KNOT INVARIANTS WITH MULTIPLE SKEIN RELATIONS CONTAINING VIRTUAL CROSSINGS

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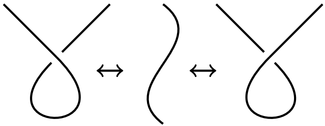
Table of Contents

Introduction

The fundamental question in Knot theory is, when two given knot diagrams are equivalent? The two knot diagrams D and D' are equivalent if and only if D can be transformed to D' using generalized Reidemeister moves. However, it is not easy to show the equivalence of two knot diagrams by Reidemeister moves. Therefore, we need a knot invariant to show either two knot diagrams are equivalent or not.



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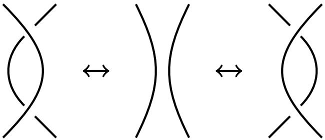

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Introduction

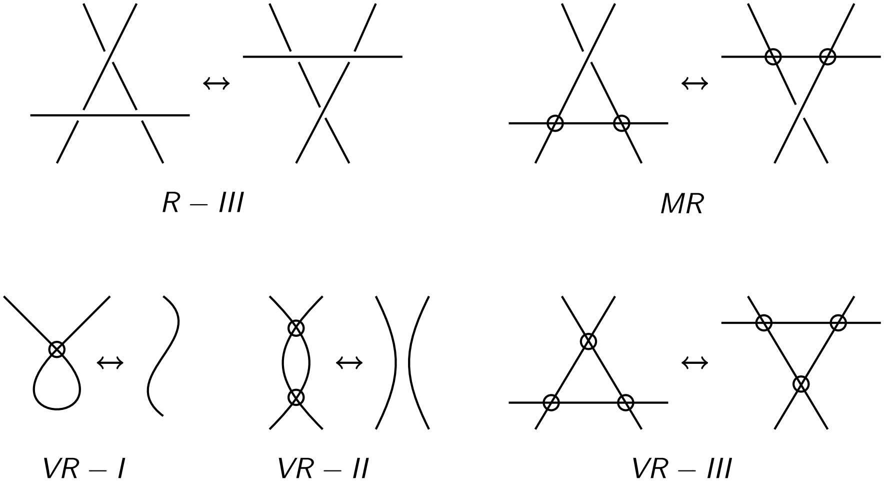


Figure 1: Generalized Reidemeister moves.

Introduction

Figure 2: The skein relation of Kauffmann polynomial

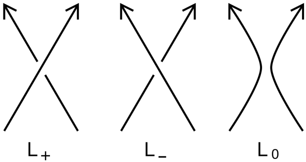
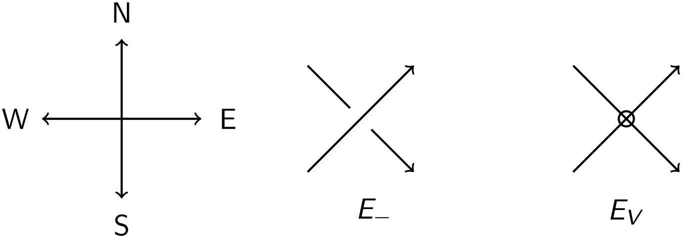


Figure 3: The skein relation ofJones polynomial

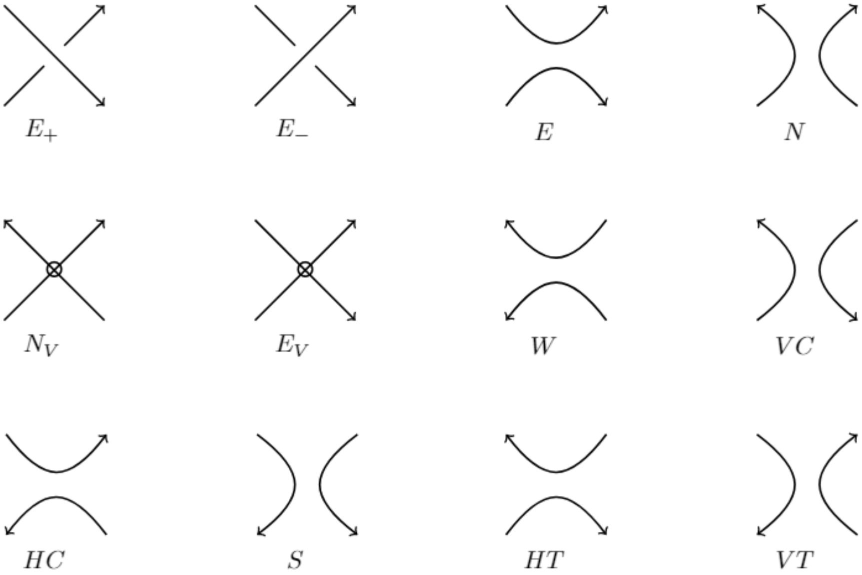
Those well-know knot invariants has only one skein relation, and the skein relation has three terms.

Introduction



The symbols E, S, W, N denote the cardinal directions, east, south, west, and north. The $+$ or $-$ sign in the subscript is representing the positive crossing or negative crossing. For example, E_- means the middle of the two arrows points toward the east direction, and it is a negative crossing. Similarly, E_V means the middle of the two arrows points toward the east direction, and the crossing is a virtual crossing.

Introduction



Introduction

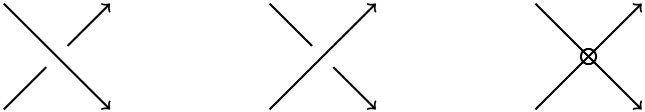
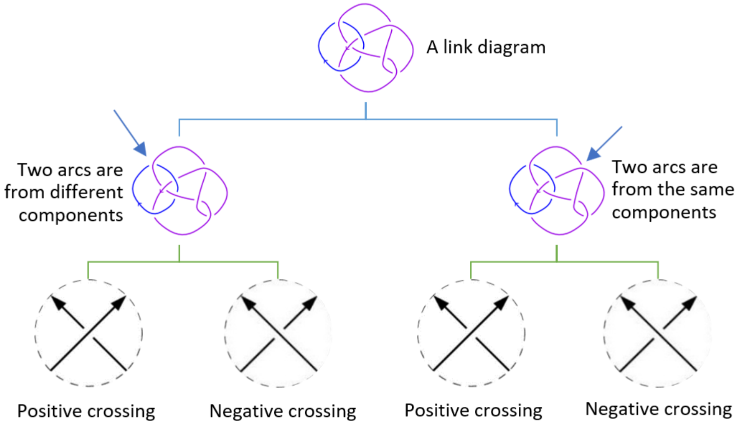


Figure 4: The classical and virtual crossings

- Crossing virtualization is a local move on a classical crossing that change the classical crossing into the virtual crossing.
- Virtualizing all classical crossings converts a knot into the unknot.
- Virtualization reduces the number of classical crossings.

Since virtualizing or smothing a classical crossings reduces the number of classical crossings, therefore, we used a system of skein relations that either smooth a classical crossing or replace it with the virtual crossing.

Introduction



We use a system of skein equations to construct a knot invariant, each skein equation has six or eight terms. The multiple skein relations use new ways to smooth or virtualize a crossing.

Theorem 1

Theorem

For oriented link diagrams, there is a link invariant f with values in X and satisfies the following skein relations: If the two arcs in the local diagram are from the same link component, then

$$f(E_+) = -\{c_1 f(E) + c_2 f(W) + c_3 f(HC) + c_4 f(HT) + d_1 f(VC) + d_2 f(VT) + ef(E_V)\}$$

$$f(E_-) = -\{\overline{c}_1 f(E) + \overline{c}_2 f(W) + \overline{c}_3 f(HC) + \overline{c}_4 f(HT) + \overline{d}_1 f(VC) + \overline{d}_2 f(VT) + \overline{e} f(E_V)\}$$

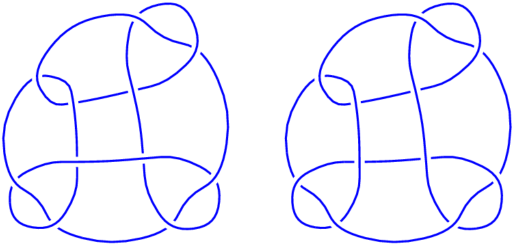
If the two arcs are from different components, then

$$f(E_+) = -\{c_1' f(E) + c_2' f(W) + d_1' f(S) + d_2' f(N) + e' f(E_V)\}$$

$$f(E_-) = -\{\overline{c}_1' f(E) + \overline{c}_2' f(W) + \overline{d}_1' f(S) + \overline{d}_2' f(N) + \overline{e}' f(E_V)\}$$

Here X denote the quotient commutative ring and $R = R_1 \cup R_2 \cup R_3$.

$$Z[c_1, c_2, c_3, c_4, d_1, d_2, \overline{c}_1, \overline{c}_2, \overline{c}_3, \overline{c}_4, \overline{d}_1, \overline{d}_2, c_1', c_2', d_1', d_2', \overline{c}_1', \overline{c}_2', \overline{d}_1', \overline{d}_2', z_{(n,k)}]/R.$$



(a) K11n73 (b) K11n74

- Is it only another complicated invariant?
- Is there any relation with existing invariants?
- Both Kauffman two-variable polynomial and HOMFLY polynomial can not distinguish the K11n73 and K11n74, but our invariant can distinguish K11n73 and K11n74 clearly and very easily.

Freedom of resolving crossing order

The first crossing p The second crossing q

Figure 6: The label of two crossings (both are positive crossings)

For two classical crossings, there must be at most four arcs. Denote them by A, B, C, and D. The arrow of each arc represent the orientation of the link component. If more than one arc is from the same link component, we write them together. For example, the three arcs A, B and D are from the same link component, such that $A \to B \to D$ along the link orientation and C is another component, then we donate this information as (ABD, C). There are several ways to join these arcs, therefore, we have many possible cases.

Case 1

$$
\begin{aligned}
&c'_2\overline{d}'_1 = \overline{d}'_1 c'_2,\ c'_2\overline{c}'_2 = \overline{d}'_1 d'_2,\\
&d'_2\overline{d}'_1 = \overline{c}'_2 c'_2,\ d'_2\overline{c}'_2 = \overline{c}'_2 d'_2,\\
&c'_1 c'_2 + c'_2\overline{d}'_2 = c'_2 c'_1 + \overline{d}'_2 c'_2,\\
&c'_1 d'_2 + c'_2\overline{c}'_1 = c'_2 d'_1 + \overline{d}'_2 d'_2,\\
&d'_1 c'_2 + d'_2\overline{d}'_2 = d'_2 c'_1 + \overline{c}'_1 c'_2,\\
&d'_1 d'_2 + d'_2\overline{c}'_1 = d'_2 d'_1 + \overline{c}'_1 d'_2,\\
&d'_2\overline{e}' = d'_2 e',\ e' c'_2 = \overline{e}' c'_2,\\
&e' d'_2 = \overline{e}' d'_2,\ c'_2\overline{e}' = c'_2 e'.
\end{aligned}
$$

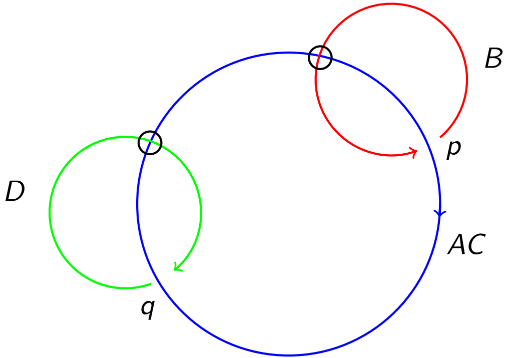


Figure 7: CASE (AC, B, D)

Case 2

$$
\begin{aligned}
&c_1'c_1 + c_2'c_2 = c_1c_1' + c_2c_2',\\
&c_1'c_2 + c_2'c_1 = c_2c_1' + c_1c_2',\\
&d_2'\overline{c}_1 + d_1'\overline{c}_2 = \overline{c}_1d_2' + \overline{c}_2d_1',\\
&d_2'\overline{c}_2 + d_1'\overline{c}_1 = \overline{c}_2d_2' + \overline{c}_1d_1',\\
&c_1'c_4 + c_2'c_4 = c_3c_1' + c_3c_2',\\
&c_1'd_1 + c_2'd_1 = \overline{d}_1d_2' + \overline{d}_1d_1',\\
&c_1'c_3 + c_2'c_3 = c_4c_1' + c_4c_2',\\
&c_1'd_2 + c_2'd_2 = \overline{d}_2d_2' + \overline{d}_2d_1',\\
&d_2'\overline{d}_1 + d_1'\overline{d}_1 = d_1c_1' + d_1c_2',\\
&d_2'\overline{c}_4 + d_1'\overline{c}_4 = \overline{c}_3d_2' + \overline{c}_3d_1',\\
&d_2'\overline{d}_2 + d_1'\overline{d}_2 = d_2c_1' + d_2c_2',\\
&d_2'\overline{c}_3 + d_1'\overline{c}_3 = \overline{c}_4d_2' + \overline{c}_4d_1',\\
&e'c_1' = ec_1',\ e'c_2' = ec_2',\\
&e'd_2' = \overline{e}d_2', e'd_1' = \overline{e}d_1',\\
&c_1'e = c_1'e', c_2'e = c_2'e',\\
&d_2'\overline{e} = d_2'e', d_1'\overline{e} = d_1'e'.
\end{aligned}
$$

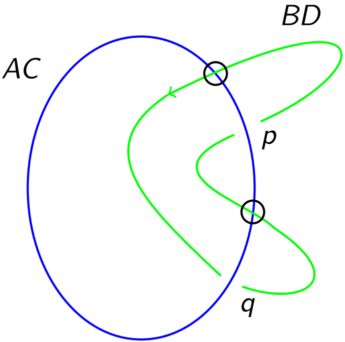


Figure 8: CASE (AC, BD)

Case 3

$c_4\overline{d}_1' = c_2c_2', \ c_4\overline{c_2}' = c_2d_2',$
$c_2\overline{d}_2' + c_3c_2' = c_1c_2' + c_2c_1',$
$c_2\overline{c_1}' + c_3d_2' = c_1d_2' + c_2d_1',$
$c_2\overline{d}_1' = c_3c_2', \ c_2\overline{c_2}' = c_3d_2',$
$c_4\overline{d}_2' + c_1c_2' = c_4c_2' + c_4c_1',$
$c_4\overline{c_1}' + c_1d_2' = c_4d_2' + c_4d_1',$
$d_1\overline{d}_2' + d_2c_2' = d_1c_2' + d_1c_1',$
$d_1\overline{c_1}' + d_2d_2' = d_1d_1' + d_1d_2',$
$d_1\overline{e}' = d_1e', \ d_1\overline{d}_1' = d_2c_2',$
$d_1\overline{c_2}' = d_2d_2', c_2\overline{e}' = c_2e',$
$c_4\overline{e}' = c_4e'.$

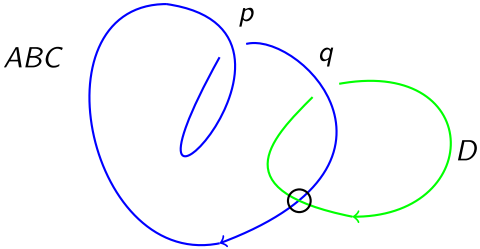


Figure 9: CASE (ABC, D)

Case 4

$c_4\overline{d}'_2 + c_3\overline{d}'_1 + c_2c'_2 + c_1c'_1 = \overline{d}'_2c_3 + \overline{d}'_1c_4 + c'_2c_2 + c'_1c_1$, $c_4\overline{d}'_1 + c_3\overline{d}'_2 + c_2c'_1 + c_1c'_2 = \overline{d}'_1c_3 + \overline{d}'_2c_4 + c'_2c_1 + c'_1c_2$, $d_1\overline{c_4} + d_2\overline{c_4} = \overline{c_1}d_2 + \overline{c_2}d_1$, $d_1\overline{c_3} + d_2\overline{c_3} = \overline{c_1}d_1 + \overline{c_2}d_2$, $d_2\overline{d}_2 + d_1\overline{d}_2 = \overline{c_1}'c_4 + \overline{c_2}'c_3 + d'_1c_2 + d'_2c_1$, $d_2\overline{d}_1 + d_1\overline{d}_1 = \overline{c_1}'c_3 + \overline{c_2}'c_4 + d'_2c_2 + d'_1c_1$, $c_3\overline{c_1}' + c_4\overline{c_2}' + c_2d'_2 + c_1d'_2 = \overline{d}_1d_2 + \overline{d}_1d_1$, $c_4\overline{c_1}' + c_3\overline{c_2}' + c_2d'_1 + c_1d'_1 = \overline{d}_2d_2 + \overline{d}_2d_1$, $d_1\overline{c}_1 + d_2\overline{c}_2 = \overline{c}_3d_1 + \overline{c}_3d_2, d_1\overline{e} = d_1e$, $d_2\overline{c}_1 + d_1\overline{c}_2 = \overline{c}_4d_1 + \overline{c}_4d_2, d_2\overline{e} = d_2e, ec_1 = e'c_1, ec_2 = e'c_2, ec_4 = \overline{e}'c_4$, $ed_1 = \overline{e}d_1, ec_3 = \overline{e}'c_3, ed_2 = \overline{e}d_2, c_1e' = c_1e, c_2e' = c_2e, c_3\overline{e}' = c_3e, c_4\overline{e}' = c_4e$.

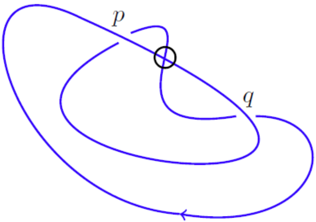


Figure 10: CASE ($ACBD$)

Case 5

$$
\begin{aligned}
&c_3c_2 = c_2c_4, c_3c_3 = c_2c_2,\\
&c_2c_1 + c_4c_2 = c_1c_2 + c_2c_3,\\
&c_2c_4 + c_4c_4 = c_1c_4 + c_2c_1,\\
&c_2d_1 + c_4d_1 = c_1d_1 + c_2d_2,\\
&c_3c_1 + c_1c_2 = c_3c_2 + c_3c_3,\\
&c_1c_4 = c_3c_1,\ c_1d_1 = c_3d_2,\\
&c_2c_2 = c_4c_4, c_2c_3 = c_4c_2,\\
&c_2d_2 = c_4d_1, d_2c_2 = d_1c_4,\\
&d_2c_3 = d_1c_2, d_2d_2 = d_1d_1,\\
&d_2c_1 + d_1c_2 = d_2c_2 + d_2c_3,\\
&d_1c_4 = d_2c_1, c_3d_2 = c_2d_1.
\end{aligned}
$$

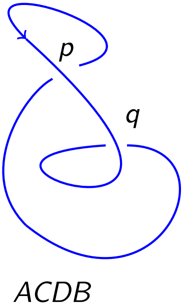


Figure 11: CASE (*ACDB*)

The complete set of relations

Suppose D has at least two classical crossings p and q, and both of them are positive crossings. First we resolve D at p, then we get many diagrams $D_1, D_2, \cdots$. We resolve each D_i at q, then we get the linear sum. On the other hand, if we resolve D at q first, we get many diagrams $D_1', D_2', \cdots$. We resolve each D_i' at p, and then we get the linear sum. By the induction hypothesis $f_{pq} = f_{qp}$.

The set of all above relations is called the complete set of relations and denoted by R_1 therefor

$$R_1 = \{\text{The complete set of above relations}\}.$$

If the variables satisfy the relations in R_1, then $f_{pq}(D) = f_{qp}(D)$

Index pair and definition of invariant

- The index pair (c, k) and induction on this index pair is used to prove the theorem. Here c denote the number of classical crossings and k denote the number of virtual crossings of the diagram.
- Let $f(E_+) = -c_1' f(E) - c_2' f(W) - d_1' f(S) - d_2' f(N) - e' f(E_V)$ then the diagram E_+ has index (c, k), and all diagrams on the right-hand side has $c - 1$ classical crossings.
- Let $S(c, k)$ denote the set of all oriented link diagrams with indices $\leq (c, k)$. Note that $S(0, 0)$ is the set of trivial diagrams.
- If the diagram D has index $(0, 0)$ then it is the trivial diagram with n components. The value for the trivial n-component link is denoted by $z_{(n,0)}$. If the diagram D has index $(0, k)$ then it is the trivial diagram with n components and k virtual crossings. The value for the n-component link with only k virtual crossings is denoted by $z_{(n,k)}$.

$f(D)$ is invariant under generalized Reidemeister moves

Classical Reidemeister Move-I

Let two diagrams D and D' are differed by a R-I.

- We can assume that there is no other classical crossing except $R - I$. If there is any other classical crossing then by induction we can reduced to the case that there is no other classical crossing.
- Since D has no classical crossing, therefore, D has index $(0, k)$ and $f(D) = z_{(n,k)}$.
- The value of two diagrams D and D' is equal if and only if the following equations are always true; If the crossing is positive, then $z_{(n,k)} = -(c_1 + c_2 + c_3 + c_4)z_{(n+1,k)} - (d_1 + d_2)z_{(n,k)} - ez_{(n,k+1)}$. If the crossing is negative, then $z_{(n,k)} = -(\overline{c}_1 + \overline{c}_2 + \overline{c}_3 + \overline{c}_4)z_{(n+1,k)} - (\overline{d}_1 + \overline{d}_2)z_{(n,k)} - \overline{e}z_{(n,k+1)}$.

Classical Reidemeister Move-II

Suppose given two diagrams D and D' are differed by a R-II. Like above we can assume that there is no other classical crossing except $R-II$. Since D has no classical crossing, therefor $f(D) = z_{(n,k)}$. Now we resolve both p and q using our skein relation. The orientation of components is such that the crossing p is positive, and the two arcs are from different link components, then resolve at p we get

$$f(E_+) = -c_1' f(E) - c_2' f(W) - d_1' f(S) - d_2' f(N) - e' f(E_V)$$

Now each diagram on the right-hand side of this equation has only one classical crossing q, so we again apply skein relation. After applying skein relation at q we get

$$z_{(n,k)} = -(c_1' + c_2' + d_1' + d_2') z_{(n-1,k)} + e' \{ (\overline{c}_1' + \overline{c}_2' + \overline{d}_1' + \overline{d}_2') z_{(n-1,k+1)} + \overline{e}' z_{(n,k+2)} \}$$

Similarly, if we resolve first q, then p we get the following equation

$$z_{(n,k)} = -(\overline{c}_1' + \overline{c}_2' + \overline{d}_1' + \overline{d}_2') z_{(n-1,k)} + \overline{e}' \{ (c_1' + c_2' + d_1' + d_2') z_{(n-1,k+1)} + e' z_{(n,k+2)} \}$$

Classical Reidemeister Move-III

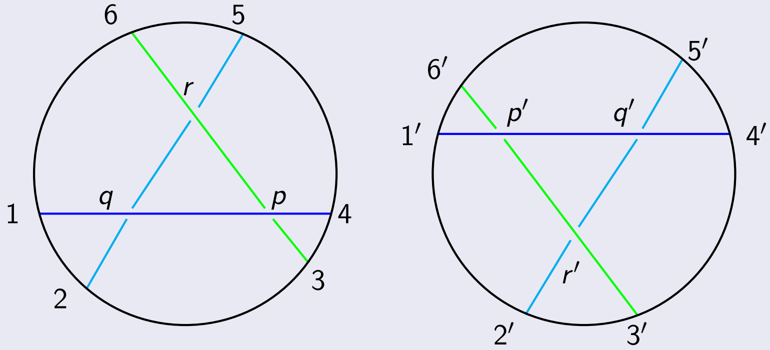


Figure 12: Classical Reidemeister move-III

Inside each disk there are three arcs and ends of these arcs are attached with the boundary of disk, there are fifteen possibilities to connect these end points with each other.

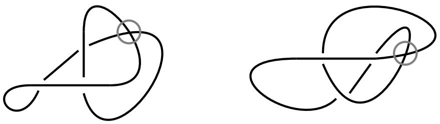


Figure 13: Two diagrams differed by R-III

- Except one set of diagrams in Fig.13 all other D and D' are either same diagrams or both D and D' can be transformed into the trivial diagrams using the R-I and R-II.
- For D and D' in Fig.13 if r is the intersection of arc m (middle) and b (bottom), then we resolve D at r and we get many new diagrams D_1 ,D_2 $\cdots$ similarly, we resolve D' at r' and get many new link diagrams D'_1, D'_2, $\cdots$.
- Each D_i is resembling to D'_i therefor for those diagrams we have $f(D_i) = f(D'_i)$. By induction hypothesis and the skein equation we get $f(D) = f(D')$.

Mixed Reidemeister Move

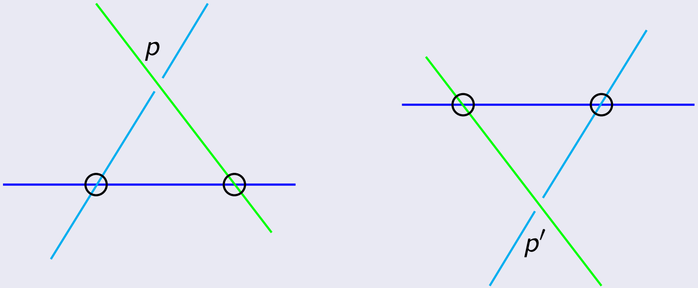


Figure 14: Mixed Reidemeister move

If we resolve D at point p we will get the linear combination of the trivial diagrams $D_1, D_2, \cdots D_n$. Similarly, if we resolve D' at point p' we will get the linear combination of the trivial diagrams $D'_1, D'_2, \cdots D'_n$. Each D_i is resembling to D'_i for those diagrams, we have $f(D_i) = f(D'_i)$, therefor, by the skein equation $f(D) = f(D')$.

Virtual Reidemeister Moves

- Suppose two diagrams D and D' are differed by a virtual Reidemeister moves.
- Since D is the trivial diagram, therefore $f(D) = z_{(n,k)}$.
- If D' is the diagram with only virtual Reidemeister move-I then the value of D' is $f(D') = z_{(n,k+1)}$.
- Similarly, If D' is the diagram with only virtual Reidemeister move-II then the value of D' is $f(D') = z_{(n,k+2)}$.
- If the two diagrams D and D' are differed by the virtual Reidemeister move-III then $f(D) = z_{(n,k+3)} = f(D')$.
- The virtual Reidemeister moves invariance is guaranteed if and only if $z_{(n,k)} = z_{(n,k+1)}$.

The set of complete equations

If the following equations are always true, then our function $f(D)$ is invariant under classical Reidemeister moves. The set of these equations is denoted by R_2.

$$z_{(n,k)} = -(c_1 + c_2 + c_3 + c_4)z_{(n+1,k)} - (d_1 + d_2)z_{(n,k)} - ez_{(n,k+1)}$$

$$z_{(n,k)} = -(\overline{c}_1 + \overline{c}_2 + \overline{c}_3 + \overline{c}_4)z_{(n+1,k)} - (\overline{d}_1 + \overline{d}_2)z_{(n,k)} - \overline{e}z_{(n,k+1)}$$

$$z_{(n,k)} = -(c'_1 + c'_2 + d'_1 + d'_2)z_{(n-1,k)} + e'\{(\overline{c}'_1 + \overline{c}'_2 + \overline{d}'_1 + \overline{d}'_2)z_{(n-1,k+1)} + \overline{e}'z_{(n,k+2)}\}$$

$$z_{(n,k)} = -(\overline{c}'_1 + \overline{c}'_2 + \overline{d}'_1 + \overline{d}'_2)z_{(n-1,k)} + \overline{e}'\{(c'_1 + c'_2 + d'_1 + d'_2)z_{(n-1,k+1)} + e'z_{(n,k+2)}\}$$

$$R_2 = \{\text{The set of above equations}\}.$$

Similarly, $f(D)$ is invariant under virtual Reidemeister moves, if the following equation is always true. This equation is denoted by R_3.

$$R_3 = \{z_{(n,k)} = z_{(n,k+1)}\} \qquad n \geq 1 \text{ and } k \geq 0$$

Theorem

For oriented link diagrams, there is a link invariant f *with values in* X_c *and satisfies the following skein relations: If the two arcs in the local diagram are from the same link component, then*

$$f(E_+) = c_1 f(E) + c_2 f(W) + c_3 f(HC) + c_4 f(HT) + d_1 f(VC) + d_2 f(VT) + ef(E_V)$$

$$f(E_-) = \overline{c}_1 f(E) + \overline{c}_2 f(W) + \overline{c}_3 f(HC) + \overline{c}_4 f(HT) + \overline{d}_1 f(VC) + \overline{d}_2 f(VT) + \overline{e} f(E_V)$$

If the two arcs are from different components, then

$$f(E_+) = c'_1 f(E) + c'_2 f(W) + d'_1 f(S) + d'_2 f(N) + e' f(E_V)$$

$$f(E_-) = \overline{c}'_1 f(E) + \overline{c}'_2 f(W) + \overline{d}'_1 f(S) + \overline{d}'_2 f(N) + \overline{e}' f(E_V)$$

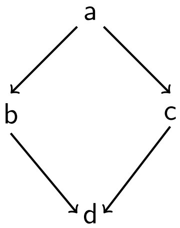
For any trivial link diagram with n *components,* $f(D) = z_{(n,k)}$ *and* $R^c = R_1 \cup R_2$.

$$X_c = Z[c_1, c_2, c_3, c_4, d_1, d_2, \overline{c}_1, \overline{c}_2, \overline{c}_3, \overline{c}_4, \overline{d}_1, \overline{d}_2, c'_1, c'_2, d'_1, d'_2, \overline{c}'_1, \overline{c}'_2, \overline{d}'_1, \overline{d}'_2, z_n]/R^c$$

Reduction of Coefficients by Diamond Lemma

The Newman's Diamond lemma is described as binary relation $\rightarrow$ on set S with the following two properties

1. A chain is a sequence of elements $x_1, x_2, x_3, \cdots \in S$ such that $x_1 \rightarrow x_2 \rightarrow x_3 \cdots \rightarrow x_n$. A chain is called terminating if it has finite length. All the possible chains on S should be terminating.



2. For all, $a, b, c, d \in S$ if there exist two possible ways to rewrite a such that $a \rightarrow b$ and $a \rightarrow c$, then there exist two terminating chains such that $b \rightarrow d$ and $c \rightarrow d$. This property is known as diamond property. If d can't be further simplified using any rewriting rule, than d is called normal form of a.

If d can't be further simplified using any rewriting rule, than d is called the normal form of a.

Let $c = c_1 = \overline{c}_1$, $c' = c'_1 = \overline{c}'_1$, $e = \overline{e} = e' = \overline{e}'$ and all other coefficients are equal to zero. Let $z_{n+1} = kz_n$ and X'_c denote the quotient commutative ring $X'_c = Z[c, c', e, z_n]/R'_c$.
Then $R'_c = \{1 + ck + e = 0, k + c' - c'e - e^2k = 0, -cc'e + e^3 + cc' + e^2 - e - 1 = 0\}$.

Theorem

For oriented link diagrams, there is a link invariant f with values in X'_c and satisfies the following skein relations: If the two arcs are from the same link component, then

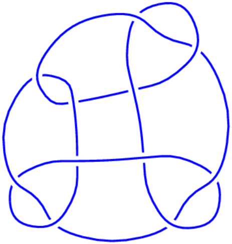
$$f(E_\pm) = -cf(E) - ef(E_V)$$

If the two arcs are from different components, then

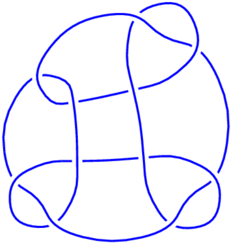
$$f(E_\pm) = -c'f(E) - ef(E_V)$$

The value for the trivial n-component link is z_n and if $z_{n+1} = kz_n$ then the invariant has a unique normal form defined by R'_c.

Comparison with Existing Invariants



(a) K11n73



(b) K11n74

This invariant still works where Kauffman two-variable polynomial and HOMFLY polynomial are fail to distinguish two different knots. For example, both Kauffman two-variable polynomial and HOMFLY polynomial can not distinguish the K11n73 and K11n74, but our invariant can distinguish K11n73 and K11n74 clearly and very easily even using the simplified version.

Invariant	K11n73	K11n74
HOMFLY	$(-4v^{-2}+11-8v^2+2v^4)+(-4v^{-2}+15-12v^2+3v^4)z^2+(-v^{-2}+7-6v^2+v^4)z^4+(1-v^2)z^6$	$(-4v^{-2}+11-8v^2+2v^4)+(-4v^{-2}+15-12v^2+3v^4)z^2+(-v^{-2}+7-6v^2+v^4)z^4+(1-v^2)z^6$
Kuffman	$(2a^{-4}+8a^{-2}+11+4a^2)z^0+(-4a^{-5}-13a^{-3}-17a^{-1}-13a-5a^3)z^1+(3a^{-6}+a^{-4}-13a^{-2}-19-8a^2)z^2+(10a^{-5}+24a^{-3}+29a^{-1}+25a+10a^3)z^3+(-4a^{-6}-a^{-4}+11a^{-2}+18+10a^2)z^4+(-9a^{-5}-16a^{-3}-15a^{-1}-14a-6a^3)z^5+(a^{-6}-3a^{-4}-6a^{-2}-8-6a^2)z^6+(2a^{-5}+3a^{-3}+2a^{-1}+2a+a^3)z^7+(a^{-4}+a^{-2}+1+a^2)z^8$	$(2a^{-4}+8a^{-2}+11+4a^2)z^0+(-4a^{-5}-13a^{-3}-17a^{-1}-13a-5a^3)z^1+(3a^{-6}+a^{-4}-13a^{-2}-19-8a^2)z^2+(10a^{-5}+24a^{-3}+29a^{-1}+25a+10a^3)z^3+(-4a^{-6}-a^{-4}+11a^{-2}+18+10a^2)z^4+(-9a^{-5}-16a^{-3}-15a^{-1}-14a-6a^3)z^5+(a^{-6}-3a^{-4}-6a^{-2}-8-6a^2)z^6+(2a^{-5}+3a^{-3}+2a^{-1}+2a+a^3)z^7+(a^{-4}+a^{-2}+1+a^2)z^8$

Invariant	K11n73	K11n74
Our Invariant	$(24ec^3c'^3 - 10e^2c^3c'^3 - 21c^3c'^3 + 50ec^2c'^2 + 12c^2c'^2 - 47e^2c^2c'^2 - 31e^2cc' - 14ecc' - 22cc' + 31e^2 + 62e + 32)z_1$	$(4e^2c^3c' + 2c^3c' - 4ec^3c' + 2e^2c^2 + 6ec^2c'^2 + 5c^2c'^2 - 2ec^2 - 4e^2c^2c'^2 - 2c^2 - 2e^2c - 13e^2cc' - 14ecc' - 35cc' + 30e^2 + 60e + 31)z_1$

Thank You