness Theorem* ect.

- **(1998) Kapovich**
  a complex hyperbolic manifold M, if $\pi_1(M)$ algebraic fibers, then the fiber-kernel is not finitely presented.

- **(1973) Scott's theorem**
  Finitely generated 3-manifold groups are all finitely presented
  Ahlfors' Finiteness Theorem, *Sullivan's Cusp Finiteness Theorem* ect.

- **(1998) Kapovich**
  a complex hyperbolic manifold M, if $\pi_1(M)$ algebraic fibers, then the fiber-kernel is not finitely presented.

⋆ ⋆ ⋆ ⋆ ⋆

We discuss here all possible "JNW-type" algebraic fibrations of 4-manifold that covering the right-angled hyperbolic 4-polytope P^4, which is dual to the Gosset polytope 0_{21}.

Presentation Outline

1 Background

2 Manifolds Constructed via Colored Right-angled Polytopes
- Right-angled Coxeter Groups
- Real toric manifolds

3 Algebraic fibration

4 our work

Definition

The **right-angled Coxeter group** $C(\Gamma)$ associated to a finite simplicial graph has finite generating set $\{g_1, \cdots g_N\}$ in one to one correspondence with the vertex set $V(\Gamma) = \{v_1, \cdots v_N\}$ and has finite presentation

$$C(\Gamma) = \langle\, v \in V(\Gamma) \mid u^2 = 1,\ [u, v] = 1 \text{ if } (u, v) \in E(\Gamma)\,\rangle,$$

where $E(\Gamma)$ denote the set of edges of Γ.

- $C(\Gamma)$ acts properly and cocompactly on a CAT(0) cube complex $\widetilde{X}$ (Davis complex of $C(\Gamma)$).

- $C(\Gamma)$ acts properly and cocompactly on a CAT(0) cube complex $\widetilde{X}$ (Davis complex of $C(\Gamma)$).
- The 1-skeleton of $\widetilde{X}$ is isomorphic to the Cayley graph of G after identifying each bigon to an edge.

  n-cubes are equivariantly added to the 1-skeleton for each collection of n pairwise commuting generators.

- $C(\Gamma)$ acts properly and cocompactly on a CAT(0) cube complex $\widetilde{X}$ (Davis complex of $C(\Gamma)$).
- The 1-skeleton of $\widetilde{X}$ is isomorphic to the Cayley graph of G after identifying each bigon to an edge.

  n-cubes are equivariantly added to the 1-skeleton for each collection of n pairwise commuting generators.
- There is a natural abelianization epimorphism $\alpha : C(\Gamma) \to \mathbb{Z}_2^{|V|}$.

L is a simple n-polytope with $\mathcal{F}(L) = \{F_1, F_2, \ldots, F_m\}$.

$$\lambda : \mathcal{F}(L) = \{F_1, F_2, \ldots, F_m\} \longrightarrow \mathbb{Z}_2^k, n \leq k \leq m$$

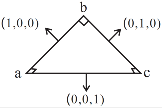
satisfies that $\langle \lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_n}) \rangle = \mathbb{Z}_2^n$ when $\cap F_{i_l} \neq \emptyset$.

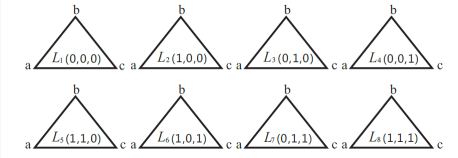
$M(L, \lambda) := L \times \mathbb{Z}_2^k / \sim$ is a **real toric manifold over the polytope L**,

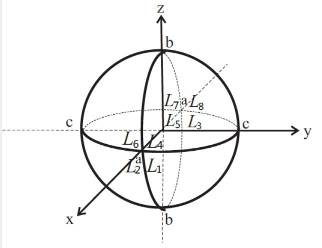
$$(x, g_1) \sim (y, g_2) \Longleftrightarrow \begin{cases} x = y \text{ and } g_1 = g_2 & \text{if } x \in \operatorname{Int} L, \\ x = y \text{ and } g_1^{-1} g_2 \in G_f & \text{if } x \in \partial L, \end{cases}$$

where $f = F_{i_1} \cap \cdots \cap F_{i_{n-r}}$ is the unique face of co-dim $n - r$ that $x \in \text{int}(f)$, $G_f = \langle \lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_{n-r}}) \rangle$.

Example


b
(1,0,0)
(0,1,0)
a
c
(0,0,1)


L1 (0,0,0)
L2 (1,0,0)
L3 (0,1,0)
L4 (0,0,1)
L5 (1,1,0)
L6 (1,0,1)
L7 (0,1,1)
L8 (1,1,1)
a
b
c


z
y
x
a
b
c
L1
L2
L3
L4
L5
L6
L7
L8

Presentation Outline

1 Background

2 Manifolds Constructed via Colored Right-angled Polytopes

3 Algebraic fibration
- Bestvina-Brady Morse Theory on Cube Complex
- Jankiewicz-Norin-Wise Admissible System

4 our work

Definition

Suppose group $G \curvearrowright^{\text{iso.}} \widetilde{X}^{\text{c.c.c}}$ [1] freely, cocompactly, properly. Let $\phi : G \to \mathbb{Z}$ be an epimorphism, and let $\mathbb{Z} \curvearrowright^{\text{trans.}} \mathbb{R}$.

Say C^1-map $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ is a ***ϕ-equivariant Morse function on $\widetilde{X}$*** if:

[1] This is short for "contratible cube complex"

Definition

Suppose group $G \curvearrowright^{\text{iso.}} \widetilde{X}^{\text{c.c.c}}$ [1] freely, cocompactly, properly. Let $\phi : G \to \mathbb{Z}$ be an epimorphism, and let $\mathbb{Z} \curvearrowright^{\text{trans.}} \mathbb{R}$.

Say C^1-map $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ is a ***ϕ-equivariant Morse function on $\widetilde{X}$*** if:

- $\widetilde{\phi} \circ g = \phi(g) \circ \widetilde{\phi}$ for all $g \in G$ (equivariant condition).

[1] This is short for "contratible cube complex"

Definition

Suppose group $G \curvearrowright^{\text{iso.}} \widetilde{X}^{\text{c.c.c}}$ [1] freely, cocompactly, properly. Let $\phi : G \to \mathbb{Z}$ be an epimorphism, and let $\mathbb{Z} \curvearrowright^{\text{trans.}} \mathbb{R}$.

Say C^1-map $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ is a ***ϕ-equivariant Morse function on $\widetilde{X}$*** if:

- $\widetilde{\phi} \circ g = \phi(g) \circ \widetilde{\phi}$ for all $g \in G$ (equivariant condition).
- $\forall$ n-cell $e \in \widetilde{X}$ with characteristic map $\chi_e : \square^n \to \widetilde{X}$, we have $\widetilde{\phi} \circ \chi_e : \square^n \to \mathbb{R} \xrightarrow{\text{extends}} \mathbb{R}^n \to \mathbb{R}$, and $\widetilde{\phi} \circ \chi_e$ is constant only for $n = 0$.

[1] This is short for "contratible cube complex"

Definition

Suppose group $G \curvearrowright^{\text{iso.}} \widetilde{X}^{\text{c.c.c}}$ [1] freely, cocompactly, properly. Let $\phi : G \to \mathbb{Z}$ be an epimorphism, and let $\mathbb{Z} \curvearrowright^{\text{trans.}} \mathbb{R}$.

Say C^1-map $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ is a ***ϕ-equivariant Morse function on $\widetilde{X}$*** if:

- $\widetilde{\phi} \circ g = \phi(g) \circ \widetilde{\phi}$ for all $g \in G$ (equivariant condition).
- $\forall$ n-cell $e \in \widetilde{X}$ with characteristic map $\chi_e : \square^n \to \widetilde{X}$, we have $\widetilde{\phi} \circ \chi_e : \square^n \to \mathbb{R} \xrightarrow{\text{extends}} \mathbb{R}^n \to \mathbb{R}$, and $\widetilde{\phi} \circ \chi_e$ is constant only for $n = 0$.
- The $\widetilde{\phi}(\widetilde{X}^{(0)})$ is discrete in $\mathbb{R}$.

[1] This is short for "contratible cube complex"

Ascending / Descending Links

Given a cell e in a cube complex $\widetilde{X}$ with $\chi_e : \square_e \to e, w \mapsto c \in e$

$$\chi_{e^*} : link(w, \square_e) \to link(v, e) \subset Lk(v, \widetilde{X}).$$

Given a cell e in a cube complex $\widetilde{X}$ with $\chi_e : \square_e \to e, w \mapsto c \in e$

$$\chi_{e^*} : link(w, \square_e) \to link(v, e) \subset Lk(v, \widetilde{X}).$$

Definition

The ascending link of $\widetilde{\phi}$:
$L_\uparrow = \bigcup \{\chi_{e^*}(link(w, \square_e)) \mid \chi_{e^*}(w) = v$ and $\widetilde{\phi}\chi_e$ has a min at w $\}$
The descending link of $\widetilde{\phi}$:
$L_\downarrow = \bigcup \{\chi_{e^*}(link(w, \square_e)) \mid \chi_{e^*}(w) = v$ and $\widetilde{\phi}\chi_e$ has a max at w $\}$

Theorem *(Bestvina-Brady: 1997; Brady: 1999)*

Let $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = Ker(\phi)$ be as above.

Theorem *(Bestvina-Brady: 1997; Brady: 1999)*

Let $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = Ker(\phi)$ be as above.

- Suppose the reduced homology of each $\mathsf{L}_{\uparrow}$ and each $\mathsf{L}_{\downarrow}$ is homological n-connected. Then $H \in FH_{n+1}$.

Theorem *(Bestvina-Brady: 1997; Brady: 1999)*

Let $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = Ker(\phi)$ be as above.

- Suppose the reduced homology of each $L_{\uparrow}$ and each $L_{\downarrow}$ is homological n-connected. Then $H \in FH_{n+1}$.
- Suppose the reduced homology of each $\text{link}_{\uparrow}$ and each $\text{link}_{\downarrow}$ is zero in all dimension 0 through n+1, except for dimension n. Then H is of type FP_n but is not of type FP_{n+1}.

Theorem *(Bestvina-Brady: 1997; Brady: 1999)*

Let $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = Ker(\phi)$ be as above.

- Suppose the reduced homology of each $\mathsf{L}_{\uparrow}$ and each $\mathsf{L}_{\downarrow}$ is homological n-connected. Then $H \in FH_{n+1}$.
- Suppose the reduced homology of each link$_{\uparrow}$ and each link$_{\downarrow}$ is zero in all dimension 0 through n+1, except for dimension n. Then H is of type FP_n but is not of type FP_{n+1}.
- If all $\mathsf{L}_{\uparrow}$s and $\mathsf{L}_{\downarrow}$s are simply connected, then H is finitely presented.

- $F_1 \Leftrightarrow$ finitely generated; $F_2 \Leftrightarrow$ finitely presented
- $FH_1(R)$ for all $R \Leftrightarrow F_1$
- $F_n \Rightarrow FH_n \Rightarrow FP_n$

Jankiewicz-Norin-Wise Admissible System

Let $\Gamma = \Gamma(V, E)$ be a simplicial graph with vertices V and edges E.

- A state of Γ is a subset $S \subset V$.

Jankiewicz-Norin-Wise Admissible System

Let $\Gamma = \Gamma(V, E)$ be a simplicial graph with vertices V and edges E.

- A state of Γ is a subset $S \subset V$.
- The state is legal if Γ_S and Γ_{V-S} are both nonempty and connected.

Let $\Gamma = \Gamma(V, E)$ be a simplicial graph with vertices V and edges E.

- A state of Γ is a subset $S \subset V$.
- The state is legal if Γ_S and Γ_{V-S} are both nonempty and connected.
- A move at $v \in V$ is an element $m_v \in 2^V$ satisfying:
    1. $v \in m_v$.
    2. $u \notin m_v$ if $\{u, v\} \in E$.
- A move system is a choice m_v of a move for each $v \in V$.

  Identify $\mathbb{Z}_2^V$ with 2^V

- The system is legal if there is an M-orbit all of whose elements are legal states. The M-orbit is called as a legal orbit. Refer such pair of move system and state as an admissible system.

Theorem *(Jankiewicz-Norin-Wise: 2019)*

Let Γ be a finite graph. Suppose there is a move system $m : V \to 2^V$ with a legal orbit. Then there is a discrete Morse function $\widetilde{\phi} : \widetilde{X} \to \mathbb{R}$ whose ascending and descending links are non-empty and connected.

Presentation Outline

1 Background

2 Manifolds Constructed via Colored Right-angled Polytopes

3 Algebraic fibration

4 our work

Hyperbolic right-angled polytope P^4

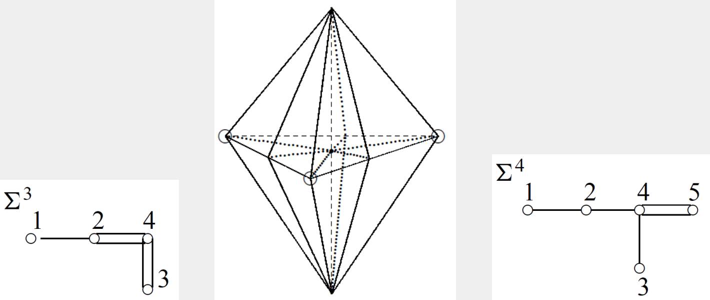


Figure 1: Figures are from L. Potyagailo-E. Vinberg (2005)

	facets	ideal	finite	Isom(P^4)	orbit χ	volume
P^4	10	5	5	A_4	1/16	$\pi^2/12$

Results

We search all the “JNW-type" algebraic fibration over P^4

$\mid$ move system $\mid$	5	6	7	8	9	10
rare move systems	6	90	145	75	15	1
factoring Isom(P^4)	1[2]	2	4	2	1	1
legal state	[1+(5)]	[1]+[1+1]	4×[0]	2×[0]	[0]	[0]

```
5   [[1, 8], [2, 10], [3, 7], [4, 5], [6, 9]]          [1, 2, 3, 4, 9]
                                                        [1, 2, 3, 4] [1, 2, 3, 6] [1, 2, 4, 6] [1, 3, 4, 6] [2, 3, 4, 6]

6   [[1, 8], [2, 7], [3, 9], [4, 6], [5], [10]]        [1, 2, 3, 6]
    [[1, 10], [2, 7], [3, 9], [4, 5], [6], [8]]        [1, 2, 3] [1, 2, 3, 4]

    [[1], [2], [3], [4], [5, 10], [6, 9], [7, 8]]      []
7
    [[1], [2, 6], [3], [4], [5, 10], [7, 8], [9]]      []
    [[1, 8], [2], [3, 7], [4], [5], [6, 9], [10]]      []
    [[1, 9], [2, 6], [3], [4], [5], [7, 8], [10]]      [1, 2, 7]

8   [[1], [2], [3], [4, 8], [5, 10], [6], [7], [9]]
    [[1, 10], [2], [3, 7], [4], [5], [6], [8], [9]]

9   [[1, 9], [2], [3], [4], [5], [6], [7], [8], [10]]

10  [[1], [2], [3], [4], [5], [6], [7], [8], [9],[10]]
```

[2]This is found by Italiano-Matelli-Migliorini (2021)

- "Different" algebraic fibrations of a given n- manifold, where $n \geq 4$.

Definition (2011 D. Calegari-H. Sun-S. Wang)

A fibered pair $(\widetilde{M}, \widetilde{\mathbb{F}})$ covers $(M, \mathbb{F})$ if there is a finite covering of manifolds $\pi : \widetilde{M} \to M$ such that $\pi^{-1}(\mathbb{F})$ is isotopic to $\widetilde{\mathbb{F}}$.

Two fibered pairs $(M_1, \mathbb{F}_1)$ and $(M_2, \mathbb{F}_2)$ are commensurable if there is a third fibered pair $(\widetilde{M}, \widetilde{\mathbb{F}})$ that covers both.

• "Different" algebraic fibrations of a given n- manifold, where $n \geq 4$.

Definition (2011 D. Calegari-H. Sun-S. Wang)

A fibered pair $(\widetilde{M}, \widetilde{\mathbb{F}})$ covers $(M, \mathbb{F})$ if there is a finite covering of manifolds $\pi : \widetilde{M} \to M$ such that $\pi^{-1}(\mathbb{F})$ is isotopic to $\widetilde{\mathbb{F}}$.

Two fibered pairs $(M_1, \mathbb{F}_1)$ and $(M_2, \mathbb{F}_2)$ are commensurable if there is a third fibered pair $(\widetilde{M}, \widetilde{\mathbb{F}})$ that covers both.

• Σ-invariant of fundamental group of high-dimensional hyperbolic manifolds

Thank You for your listening!