# Region crossing change on surfaces 

Zhiyun Cheng<br>Beijing Normal University

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This is a joint work with
Jiawei Cheng, Jinwen Xu and Jieyao Zheng.


Some part of this talk is based on an undergraduate research project of Beijing Normal University.
"Region Select" ${ }^{1}$ is an interesting puzzle game developed by Akio Kawauchi, Ayaka Shimizu and Kengo Kishimoto from OCAMI.


- There is a light bulb on each crossing.
- Click one region to switch the light bulbs on the boundary.
- Light up all light bulbs!

[^0]What is region crossing change?

- An unknotting operation is a local operation on a tangle in $S^{3}$, which allows us to untie any knot in finitely many steps.
- The simplest example of unknotting operation is crossing change (H. Wendt 1937).


Some other well studied unknotting operations:

- \#-operation (H. Murakami 1985)
- $\triangle$-operation (H. Murakami, Y. Nakanishi 1989)
- n-gon move (Haruko Aida 1992)


$n$-gon move


## Remark

Reidemeister moves are permitted during the transformation.

Two examples of $\sharp$-operation.

(from H. Murakami's paper)

- In 2010, Kishimoto proposed a new local operation region crossing change on "Friday Seminar on Knot Theory" in Osaka City University.
- Applying region crossing change on a region of a diagram means to switch all the crossing points on the boundary of this region.



## Question (Kishimoto 2010)

Is region crossing change an unknotting operation?
Remark
Reidemeister moves are forbidden during the transformation, since \#-operation and $n$-gon move are both special cases of region crossing change.

Region crossing change on $R^{2}\left(S^{2}\right)$

- G a link diagram.
- A set of crossing points of $G$ are called region crossing change admissible if one can switch these crossing points and preserve the others by a sequence of region crossing changes.

Theorem (Ayaka Shimizu 2010)
Let $D$ be a knot diagram and $p$ a crossing point of $D$, then $p$ is region crossing change admissible.

## Corollary (Ayaka Shimizu 2010)

Region crossing change is an unknotting operation for knot diagrams.

## Remark

- As an unknotting operation, we can consider the corresponding "unknotting number". A natural idea is defining the "unknotting number" of a knot $K$ to be the minimal number of region crossing changes needed to unknot $K$ among all knot diagrams representing $K$. However, under this definition each nontrivial knot has "unknotting number" one. (Aida 1992)
- Ayaka Shimizu introduced another definition of "unknotting number" corresponding to region crossing change.
- Ayaka Shimizu's result has been extended to spatial-graph diagrams (Hayano-Shimizu, 2015)/region freeze crossing change (Inoue-Shimizu, 2016).


A simple observation shows that region crossing change is NOT always an unknotting operation for link diagrams.

## Question

For which kind of link diagrams, region crossing change is an unknotting operation?

More precisely,
Question
Given a link diagram, which crossing points are region crossing change admissible?

Conversion between link diagram and signed planar graph (Tait graph)
(1) color all the regions in checkerboard fashion.
(2) put one vertex in each black region, connect each pair of vertices that share a common crossing point.
(3) assign a sign to each edge according to the figure below.


Example


From a link diagram $D$ we can obtain a signed planar graph $G$ and its dual graph $G^{\prime}$. In graph theory, the incidence matrix of $G$ is a $v \times e$ matrix, here $v, e$ denote the order and size of $G$ respectively:

$$
M(G)=\left(m_{x}(y)\right), \quad x \in V(G) \text { and } y \in E(G)
$$

and

$$
m_{x}(y)= \begin{cases}1 & \text { if } y \text { is incident with } x \\ 0 & \text { otherwise }\end{cases}
$$

With the 1-1 correspondence between the edge set of $G$ and $G^{\prime}$, we have a new matrix $M(D)$ (we name it the incidence matrix of $D)$ as below:

$$
M(D)=\left[\begin{array}{c}
M(G) \\
M\left(G^{\prime}\right)
\end{array}\right]
$$

An example of the incidence matrix of a link diagram (shadow).


|  | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| A | 1 | 0 | 0 | 0 | 1 | 1 |
| B | 1 | 0 | 1 | 0 | 0 | 1 |
| C | 1 | 1 | 0 | 0 | 1 | 0 |
| D | 1 | 1 | 1 | 0 | 0 | 0 |
| E | 0 | 1 | 1 | 1 | 0 | 0 |
| F | 0 | 1 | 0 | 1 | 1 | 0 |
| G | 0 | 0 | 1 | 1 | 0 | 1 |
| H | 0 | 0 | 0 | 1 | 1 | 1 |

## Remark

- $M(D)$ is a $(c+2) \times c$ matrix, if $D$ has $c$ crossing points.
- Ayaka Shimizu's result $\Longrightarrow \mathbb{Z}_{2}$-rank $M(D(K))=c$.

As a generalization, we have

## Theorem (C-Gao 2011)

Let $D$ be a n-component link diagram, then the $\mathbb{Z}_{2}$-rank of $M(D)$ equals to $c-n+1$, here $c$ denotes the crossing number of $D$.

## Remark

This theorem provides an algorithm to calculate the number of components of a link from the Tait graph.


A Tait graph corresponding to a 4-component link diagram

## Remark

- The number of components of the associated link can be also read from the mod-2 Laplacian matrix of the Tait graph.
- Let $G$ be a Tait graph, the mod-2 Laplacian matrix $Q_{2}(G)=\left(q_{i j}\right)$, where $q_{i i}$ is the degree of the $i$-th vertex and $q_{i j}$ is the number of edges between the $i$-th and $j$-th vertices.
- $Q_{2}(G)$ is also the presentation matrix for $H_{1}\left(M_{2}(L) ; \mathbb{Z}_{2}\right) \oplus \mathbb{Z}_{2}$, where $M_{2}(L)$ denotes the double branched cover of $S^{3}$ branched over $L$.

Theorem (Godsil-Royle 2001, Silver-Williams 2015)
Let $L$ be a n-component link diagram and $G$ the associated Tait graph. Then the nullity of $Q_{2}(G)$ equals $n$.

Theorem (C 2012)
Given a link diagram $L=\bigcup_{i=1}^{k} K_{i}$, if $K_{n_{1}} \cap K_{n_{2}} \neq \varnothing$, $K_{n_{2}} \cap K_{n_{3}} \neq \varnothing, \cdots, K_{n_{m-1}} \cap K_{n_{m}} \neq \varnothing$ and $K_{n_{m}} \cap K_{n_{1}} \neq \varnothing$, then for any crossing point $p_{1} \in K_{n_{1}} \cap K_{n_{2}}, \cdots, p_{m-1} \in K_{n_{m-1}} \cap K_{n_{m}}$, $p_{m} \in K_{n_{m}} \cap K_{n_{1}}$, the set $\left\{p_{1}, p_{2}, \cdots, p_{m}\right\}$ are region crossing change admissible.
Conversely, a set of crossing points are region crossing change admissible only if it is the union of some "cyclic crossing points" above.

As a corollary, we give a complete answer to Kishimoto's question.
Theorem (C 2012)
$L=K_{1} \cup K_{2} \cup \cdots \cup K_{n}$ and $L^{\prime}=K_{1}^{\prime} \cup K_{2}^{\prime} \cup \cdots \cup K_{n}^{\prime}$ are related by a sequence of region crossing changes if and only if

$$
\sum_{j \neq i} l k\left(K_{i}, K_{j}\right)=\sum_{j \neq i} I k\left(K_{i}^{\prime}, K_{j}^{\prime}\right)(\bmod 2)
$$

for all $1 \leq i \leq n$.
In particular, region crossing change is an unknotting operation on a diagram of $L=K_{1} \cup K_{2} \cup \cdots \cup K_{n}$ if and only if $L$ is proper, i.e.

$$
\sum_{j \neq i} l k\left(K_{i}, K_{j}\right)=0(\bmod 2)
$$

for all $1 \leq i \leq n$.

The Arf invariant of proper links

- K a knot in $S^{3}$
- $(M, \partial M) \subset S^{3}$ a Seifert surface of $K$
- $V$ the associated Seifert matrix
- $\left\{a_{i}, b_{i}\right\}$ a symplectic basis for the intersection form on $M$

The Arf invariant of $K$ can be defined as

$$
\operatorname{Arf}(K)=\sum_{i=1}^{n} V\left(a_{i}, a_{i}^{+}\right) V\left(b_{i}, b_{i}^{+}\right)(\bmod 2)
$$

where $a_{i}^{+}$denotes the positive pushoff of $a_{i}$.

## Remark

- $\operatorname{Arf}(K)=a_{2}(\bmod 2)$, here $a_{2}$ denotes the coefficient of $z^{2}$ in Conway polynomial.
- $\operatorname{Arf}(K)=\frac{1}{2} \triangle_{K}^{\prime \prime}(1)(\bmod 2)$.
- $\operatorname{Arf}(K)=V_{K}(i)$.

$$
\operatorname{Arf}(K)= \begin{cases}0 & \text { if } K \text { is pass-equivalent to the unknot; } \\ 1 & \text { if } K \text { is pass-equivalent to the trefoil. }\end{cases}
$$

The Arf invariant of proper links:

- $M=S^{3} \times[0,1], \partial M=S^{3} \times\{0\} \cup S^{3} \times\{1\}=\partial M_{+} \cup \partial M_{-}$.
- A proper link $L \subset \partial M_{+}$, a knot $K \subset \partial M_{-}$.
- A regularly embedded 2 -manifold $N$ of genus zero such that $\partial N \cap \partial M_{+}=L$ and $\partial N \cap \partial M_{-}=K$, then we say $K$ is related to $L$.

Theorem (R. Robertello 1965)
If $K$ and $K^{\prime}$ are two knots related to the same proper link $L$, then $\operatorname{Arf}(K)=\operatorname{Arf}\left(K^{\prime}\right)$.

Definition
Define $\operatorname{Arf}(L) \triangleq \operatorname{Arf}(K)$, where $K$ is related to the proper link $L$.

In practice, given a proper link $L=K_{1} \cup K_{2} \cup \cdots \cup K_{n}$, choose a crossing point between $K_{i}$ and $K_{j}$, then smooth it according to the orientations of $K_{i}$ and $K_{j}$.


Repeating this process until we get a knot $K$. Then $\operatorname{Arf}(L)=\operatorname{Arf}(K)$.

Question
What is the relation between region crossing change and Arf invariant?

For a given proper link diagram $L$ and a region $R$, there exists a proper link $L_{R}$ such that

$$
\operatorname{Arf}(L)+\operatorname{Arf}\left(L^{\prime}\right)=\operatorname{Arf}\left(L_{R}\right)(\bmod 2)
$$

Here $L^{\prime}$ denotes the link diagram obtained by taking region crossing change on $R$.



Consider a region $R,\left\{c_{1}, \cdots, c_{n}\right\} \subset \partial R$. Color the regions in checkerboard fashion, such that $R$ is colored white.


Theorem (C 2012)
Let $L$ be a reduced diagram of a proper link, $L^{\prime}$ is obtained by taking region crossing change on region $R$ of $L$, then

$$
\operatorname{Arf}(L)-\operatorname{Arf}\left(L^{\prime}\right)=\frac{1}{4} \sum_{i=1}^{n}\left(a\left(c_{i}\right)-w\left(c_{i}\right)\right) \bmod 2
$$

## Region crossing change on $\Sigma_{g}$ : knot case

Warning From now on, we will make a little modification on the definition of region crossing change.

## Definition

Let $\Sigma_{g}$ be an orientable closed surface with genus $g$ and $L$ a link diagram on it. Suppose $R$ is a region of $\Sigma_{g} \backslash L$ and $c$ is a crossing point on $\partial R$, if $R$ appears $k(1 \leq k \leq 4)$ times around $c$ then applying region crossing change on $R$ switches the crossing point $c$ $k$ times.


- $\Sigma_{g}$ an orientable closed surface with genus $g$.
- $L$ a flat link diagram on $\Sigma_{g}$ with $c$ flat crossing points and $r$ regions.
- Replacing each flat crossing points with an overcrossing point or an undercrossing point, one obtains $2^{c}$ link diagrams on $\Sigma_{g}$.
- Define a graph $G(L)$ such that each link diagram corresponds to a vertex, if two vertices are related by one region crossing change then we add an edge connecting them.


## Remark

- Any pair of connected components of $G(L)$ are isomorphic.
- $\forall u, v \in V(G(L))$, there exists an isomorphism
$h: G(L) \rightarrow G(L)$ such that $h(u)=v$.


## Question

- How many connected components does $G(L)$ have?
- In particular, when is $G(L)$ connected?
- One can define the incidence matrix $M(L)=\left(a_{i j}\right)_{r \times c}$, where $a_{i j}=k(\bmod 2)$ if the $i$-th region appears $k$ times around the $j$-th crossing point.
- The number of components of $G(L)$ equals $2^{c-\left(\mathbb{Z}_{2}-\operatorname{rank}(M(L))\right)}$.

Lemma (Cheng-C-Xu-Zheng 2019)
$r-\left(\mathbb{Z}_{2}\right.$-rank $\left.(M(L))\right)$ is invariant under homotopy of $L$.

## Corollary

- Let $K$ be a knot diagram on $R^{2}\left(S^{2}\right)$, then every crossing point of $K$ is (new) region crossing change admissible.
- (New) region crossing change is an unknotting operation for knot diagrams on the plane.


## Proof.

Any flat knot diagram is homotopy equivalent to a trivial circle and $r=c+2$, therefore $\mathbb{Z}_{2}$-rank $M(K)=r-2=c(K)$.

## Remark

- This result can be found in the original proof of Shimizu.
- It is not true if $g \geq 1$.


A knot diagram on $\Sigma_{1}$


A knot diagram on $\Sigma_{2}$

Theorem (Cheng-C-Xu-Zheng 2019)
Let $\Sigma_{g}$ be an orientable closed surface with genus $g$ and $K$ a knot diagram on it,

- if $\Sigma_{g} \backslash K$ admits a checkerboard coloring, then

$$
\mathbb{Z}_{2} \text {-rankM }(K)=r-2 ;
$$

- if $\Sigma_{g} \backslash K$ does not admit a checkerboard coloring, then

$$
\mathbb{Z}_{2}-\operatorname{rank} M(K)=r-1 .
$$

Corollary
Let $\Sigma_{g}$ be an orientable closed surface with genus $g$ and $K$ a knot diagram on it,

- $G(L)$ has $2^{c+2-r}$ connected components if $\Sigma_{g} \backslash K$ admits a checkerboard coloring;
- $G(L)$ has $2^{c+1-r}$ connected components if $\Sigma_{g} \backslash K$ does not admit a checkerboard coloring.

Region crossing change on $\Sigma_{g}$ : link case

- $\Sigma_{g}$ an orientable closed surface with genus $g$
- Denote the standard generators of $H_{1}\left(\Sigma_{g}, \mathbb{Z}\right)$ by $\left\{\alpha_{i}\right\}_{1 \leq i \leq 2 g}$
- $L=K_{1} \cup \cdots \cup K_{n}$ a link diagram on $\Sigma_{g}$
- $\left[K_{i}\right]=\sum_{j=1}^{2 g} a_{i j} \alpha_{j} \in H_{1}\left(\Sigma_{g}, \mathbb{Z}\right)$
- Define a matrix $N(L)=\left(a_{i j}\right)_{n \times 2 g}$


## Remark

Notice that when $n=1, \mathbb{Z}_{2}$-rank $N(K)=0$ if and only if $\Sigma_{g} \backslash K$ admits a checkerboard coloring.

Theorem (Cheng-C-Xu-Zheng 2019)
Let $\Sigma_{g}$ be an orientable closed surface with genus $g$ and $L a$ n-component link diagram on it, then

$$
\mathbb{Z}_{2}-\operatorname{rank} M(L)=r-n-1+\mathbb{Z}_{2}-\operatorname{rank} N(L) .
$$

## Thank you!


[^0]:    ${ }^{1}$ http://www.sci.osaka-cu.ac.jp/math/OCAMI/news/gamehp/etop/gametop.html

