An infinite family of nongeometric embeddings of braid groups into MCG and the homology triviality

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Joint work with Byung Chun Kim

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The scheme of the talk

- We introduce a new family of nongeometric embeddings of braid groups into mapping class groups induced by n-**fold coverings** over a disk with some ramified points. The construction of these embeddings will be expalined by some pictures and in terms of kind of 'groupoid.'
- Each standard generator of braid groups is mapped to the **lift** of the half Dehn twist with respect to n-fold covering. We should show that these lifts satisfy the braid relations. The injectivity is proved by Birmann-Hilden theory.
- We show, by using the theory of iterated loop spaces, that these embeddings induce trivial homomorphim on homology groups in any constant coefficients.(Harer conjecture)
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- Kim-S. found an affirmative answer. After posting their paper at arXiv last year, the second author received an e-mail from Ghaswala (and McLeay) that they also got the same result. About two months later they posted their paper at arXiv, stating the result for $n \geq 3$ which is an extension of the case $n = 3$. Surprisingly, Callegaro and Salvetti, a few months later, also posted their paper at arXiv stating the same result.
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Configuration space

Let $\mathcal{C}_k = \mathrm{Conf}_k(S_{0,1})/\Sigma_k$ be the **configuration space** of unordered k distinct interior points on the disk.

Then we have

$$\mathcal{C}_k \simeq \mathrm{B}\,\Gamma_{0,1,(k)} \simeq \mathrm{B}\,B_k.$$

Here $\Gamma_{g,b,(k)}$ denotes the mapping class group which the group of isotopy classes of self-homeomorphisms of genus g surface with b boundary components which fix the boundary pointwise, allowing the marked points to be permuted.

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We are interested in the embeddings of the braid groups into the mapping class groups : $B_k \hookrightarrow \Gamma_{g,b}$.

We will investigate the maps in the level of classifying space :

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An embedding $\phi : B_k \hookrightarrow \Gamma_{g,b}$ is said to be **geometric** if each standard generator of B_k is mapped to a *Dehn twist* in $\Gamma_{g,b}$.

Let $\beta_1, \ldots, \beta_{2g-1}$ be the standard generators of the braid group B_{2g} and $\alpha_1, \ldots, \alpha_{2g-1}$ be the Dehn twist generators in $\Gamma_{g,1}$ given as follows:

Let $\phi : B_{2g} \to \Gamma_{g,1}$ be the *standard* geometric embedding defined by $\beta_i \mapsto \alpha_i$. Then ϕ is extended to a homomorphism of stable groups from $B_\infty := \lim_{g\to\infty} B_g$ to $\Gamma_\infty := \lim_{g\to\infty} \Gamma_{g,1}$.

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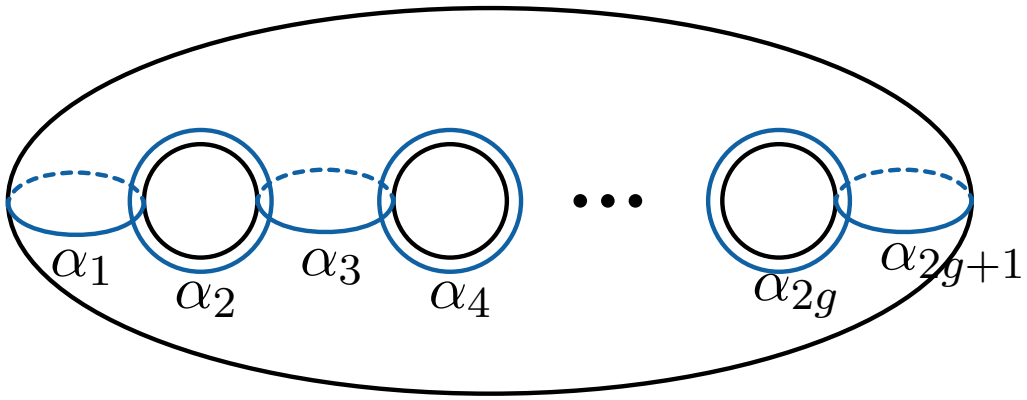

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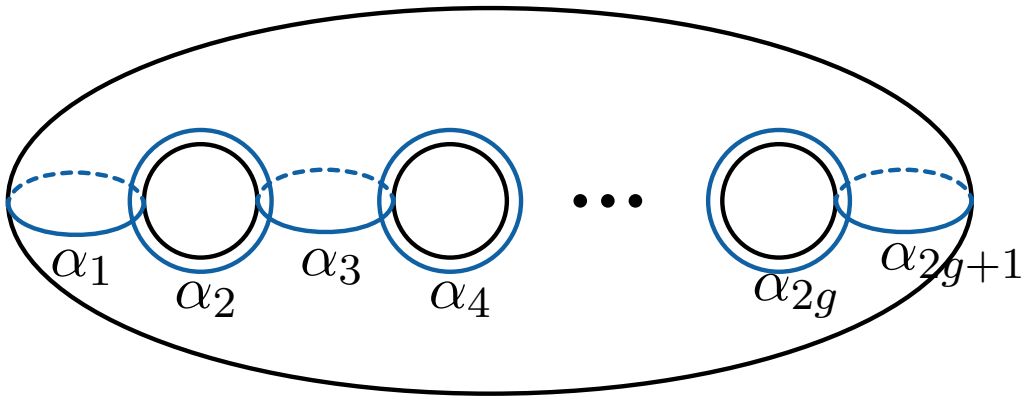

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Theorem (Harer conjecture, proved by S. and Tillmann (2007))

The map $\phi_* : H_*(B_\infty; \mathbb{Z}/2\mathbb{Z}) \to H_*(\Gamma_\infty; \mathbb{Z}/2\mathbb{Z})$ *is trivial.*

The key part of the proof of Harer conjecture is to show that the embedding of braid group into mapping class group preserves the 2-fold loop space structure in the level of classifying spaces.

- The proof of S. and Tillmann used the idea of lifting the embedding (group homomorphism) to a functor between two monoidal 2-categories.
- Later, Segal and Tillmann raised an idea of lifting the embedding to the space map $\mathcal{C}_k \to \mathcal{M}_{g,b}$ and showed that it is compatible with the action of little 2-cube operads.

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Other embeddings

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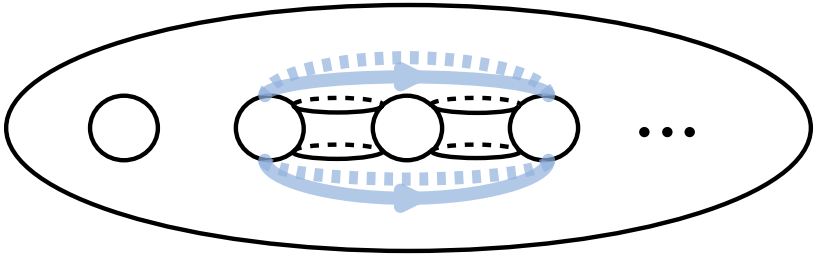

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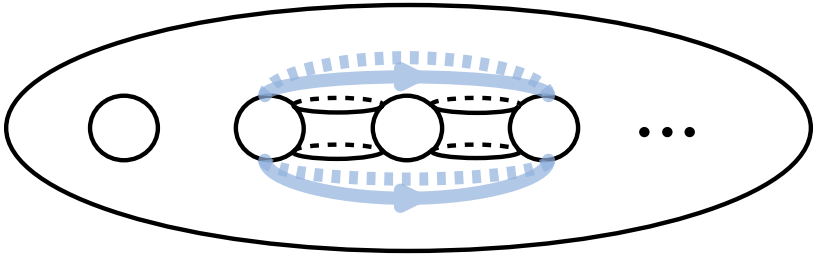

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Groupoid with intersection cycles

- We will interpret the n-fold (branched) covering of a disk D with k branch points in terms of a "groupoid with intersection cycles" (will be defined later).
- With this interpretation, we show that the "lift of half Dehn twists on D" preserves the braid relation; i.e. we construct an injective homomorphism $\phi : B_k \to \Gamma_{g,b}$, where the surface $S_{g,b}$ is the n-fold covering space over a disk with k branch points.

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The injectivity of the map $\phi : B_k \to \Gamma_{g,b}$ is proved by the Birman-Hilden theory.

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Let $p : S_{g,b} \to S_{0,1,(k)}$ be a n-fold branched covering. Assume that every half Dehn twist β_i corresponding to a generator of B_k can be lifted to a homeomorphism $\tilde{\beta}_i$ of $S_{g,b}$ for $1 \leq i \leq (k-1)$. Then the assignment $\beta_i \mapsto \tilde{\beta}_i$ defines an injective homomorphism $B_k \to \Gamma_{g,b}$

By showing that space map $\Phi : \mathcal{C}_k \simeq \mathrm{B}\, B_k \to \mathrm{B}\, \Gamma_{g,n} \simeq \mathcal{M}_{g,n}$ is compatible with the action of framed little 2-cube operad, we have the main theorem

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2-fold covering

Segal and Tillmann considered the following map:

$$\Phi : \mathcal{C}_{2g+2} \to \mathcal{M}_{g,2}$$

which takes a subset $P = \{a_1, \ldots, a_{2g+2}\} \subset \mathring{D}$ to the part of the Riemmann surface Σ_P of the function

$$f_P(z) = ((z - a_1) \cdots (z - a_{2g+g}))^{1/2}.$$

On the level of fundamental groups we get $\phi : B_{2g+2} \to \Gamma_{g,2}$ which maps the standard generators of braid group to Dehn twists on the surface.

$S_{g,2}$ is regarded as a 2-fold (branched) covering over the unit disk D with marked points P and is formed by gluing $2g + 1$ annuli. And then the generator of braid group which is half Dehn twist on the disk switching two points p_i and p_{i+1} in P is lifted by the 2-fold covering to a self-homeomorphim on $S_{g,2}$.

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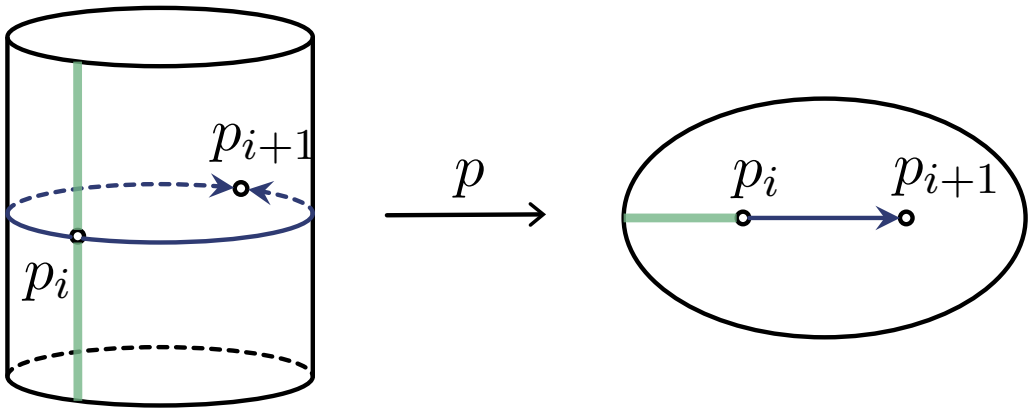


Figure: 2-fold branched covering over a disk with two branch points.

We may regard the 2-fold covering over a disk as a functor between two subcategories of the fundamental groupoids.

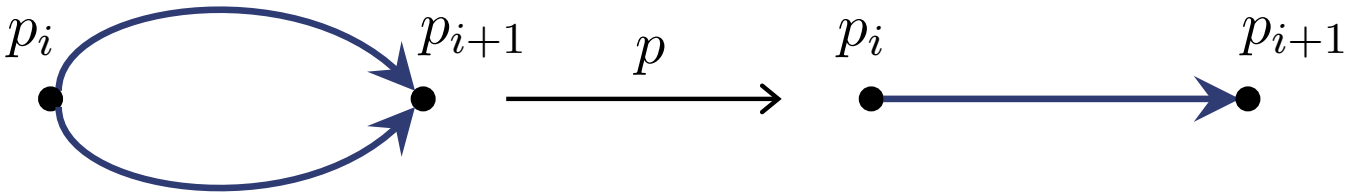


Figure: As a functor of groupoids (generated by the depicted graphs).

The lift of a half Dehn twist

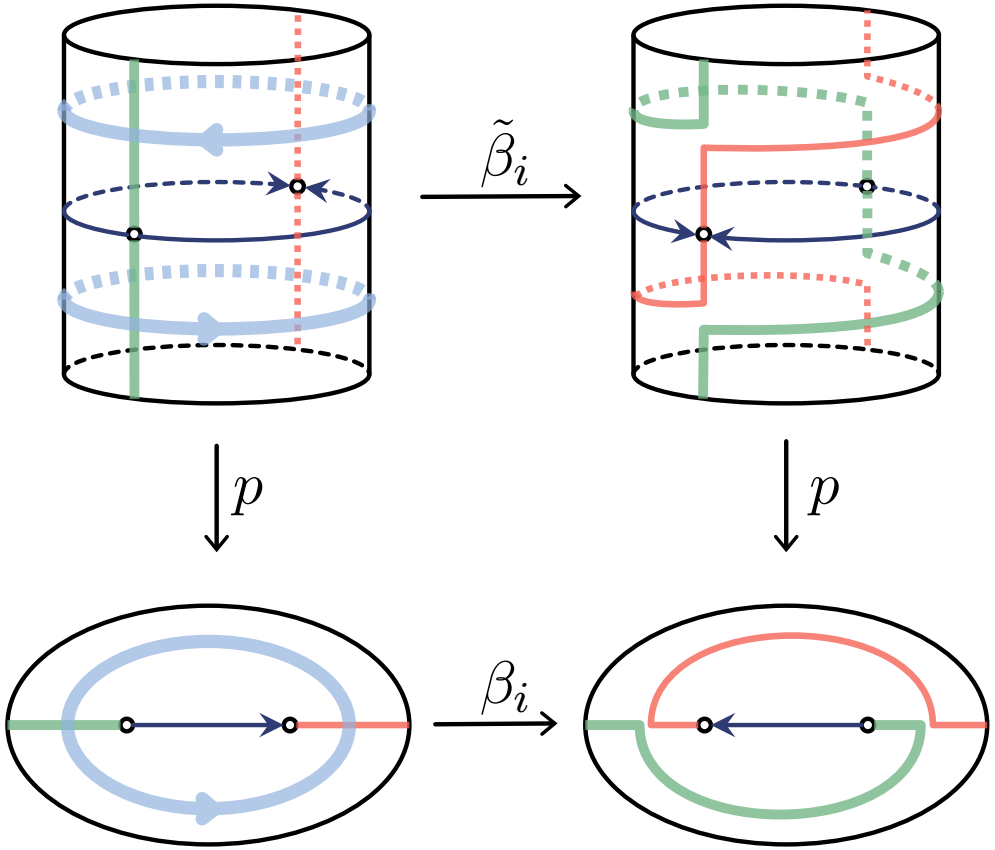
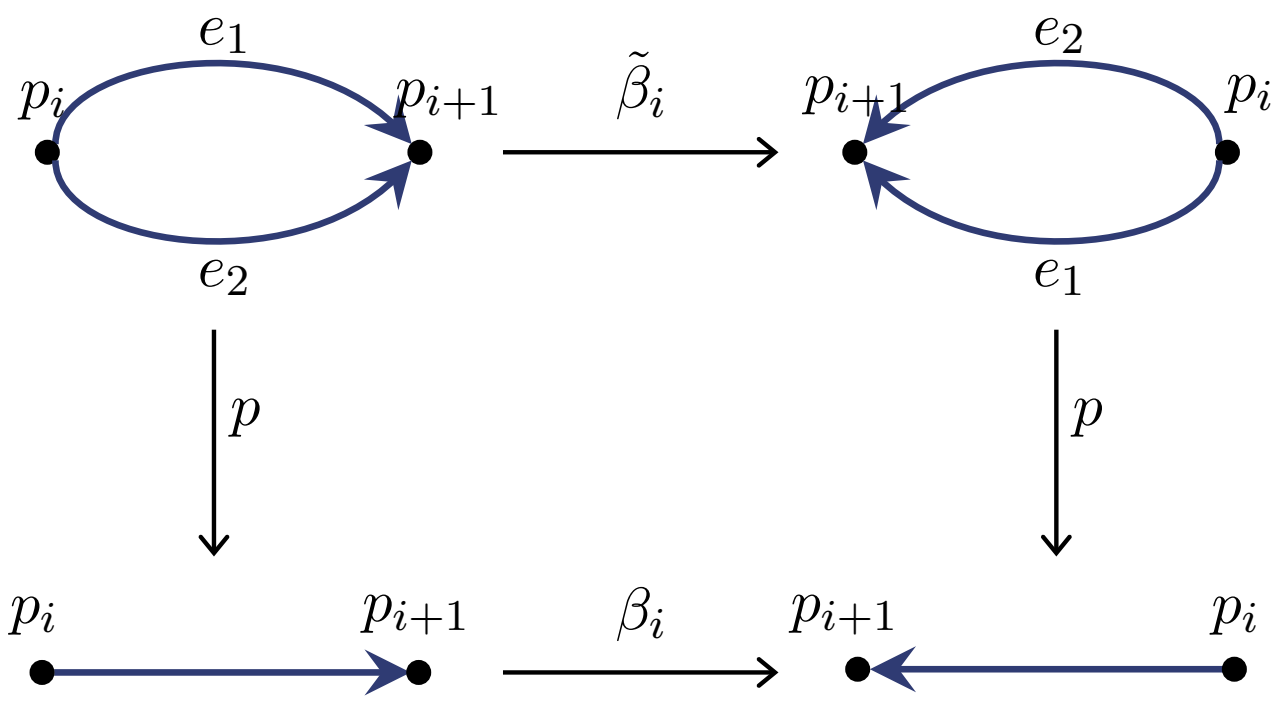


Figure: The half twist on disk is lifted to a full Dehn twist on annulus.

The lift of a half Dehn twist regarded as a functor



$$\tilde{\beta}_i : \begin{cases} p_i & \mapsto & p_{i+1} \\ p_{i+1} & \mapsto & p_i \end{cases} \quad \& \quad \begin{array}{ccc} e_1 & \mapsto & e_2^{-1} \\ e_2 & \mapsto & e_1^{-1} \end{array}$$

The corresponding surface

Given a groupoid $\mathcal{G}$ with intersection cycles at all vertices, we can construct the corresponding oriented surface.

1. To each vertex p_i assign a small disk centered at the marked point p_i.
2. To each edge $e_j^{(i)}$ assign a strip with two distinct marked points p_i and p_{i+1} on the boundary in such way that neighborhoods of the marked points p_i and p_{i+1} are of the form of circular sectors with the central angles $\frac{2\pi}{m_i+m_{i+1}}$ and $\frac{2\pi}{m_{i+1}+m_{i+2}}$, respectively.
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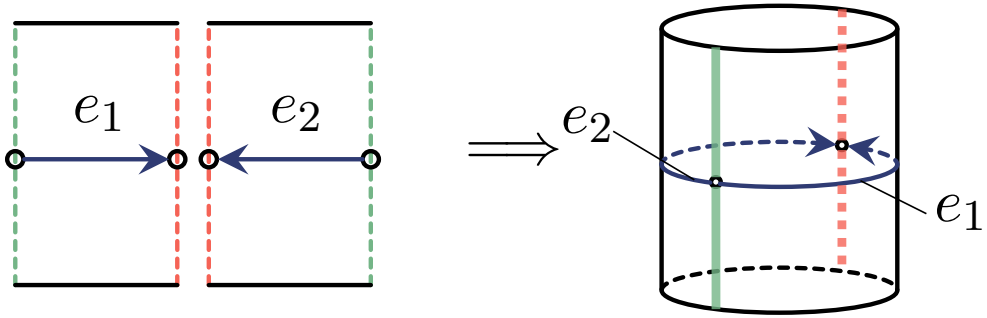

3-fold covering (two branch points)

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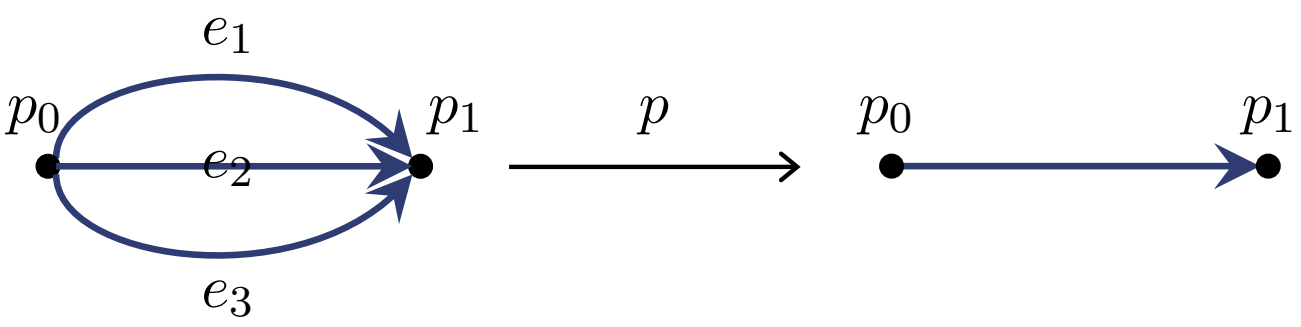

There are two cases. If the two intersection cycles at p_0 and p_1

- are different, then we get $S_{0,3}$.
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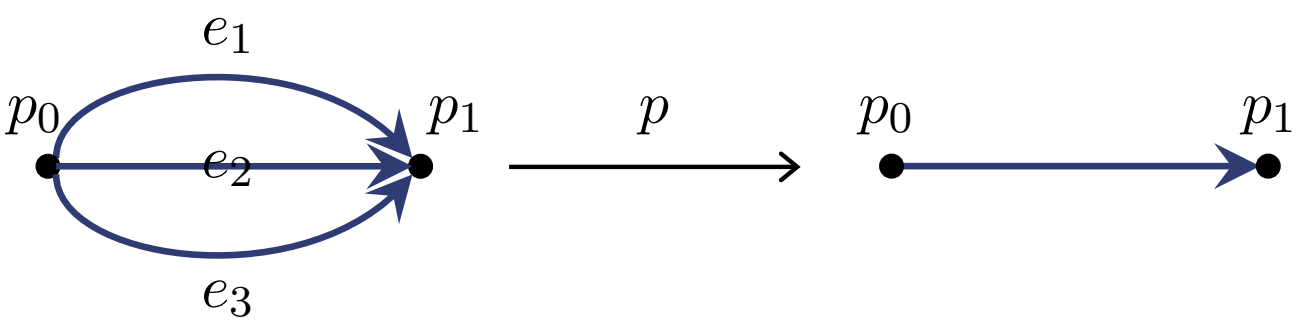

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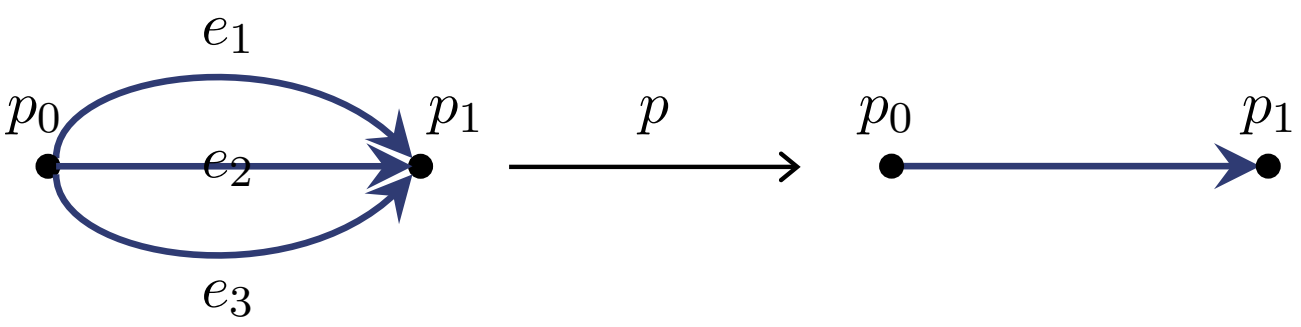

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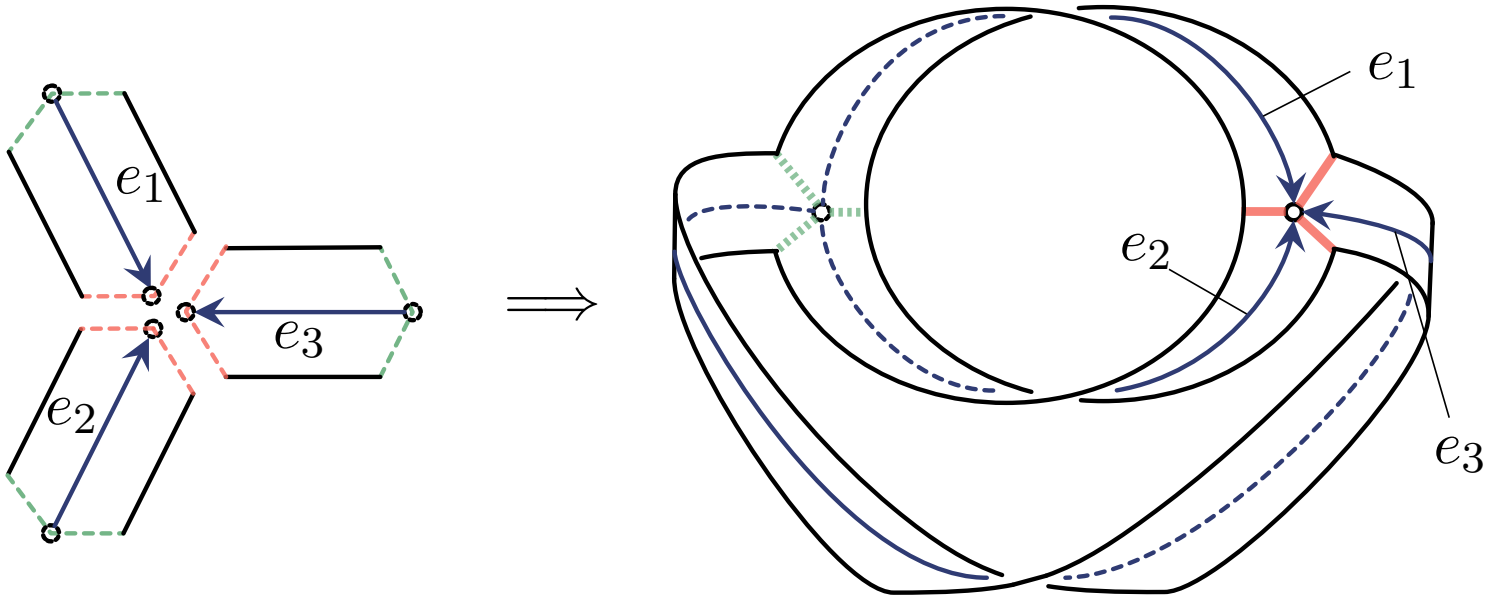
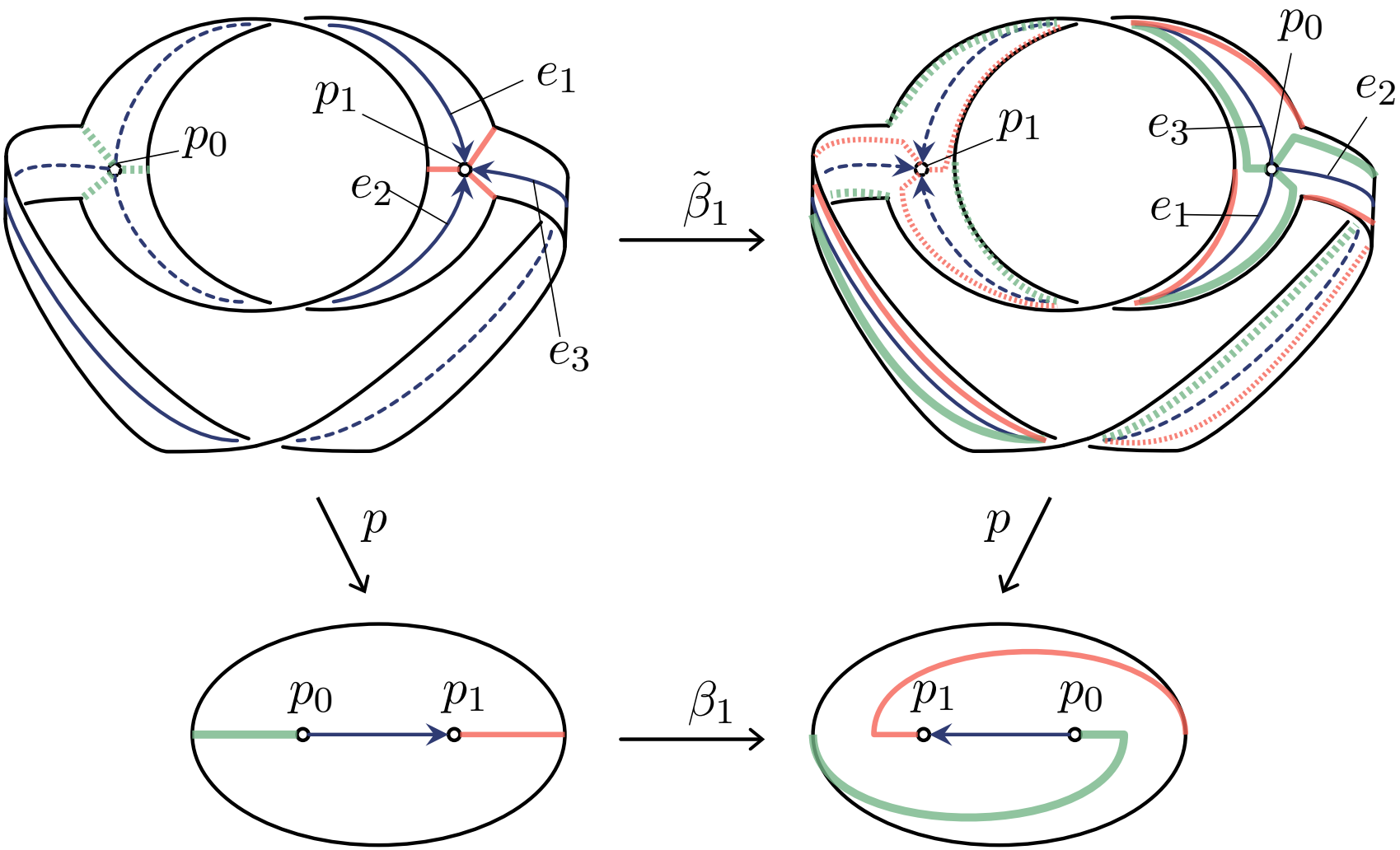


Figure: $S_{1,1}$: 3-fold branched covering of a disk with two branch points.

Lift of the half Dehn twist



1/6 Dehn twist

The lift of the half Dehn twist may be regarded as a $1/6$ Dehn twist.

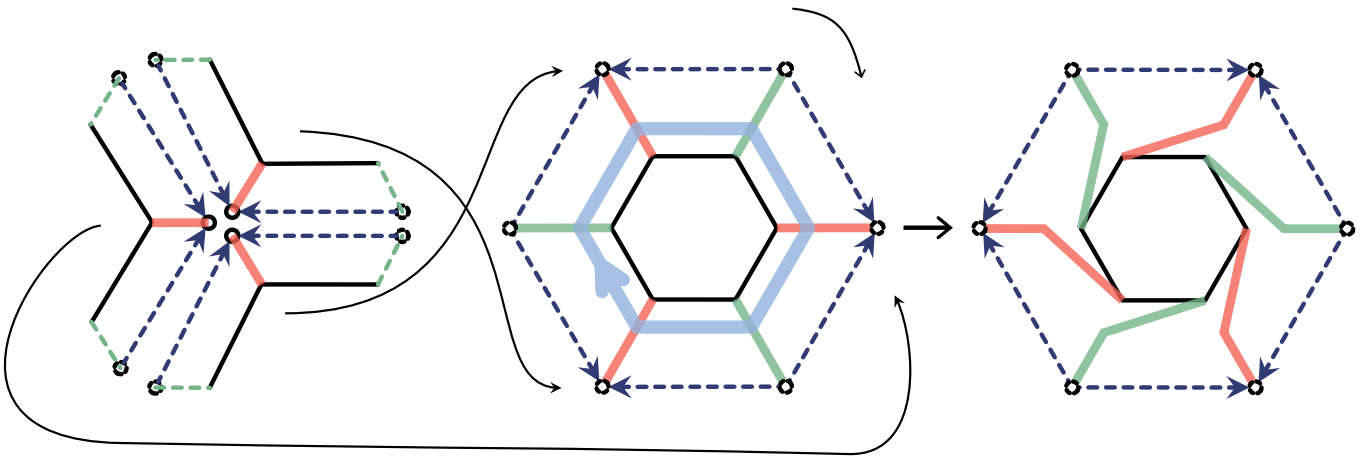
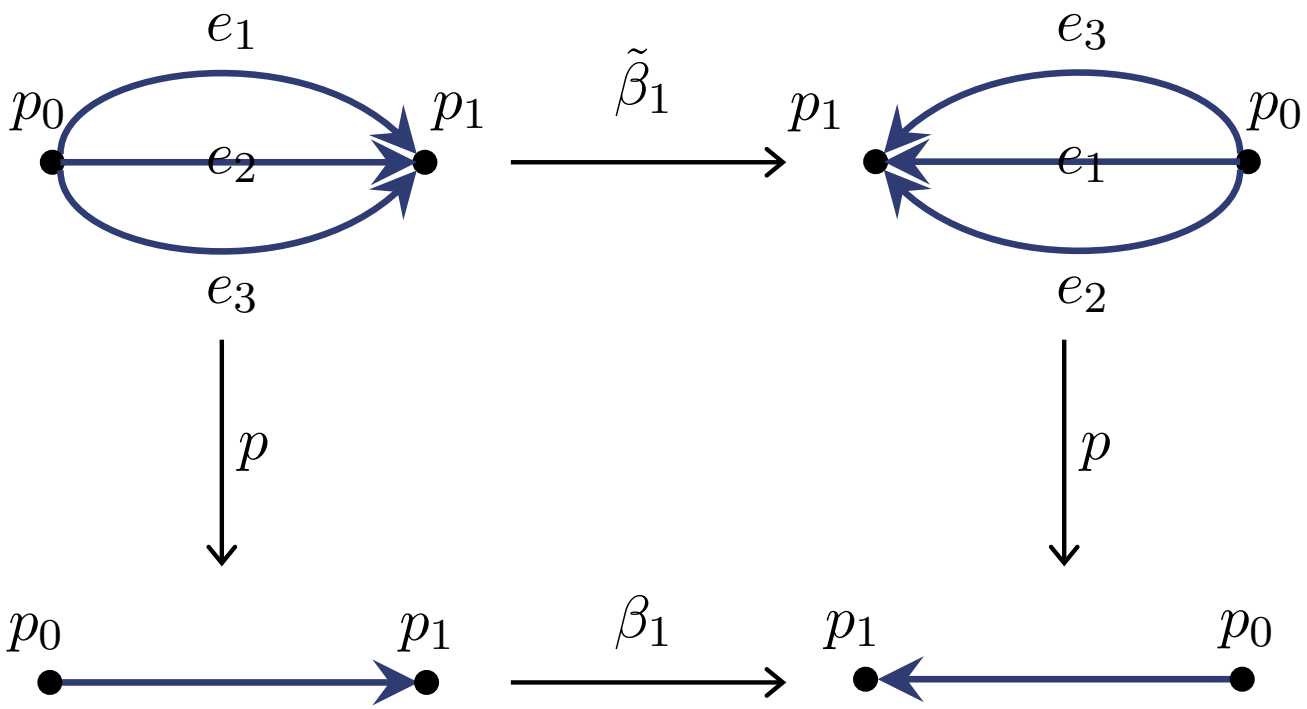


Figure: 1/6 Dehn twist.

Lift of the half Dehn twist regarded as a functor



$$\tilde{\beta}_1 : \begin{cases} p_0 & \mapsto & p_1 \\ p_1 & \mapsto & p_0 \end{cases} \quad \& \quad \begin{array}{ccc} e_1 & \mapsto & e_2^{-1} \\ e_2 & \mapsto & e_3^{-1} \\ e_3 & \mapsto & e_1^{-1} \end{array}$$

4-fold covering

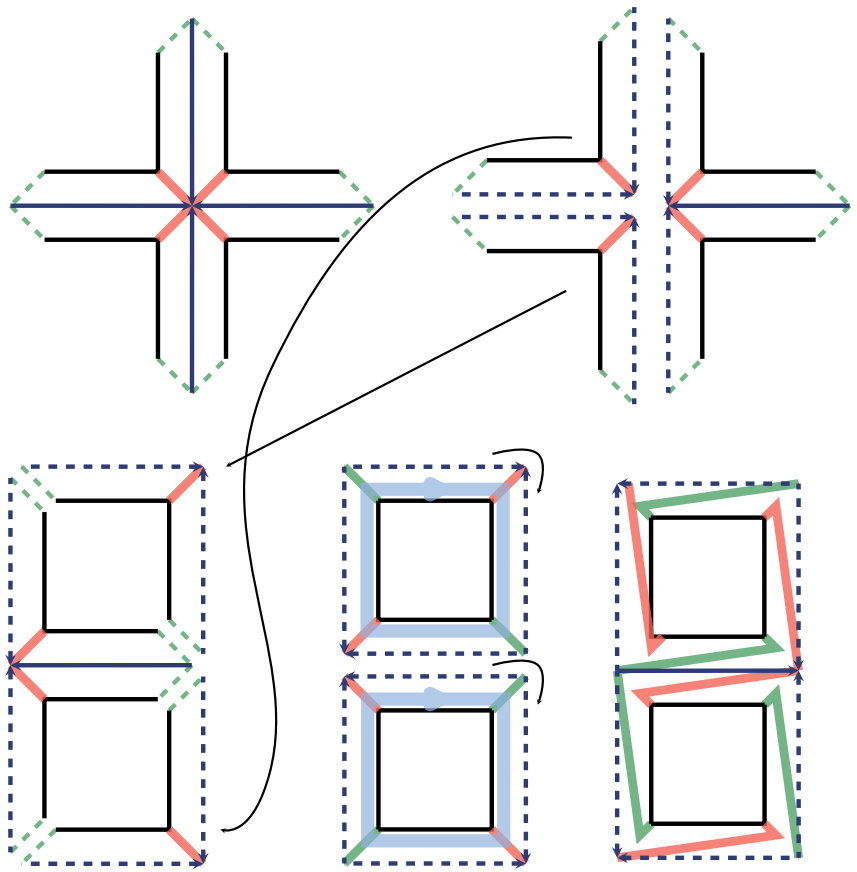


Figure: Two 1/4 Dehn twists.

4-fold covering

$$\tilde{\beta}_1 : \begin{cases} p_0 & \mapsto & p_1 \\ p_1 & \mapsto & p_0 \end{cases} \quad \& \quad \begin{array}{ccc} e_1 & \mapsto & e_2^{-1} \\ e_2 & \mapsto & e_3^{-1} \\ e_3 & \mapsto & e_4^{-1} \\ e_4 & \mapsto & e_1^{-1} \end{array}$$

n-fold branched covering

In general, for n-fold covering, a half Dehn twist on the disk is lifted to the self-homeomorphim on the surface as follows:

$$\begin{cases} 1/2\text{n Dehn twist} & \text{if } n \text{ is odd,} \\ \text{two } 1/\text{n Dehn twists} & \text{if } n \text{ is even.} \end{cases}$$

In the groupoid language,

$$\tilde{\beta}_1 : \begin{cases} p_0 & \mapsto & p_1 \\ p_1 & \mapsto & p_0 \end{cases} \quad \& \quad e_j^{(1)} \mapsto \left(e_{j+1}^{(1)}\right)^{-1}$$

where the indicies of edges read modulo n.

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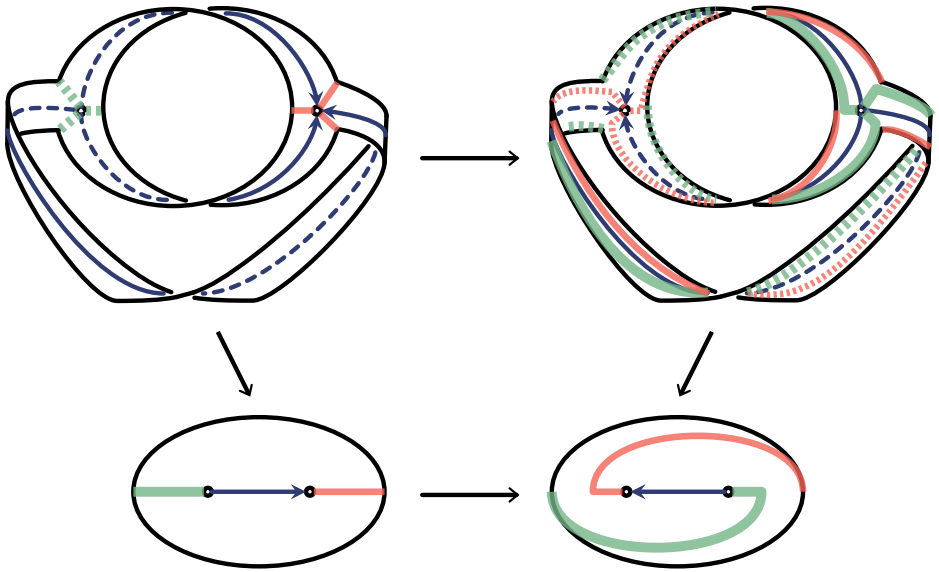
More branch points



Then we can summarize the lift of a half Dehn twist as:

$$\tilde{\beta}_i : \begin{cases} p_i \mapsto p_{i+1} \\ p_{i+1} \mapsto p_i \end{cases} \quad \& \quad \begin{aligned} e_j^{(i)} &\mapsto \left(e_{j+1}^{(i)}\right)^{-1} \\ \mathbf{e_j^{(i-1)}} &\mapsto \mathbf{e_j^{(i-1)} e_j^{(i)}} \\ \mathbf{e_j^{(i+1)}} &\mapsto \mathbf{e_{j+1}^{(i)} e_j^{(i+1)}} \end{aligned}$$

where the indices of edges read modulo n.

Braid relation

We check the braid relation $\tilde{\beta}_i\tilde{\beta}_{i+1}\tilde{\beta}_i = \tilde{\beta}_{i+1}\tilde{\beta}_i\tilde{\beta}_{i+1}$ is satisfied in the groupoid level:

$$p_i \xmapsto{\tilde{\beta}_i} p_{i+1} \xmapsto{\tilde{\beta}_{i+1}} p_{i+2} \xmapsto{\tilde{\beta}_i} p_{i+2}$$

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$$p_{i+1} \xmapsto{\tilde{\beta}_i} p_i \xmapsto{\tilde{\beta}_{i+1}} p_i \xmapsto{\tilde{\beta}_i} p_{i+1}$$

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$$e_j^{(i-1)} \xmapsto{\tilde{\beta}_i} e_j^{(i-1)} e_j^{(i)} \xmapsto{\tilde{\beta}_{i+1}} e_j^{(i-1)} e_j^{(i)} e_j^{(i+1)} \xmapsto{\tilde{\beta}_i} e_j^{(i-1)} e_j^{(i)} e_j^{(i+1)}$$

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$$e_j^{(i)} \xmapsto{\tilde{\beta}_i} \left(e_{j+1}^{(i)}\right)^{-1} \xmapsto{\tilde{\beta}_{i+1}} \left(e_{j+1}^{(i+1)}\right)^{-1} \left(e_{j+1}^{(i)}\right)^{-1} \xmapsto{\tilde{\beta}_i} \left(e_{j+1}^{(i+1)}\right)^{-1}$$

$$e_j^{(i)} \xmapsto{\tilde{\beta}_{i+1}} e_j^{(i)} e_j^{(i+1)} \xmapsto{\tilde{\beta}_i} e_j^{(i+1)} \xmapsto{\tilde{\beta}_{i+1}} \left(e_{j+1}^{(i+1)}\right)^{-1}$$

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We have constructed a homomorphism $\phi_n : B_k \simeq \Gamma_{0,1,(k)} \to \Gamma_{g,b}$ induced by the n-fold (branched) covering. By the Birman-Hilden theory, this map is proved to be injective.

Proposition

The mapping $\phi_n : \beta_i \mapsto \tilde{\beta}_i$ defines an embedding $B_k \hookrightarrow \Gamma_{g,b}$.

These new maps are all nongeometric embeddings of braid groups for $n > 2$. The image $\tilde{\beta}_i$ of β_i under ϕ_n turns out to be the product of $n-1$ Dehn twists along consecutive closed curves.

For these new embeddings we can also prove the Harer conjecture, that is, the homology homomorphisms induced by ϕ_n is trivial.

Theorem

The homology homomorphism

$$(\phi_n)_* : H_*(B_\infty; R) \to H_*(\Gamma_\infty; R)$$

induced by ϕ_n is zero for all $* \geq 1$ in any constant coefficient R

Proof of Theorem; little 2-cubes operad actions

Let $\mathcal{X}(m) = \mathrm{Conf}_m$ and let $\mathcal{Y}(0) = \mathcal{M}_{0,1} \sqcup \mathcal{M}_{0,1}$ and $\mathcal{Y}(m) = \mathcal{M}_{m-1,2}$ if $m \geq 1$. Then $\mathcal{X} = \coprod \mathcal{X}(m)$ and $\mathcal{Y} = \coprod \mathcal{Y}(m)$ are $\mathcal{C}_2$-algebras.

Each surface $S \in \mathcal{Y}(m) = \mathcal{M}_{m-1,2}$ has two parametrized boundaries. For $f \in \mathcal{C}_2(k)$, let $D_f = I^2 \setminus f({}^kJ^2)$. That is, D_f is a surface with $k+1$ parametrized boundary components. Then $\mathcal{Y}$ is a $\mathcal{C}_2$-algebra with an operad action

$$\gamma_{\mathcal{Y}} : \mathcal{C}_2(k) \times (\mathcal{Y}(m_1) \times \cdots \times \mathcal{Y}(m_k)) \to \mathcal{Y}\left(\sum m_i\right)$$

defined by

$$(f; S_1, \ldots, S_k) \mapsto ({}^2D_f \cup S_1 \cup \cdots \cup S_k)/\equiv .$$

Here two parametrized boundaries of S_i and the i-th boundaries of two D_f's are identified for each $1 \leq i \leq k$.

Uniform embeddings

We have considered three types of braid embeddings into mapping class groups.

- The standard geometric embedding
  (in fact, induced by 2-fold branched coverings).
- The (nongeometric) embeddings induced by n-fold branched coverings. $(n \geq 3)$
- The pillar switchings.

These embeddings $\phi : B_k \to \Gamma_{g,b}$ have the following common characteristics:

- The (minimum) genus g and the (minimum) number of boundary components b are determined by k.
- For such k's, $\Phi : \coprod \mathrm{Conf}_k \to \coprod \mathcal{M}_{g(k),b'}$ preserves little 2-cubes operad action.
- $\mathrm{supp}\, \phi(\beta_i)$'s are all homeomorphic.

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Uniform embeddings

Let $\phi : B_k \hookrightarrow \Gamma_{g(k),b(k)}$ be an embedding. If $\operatorname{supp}\phi(\beta_i)$'s are all homeomorphic, then we call ϕ a **uniform embedding**.

Suppose that ϕ is a uniform embedding and there exists $n \in \mathbb{Z}_+$ such that

$$b' = b(n) = b(2n) = b(3n) = \cdots$$

and $g(m_1 n) + g(m_2 n) + b' - 1 = g(m_1 n + m_2 n)$ for all $m_1, m_2 \in \mathbb{Z}_+$.

The last equation guarantees that $({}^{b'}D \cup S_{g(m_1 n),b'} \cup S_{g(m_2 n),b'})/\equiv$ is homeomorphic to $S_{g(m_1 n + m_2 n),b'}$, where D is a sphere with three ordered and parametrized boundary components.

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Conjectures

All embeddings, geometric and nongeometric, found so far are all uniform embeddings.

Conjecture

Every embedding $\phi : B_k \hookrightarrow \Gamma_{g,b}$ is (essentially or conjugately equivalent to) a uniform embedding.

Question: Is there a classification of all possible conjugacy classes of embeddings of braid groups in the mapping class groups?

Conjecture (Generalized Harer Conjecture)

For any embedding $\phi : B_k \hookrightarrow \Gamma_{g,b}$, the homology homomorphism

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Thank you