On volumes of hyperbolic right-angled polyhedra

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Right-angled polyhedra

Let H^3 denote the 3-dimensional Lobachevsky space.

A convex polyhedra in H^3 will be called right-angled if all its dihedral angles are equal to $\pi/2$.

Compact polyhedra	Mixed case	Ideal polyhedra
All vertices are finite	There are finite and ideal vertices	All vertices are ideal

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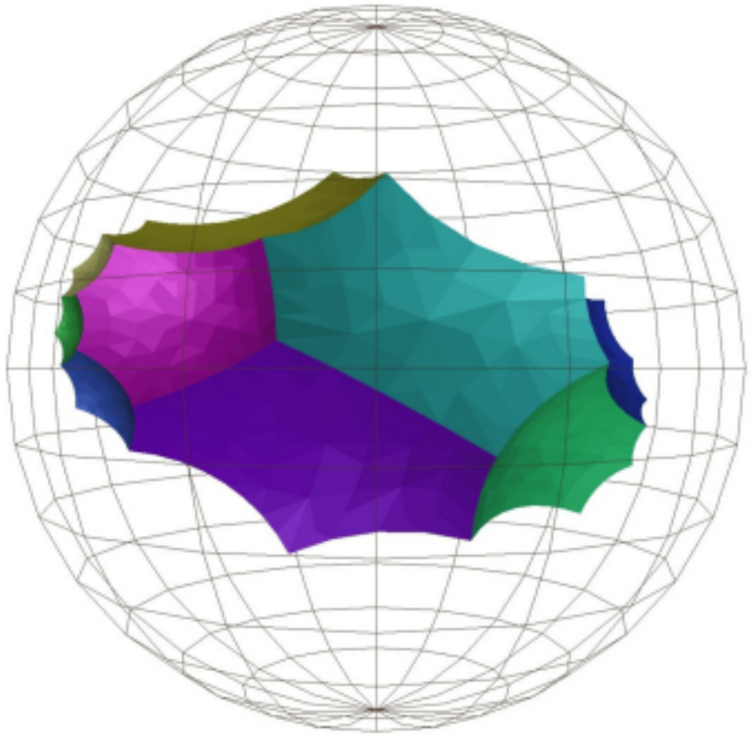


Compact polyhedra

A polyhedron is uniquely defined by combinatorics and dihedral angles

Theorem (E.M. Andreev, 1970) A bounded acute-angled polyhedron in H^n, $n \geq 3$,is uniquely defined by its combinatorial type and dihedral angles.

Andreev's theorem for compact case

Theorem (Andreev, 1970) A polyhedron P can be realized in H^3 as a compact right-angled polyhedron if and only if

1. any vertex is incident to 3 edges;
2. any face has at least 5 sides;
3. if a simple closed circuit on the surface of the polyhedron separates two faces (prismatic circuit), then it intersects at least 5 edges

The realization is unique up to isometry.

The number of compact $\pi/2$-polyhedra with a fixed number of faces

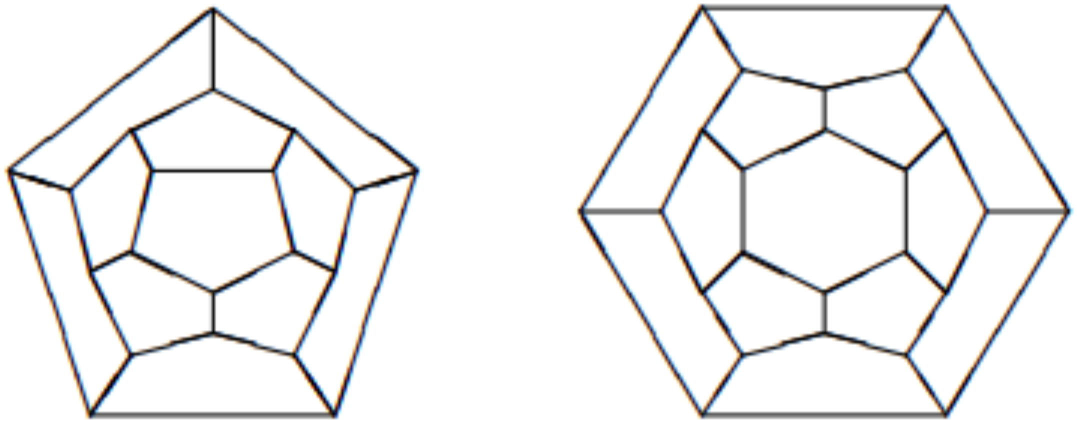
12	13	14	15	16	17	18	19	20	21
1	0	1	1	3	4	12	23	71	187

22	23	24	25
627	1970	6833	23384

Löbell polyhedra

For any $n \geq 5$, there exists a right-angled hyperbolic polyhedron $L(n)$ having $(2n+2)$ faces.

$L(5)$ and $L(6)$ look like this



Theorem (A.Vesnin, 1987)

$$vol(L(n)) = \frac{n}{2}(\Lambda(\theta_n) + \Lambda(\theta_n + \frac{\pi}{n}) + \Lambda(\theta_n - \frac{\pi}{n}) + \Lambda(\frac{\pi}{2} - 2\theta_n)),$$

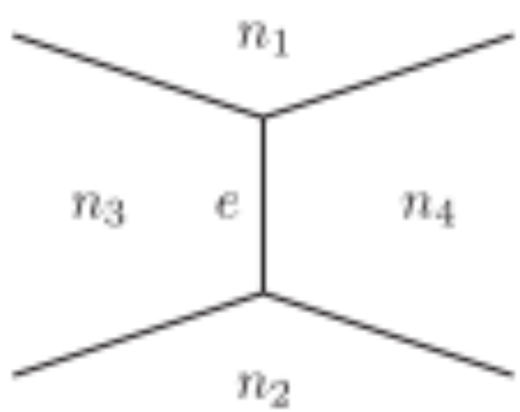
where $\Lambda(\theta) = -\int_0^\theta \log|2\sin(t)|\,dt\,.$

Composition of two polyhedra

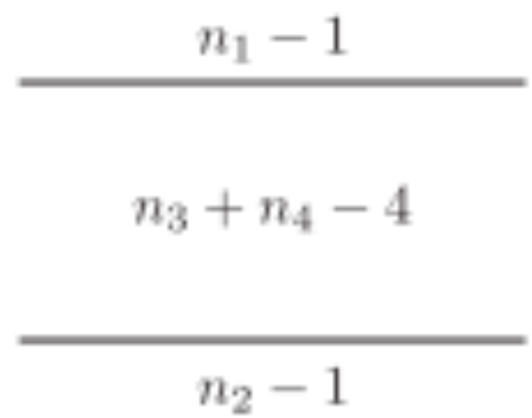
Let P_1 and P_2 be two combinatorial polyhedra with k-gonal faces F_1 and F_2. Then a composition is $P = P_1 \bigcup_{F_1=F_2} P_2$.

If P_1 and P_2 are both realized as compact right-angled polyhedra in H^3, then P is realized as a compact right-angled polyhedron in H^3.

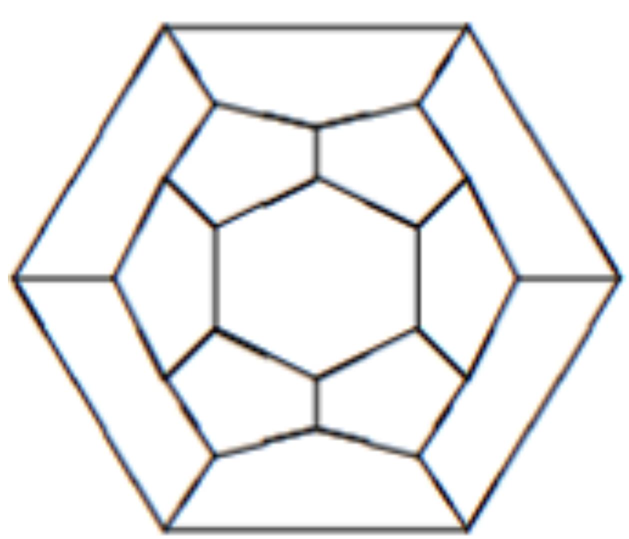
Adding an edge/deleting an edge



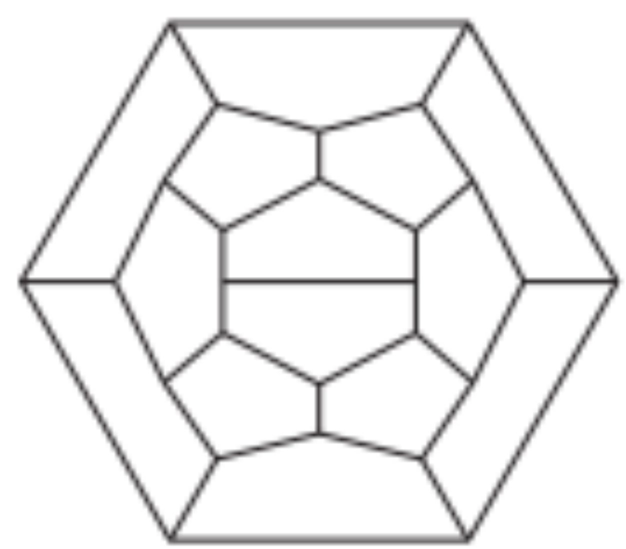
R



$R - e$



L(6)



L(6)+e

Obtaining polyhedra from Löbell polyhedra

Theorem (T.Inoue, 2008) For any compact right-angled hyperbolic polyhedron P_0 there exists a sequence of unions of right-angled hyperbolic polyhedra $P_1, \ldots, P_k$ such that each set P_i is obtained from P_{i-1} by decomposition or edge deleting, and Pk consists of Löbell polyhedra. Moreover,

$$vol(P_0) \geq vol(P_1) \geq vol(P_2) \geq \ldots \geq vol(P_k)\,.$$

Volume Bounds for compact polyhedra

Theorem (Atkinson, 2009) Let P be a compact right-angled hyperbolic 3-polyhedron with V vertices. Then the following inequalities hold.

$$\frac{v_8}{32} \cdot V - \frac{v_8}{4} \leq Vol(P) < \frac{5v_3}{8} \cdot V - \frac{25}{4} v_3 \,.$$

Theorem (E., A.Vesnin, 2020) Let P be a compact right-angled hyperbolic 3-polyhedron with V vertices. If P is not a dodecahedron, then the following inequalities hold.

$$Vol(P) \leq \frac{5v_3}{8} \cdot V - \frac{35}{4} v_3 \,.$$

$v_3 = 3\Lambda(\frac{\pi}{3}) = 1.01494160640965\ldots$, $\quad v_8 = 8\Lambda(\frac{\pi}{4}) = 3.66386237670887\ldots$

Volume Bounds for compact polyhedra

Theorem (S.Alexandrov, N.Bogachev. E., A.Vesnin, 2021) Let P be a compact right-angled hyperbolic 3-polyhedron with V vertices. Then the following inequalities hold.

(1) If $V > 80$, then

$$Vol(P) \leq \frac{5v_3}{8} \cdot V - 10v_3 \, .$$

(2) If P has a k-gonal face, $k \geq 5$, then

$$Vol(P) \leq \frac{5v_3}{8} \cdot V - \frac{5k+35}{8} v_3 \, .$$

Ideal polyhedra

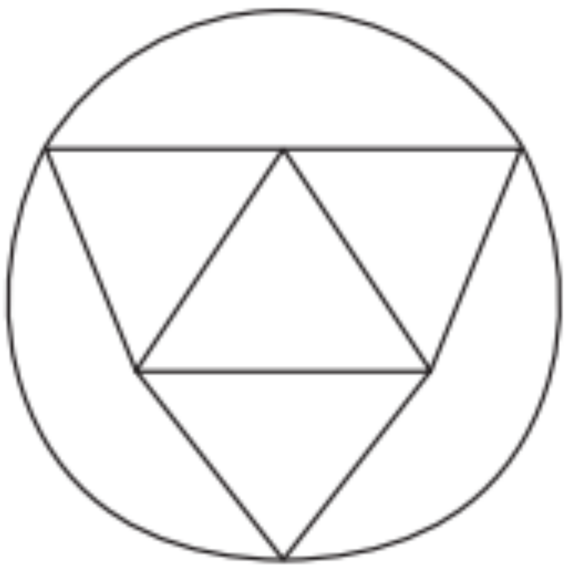
Combinatorics of ideal polyhedra

Theorem (Rivin, 1992)

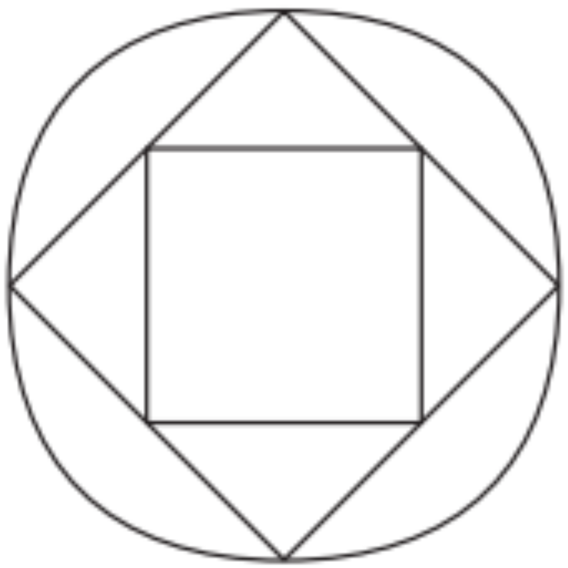
A polyhedral graph is realized as a graph of an ideal right-angled polyhedron in H^3 if and only if it is 4-valent and cyclically 6 connected. Moreover, such a realization is unique.

Same conditions can be obtained from Andreev's theorem.

Ideal antiprisms



A(3)

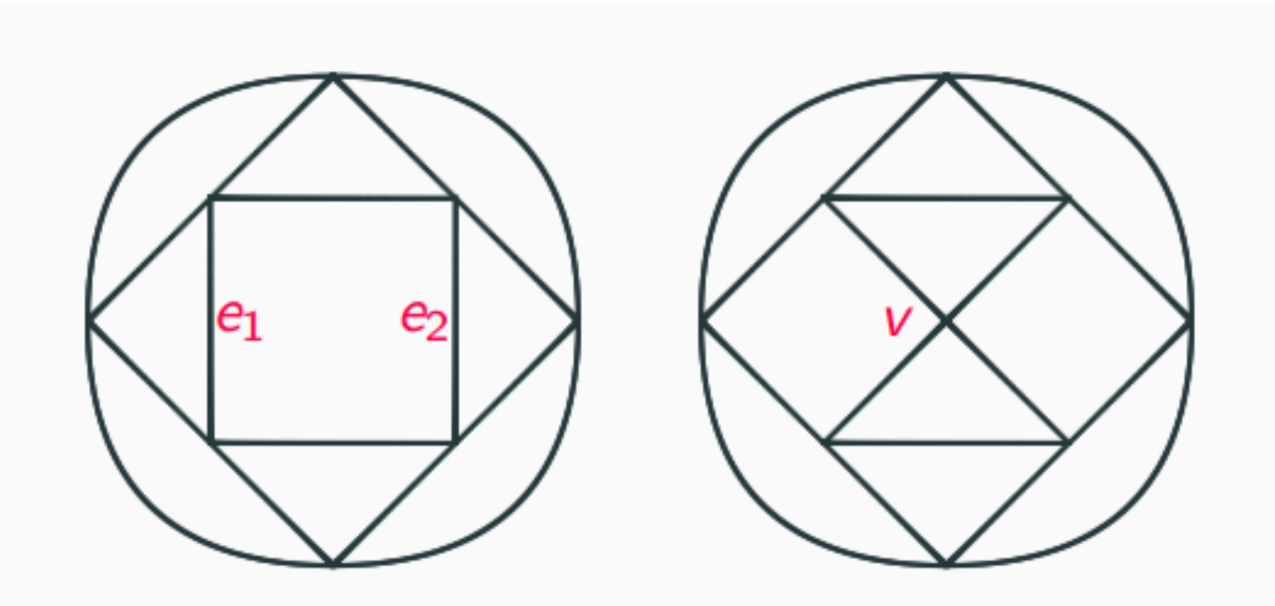


A(4)

Theorem (W. Thurston, 1980)

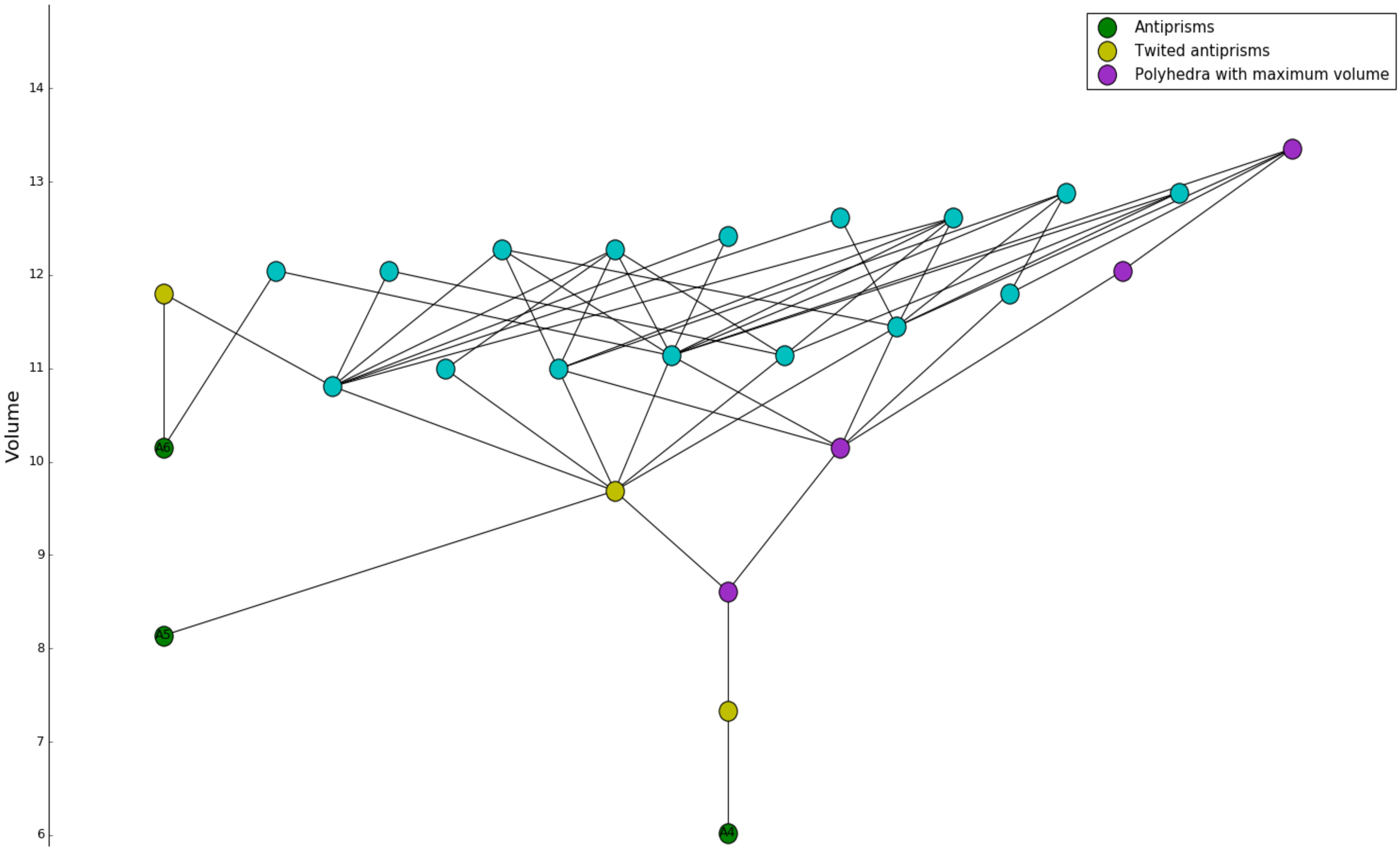
$$vol(A(n)) = 2n(\Lambda(\frac{\pi}{4} + \frac{\pi}{2n}) + \Lambda(\frac{\pi}{4} - \frac{\pi}{2n}))\,.$$

Edge twist



Edge-twist operation

Theorem (N. Yu. Erokhovets, 2019) A polyhedron is an ideal right-angled polyhedron if and only if either it is $A(n)$, $n \geq 3$, or it is obtained from $A(4)$ by an edge twist operation.



Antiprisms
Twited antiprisms
Polyhedra with maximum volume
Volume
14
13
12
11
10
9
8
7
6
A6
A5
A4

Volume Bounds for ideal polyhedra

Theorem (Atkinson, 2009) Let P be an ideal right-angled hyperbolic 3-polyhedron with V vertices. Then the following inequalities hold.

$$\frac{v_8}{4} \cdot V - \frac{v_8}{2} \leqslant Vol(P) \leqslant \frac{v_8}{2} \cdot V - 2v_8 .$$

Theorem (E., A.Vesnin, 2020) Let P be an ideal right-angled hyperbolic 3-polyhedron with V vertices. If P is not an octahedron, then the following inequalities hold.

$$Vol(P) \leqslant \frac{v_8}{2} \cdot V - \frac{5v_8}{2} .$$

Volume Bounds for ideal polyhedra

Theorem (S.Alexandrov, N.Bogachev. E., A.Vesnin, 2021) Let P be an ideal right-angled hyperbolic 3-polyhedron with V vertices. Then the following inequalities hold.

(1) If $V > 24$, then

$$Vol(P) \leqslant \frac{v_8}{2} \cdot V - 3v_8 \,.$$

(2) If P has only triangular and quadrilateral faces with $V > 72$, then

$$Vol(P) \leqslant \frac{v_8}{2} \cdot V - \left(9v_8 - 20v_3\right) .$$

(3) If P has a k-gonal face, $k \geq 3$, then

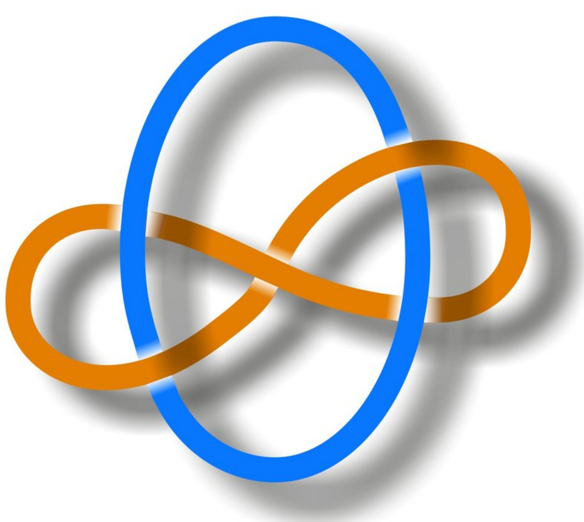
$$Vol(P) \leqslant \frac{v_8}{2} \cdot V - \frac{k+5}{4} v_8 \,.$$

Conjecture on right-angled knots

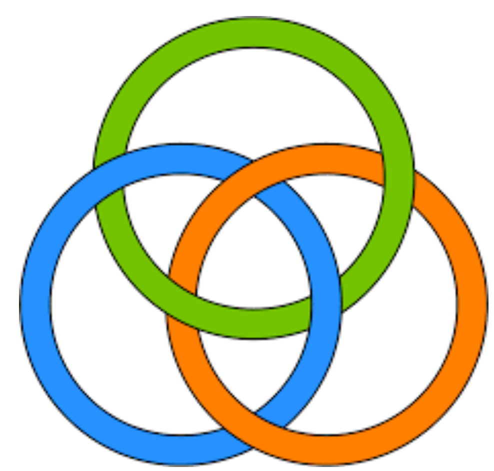
Right-angled links

Definition The link L is said to be right-angled if its compliment $S^3 \setminus L$ equipped with a hyperbolic structure can be decomposed into right-angled polyhedra.

There are known examples of right-angled links: Whitehead link can be decomposed into one copy of $A(3)$, Borromean link can be decomposed into two copies of $A(3)$.



Whitehead link



Borromean link

Right-angled knot conjecture

Conjecture (A.Champanerkar, I.Kofman, JS. Purcell, 2019) There are no right-angled knots.

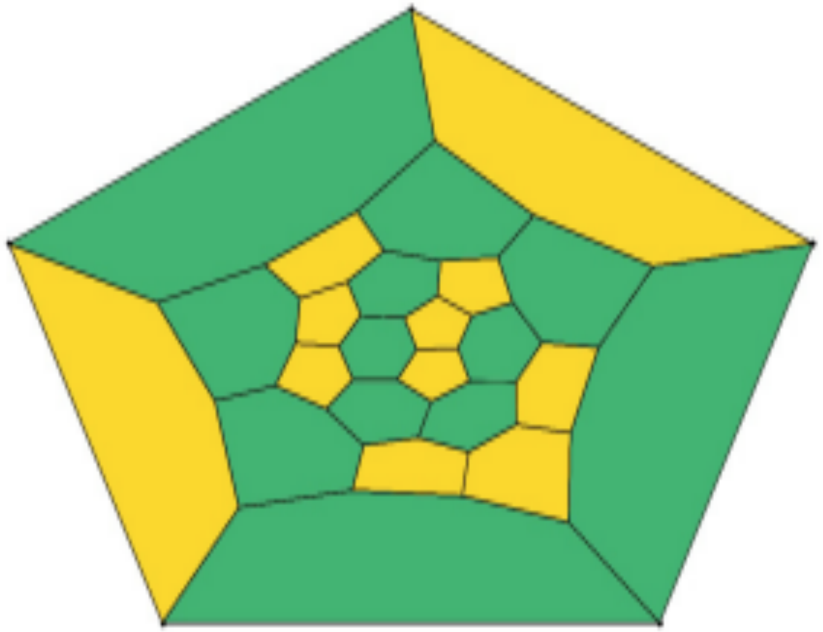
The conjecture is verified for all knots up to 11 crossings.

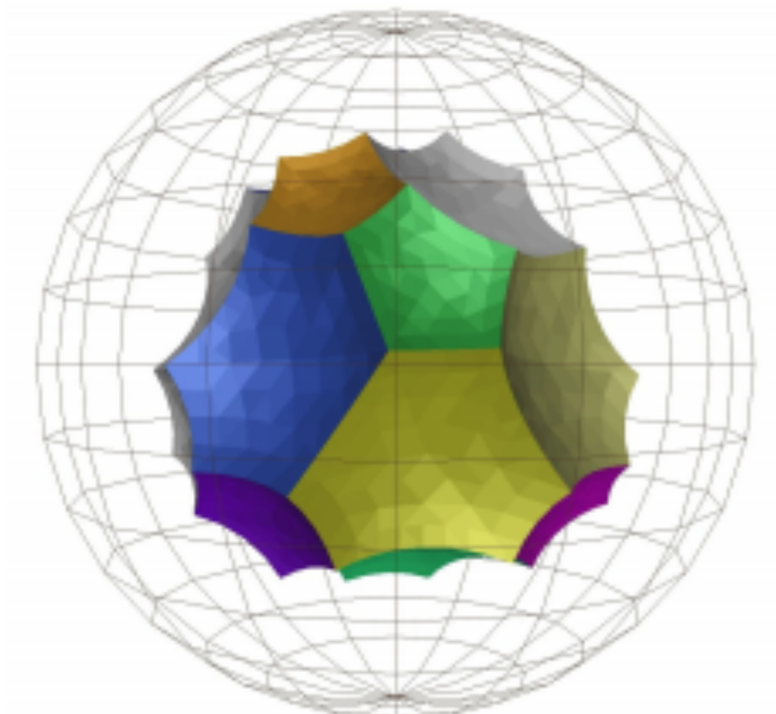
Fullerenes

Fullerenes

Fullerene is a spherically shaped molecule built entirely from carbon atoms.

Fullerene is a polyhedron with trivalent graph whose faces are 5 and 6-gons.



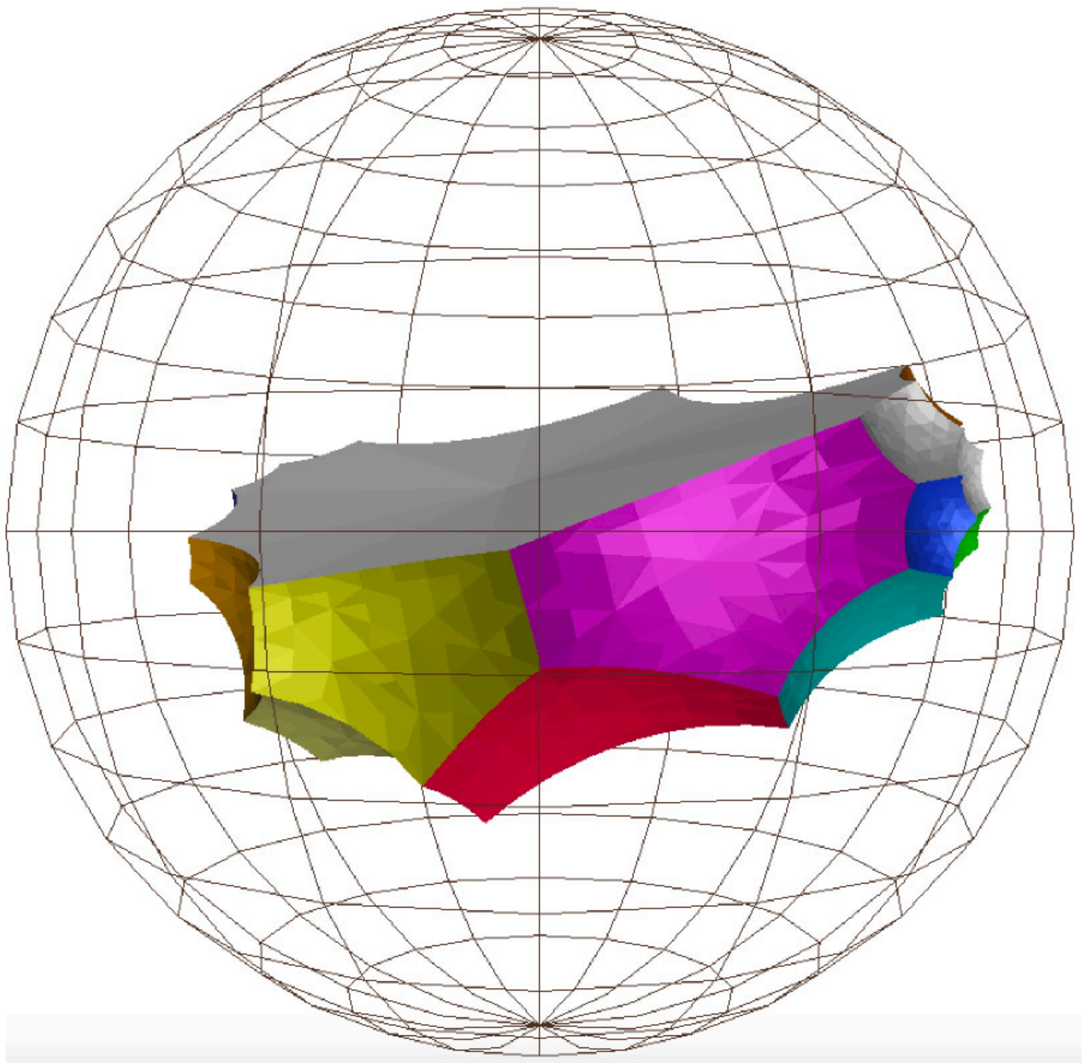


Theorem (T.Doslic, 2003)
If P is fullerene, then each prismatic cycle contains at least 5 edges.

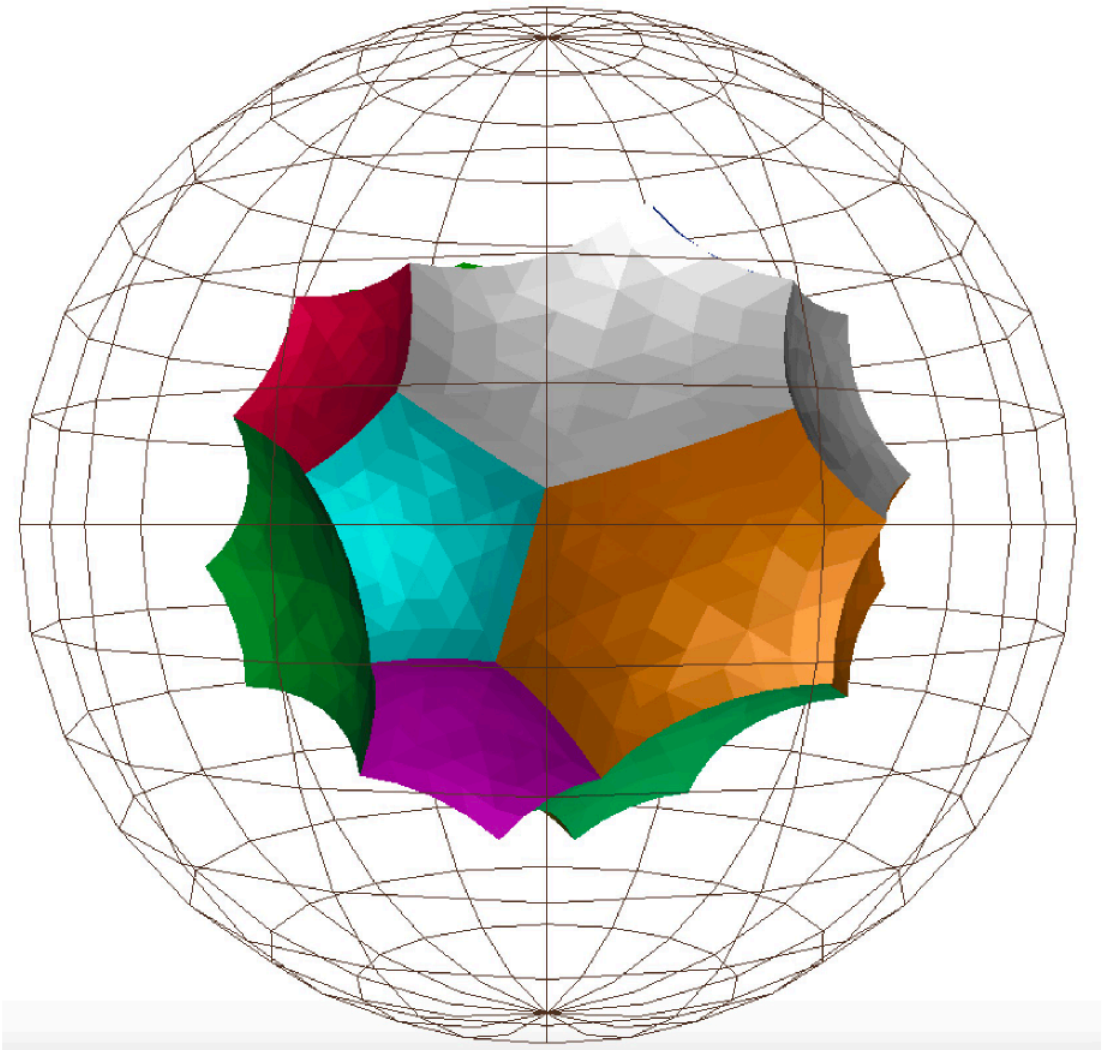
Corollary
Every fullerene can be realized as compact right-angled hyperbolic polyhedra.

Fullerenes have the biggest volume

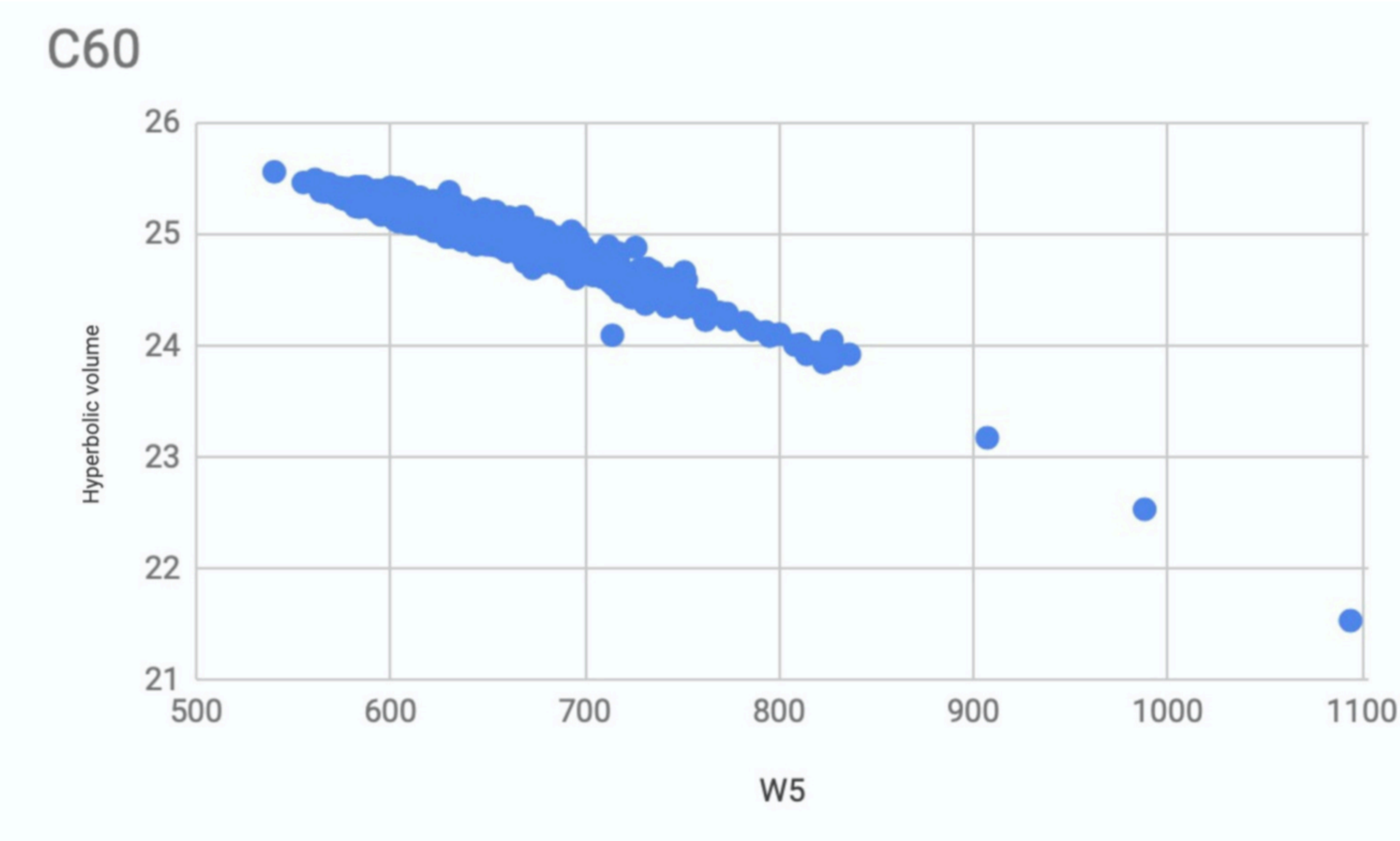
Among all compact polyhedra with fixed number of faces

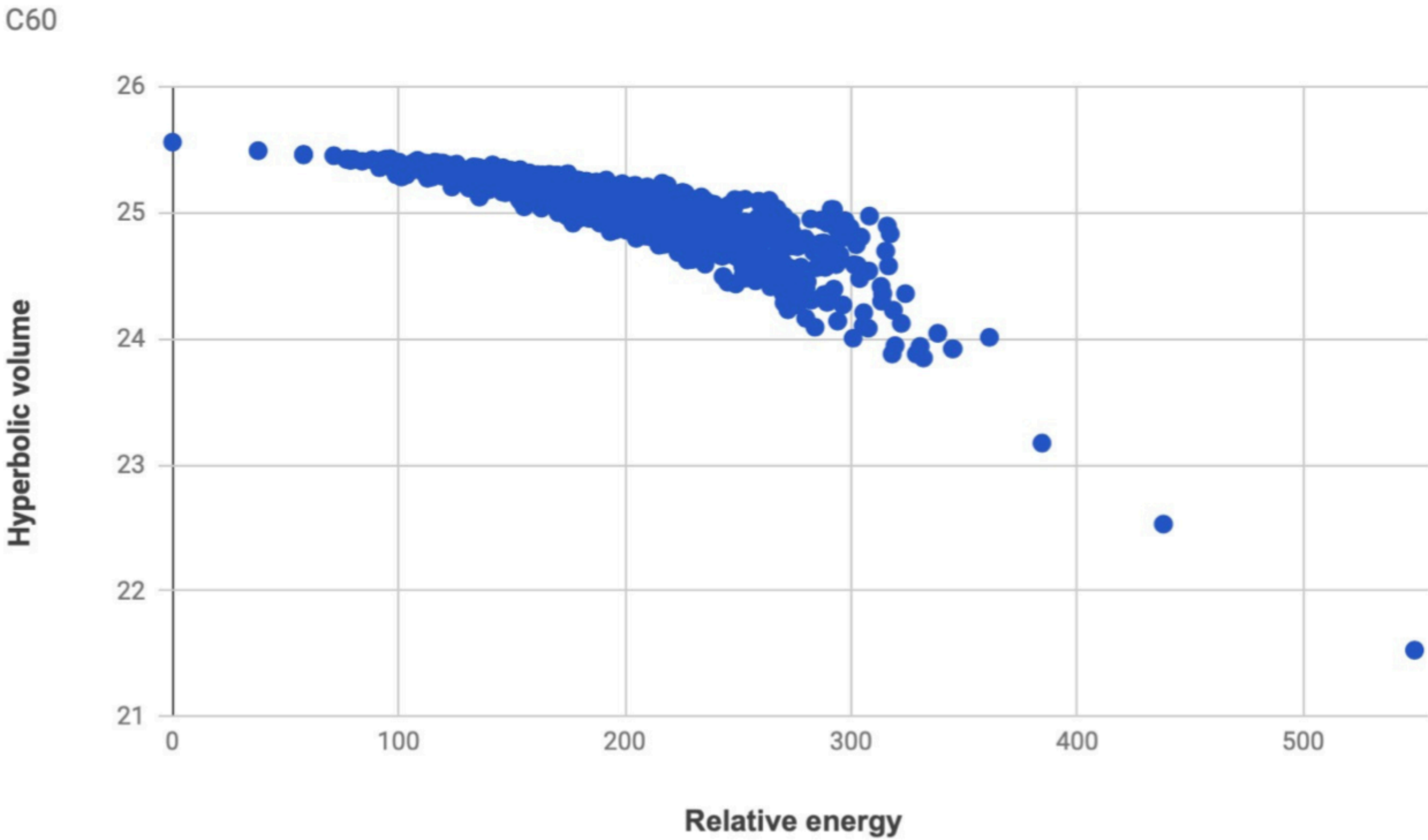


Polyhedron with 24 face and min volume
Not fullerene



Polyhedron with 24 face and max volume
Fullerene


C60
26
25
24
23
22
21
Hyperbolic volume
500
600
700
800
900
1000
1100
W5



C60
26
25
24
23
22
21
0
100
200
300
400
500
Hyperbolic volume
Relative energy

Correlation of hyperbolic volume with some invariants of fullerenes

	Wiener index	W_5	N_p	H_6	Relative energy
Hyperbolic volume	-0.831511	-0.960028	-0.884936	-0.914185	-0.862611

Here we take the list of hyperbolic volumes of all fullerenes with 60 vertices and find its correlation with lists for some other invariants.

Mixed case

Andreev's theorem

Theorem (Andreev's theorem for π/2–equiangular polyhedra)

An abstract polyhedron P is realizable as a hyperbolic π/2–equiangular polyhedron if and only if the following conditions hold:

(1) P has at least 6 faces.

(2) P each vertex has degree 3 or degree 4.

(3) For any triple of faces (F_i, F_j, F_k), such that $Fi \cap Fj$ and $Fj \cap Fk$ are edges of P with distinct endpoints, $Fi \cap Fk = \varnothing$.

(4) P has no prismatic 4 circuits.

Furthermore, each degree 3 vertex in P corresponds to a finite vertex in P, each degree 4 vertex in P corresponds to an ideal vertex in P, and the realization is unique up to isometry

Volume Bounds for polyhedra with finite and ideal vertices

Theorem (Atkinson, 2009) Let P be a right-angled hyperbolic 3-polyhedron with V_∞ ideal vertices and V_F finite vertices. Then the following inequalities hold.

$$\frac{v_8}{8} \cdot V_\infty + \frac{v_8}{32} \cdot V_F - \frac{v_8}{4} \leq Vol(P) < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \frac{v_8}{2}.$$

Theorem (S.Alexandrov, N.Bogachev. E., A.Vesnin, 2021) Let P be a finite-volume right-angled hyperbolic 3-polyhedron with V_∞ ideal vertices and V_F finite vertices. If $V_\infty + V_F > 17$, then

$$Vol(P) < \frac{v_8}{2} \cdot V_\infty + \frac{5v_3}{8} \cdot V_F - \left(v_8 + \frac{5v_3}{2}\right).$$