Decomposition of Homologically Trivial Knots in $F \times I$

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- In recent years, the interest in knots in direct products of surfaces and intervals has increased.
- This can be explained by the fact that manifolds of the form $F \times I$ constitute the next to simplest class of 3 manifolds, after the sphere S^3.
- Knots in such manifolds are described by diagrams similar to spherical diagrams of classical knots. The Reidemeister moves play the same role: they implement knot isotopies.
- The theory of knots in products of surfaces and intervals is close to virtual knot theory and dominates it in some sense.

- Let F be a connected closed orientable surface. A knot in $F \times I$ is defined as an arbitrary simple closed curve $K \subset F \times I$.

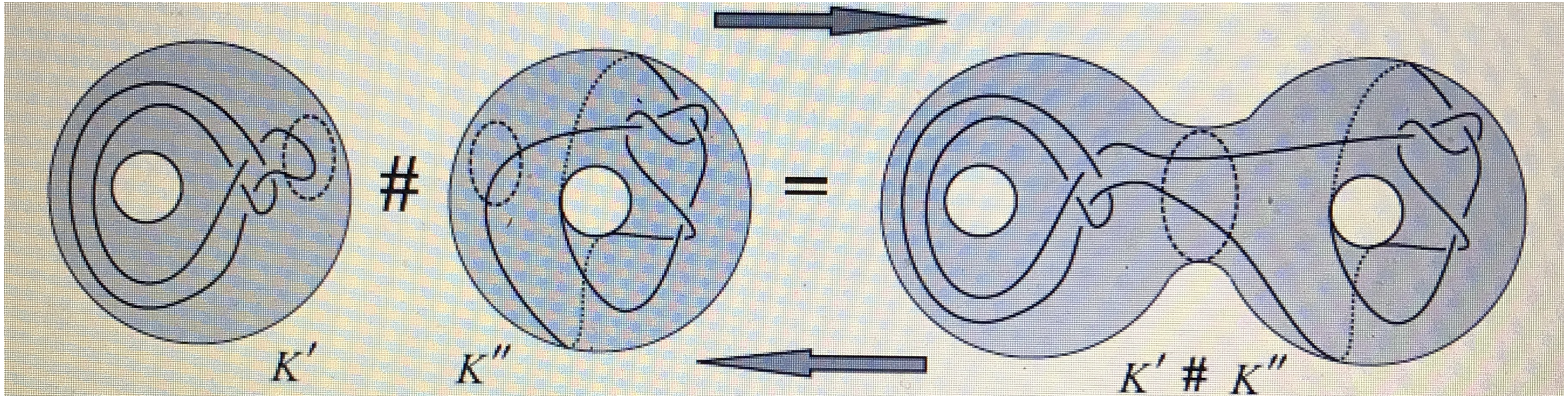
- Two knots $K \subset F \times I$ and $K' \subset F' \times I$ are said to be *equivalen*t if the pair $(F \times I, K)$ is homeomorphic to the pair $(F' \times I, K')$.

- Consider two knots $K_i \subset F_i \times I$, where $i = 1, 2$. To define their annular connected sum $K_1 \# K_2 \subset (F_1 \# F_2) \times I$, for each i we choose a disk $D_i \subset F_i$ and isotopically deform *ii* K_i so that the intersection

$l_i = K_i \cap (D_i \times I)$ is a trivial arc in the ball $D_i \times I$.

Definition 1.

- The knot $K = K_1 \# K_2$ in the product of the surface $F = F_1 \# F_2$ and the interval obtained by gluing together the manifolds $(F_i \setminus \mathrm{Int} D_i) \times I$, where $i = 1,2$, by a homeomorphism $\varphi : \partial D_1 \times I \to \partial D_2 \times I$ such that $\varphi(\partial\, l_1) = \partial l_2$ is called an *annular connected sum* of the knots $K_1 \subset F_1 \times I$ and $K_2 \subset F_2 \times I$.



- If this definition is reformulated in the language of diagrams, then we obtain the same definition as in classical case.
- We need to cut from the surfaces F_1 and F_2 a pair disks intersecting the diagram in the single simple arc each, and then glue the resulting surfaces together by a homomorphism of the boundaries preserving the endpoints of the above arcs.

- The connected sum $K_1 \# K_2$ depends both on the choice of the disks D_1 and D_2 and of the isotopic deformations of knots and on the choice of the homeomorphism φ. In the general case, the number of different connected sums of two given knots is infinite.

- However, if one of the knots is a trivial knot in $S^2 \times I$, then the knot $K_1 \# K_2$ is equivalent to the second knot. Such a summation is said to be *trivial*.

- A knot in $F \times I$ is called *prime* if it cannot be represented as a nontrivial connected sum of two other knots.

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Theorem 1.

- *If a knot* $K \subset F \times I$ *is not a trivial knot in* $S^2 \times I$*, then this knot either is prime or decomposes into a connected sum of prime knots.*

- This theorem says nothing about uniqueness of prime summands.

Definition 2.

- A knot $K \subset F \times I$ is said to be *homologically trivial* if it determines the trivial element in the first homology group
- $H_1(F \times I; \mathbb{Z}_2) = H_1(F; \mathbb{Z}_2)$

- The class of homologically trivial knots is closed with respect to connected sums: a connected sum of two knots is homologically trivial if and only if both summands are homologically trivial.

Theorem 2.

- *The summands of any homologically trivial knot represented as a connected sum of prime knots are determined uniquely up to equivalence.*

- This theorem is an analogue (for knots in thickened surfaces) of the Shubert theorem on prime decompositions of classical knots.

- To present the basic ideas in the proofs of both Theorems we define four types of reductions of knots in thickened surfaces (one is main type, and the other three are arbitrary) by analogy with annular reductions of manifolds.

- To describe the basic ideas of the proofs of these theorems, it is convenient to the use the operation of annular reduction, which is inverse to connected summation. We say that a separating annulus $A \subset F \times I$ is *admissible* (with respect to a given knot $K \subset F \times I$) if it intersects K in two points, is isotopic to a vertical annulus, and is not trivial, i.e., does not cut out from $F \times I$ a ball of the form

$D_2 \times I$ containing inside a trivial arc of K. The annular reduction consists in cutting $F \times I$ along an admissible annulus $A \subset F \times I$ and pasting the two copies of this annulus thus obtained by two handles of index 2 containing trivial arcs.

Using Diamond Lemma

- Consider the abstract graph Γ constructed as follows. Each of its vertices is either a pair of the form $(F\times I, K)$ not being a trivial knot in $S^2\times I$ or a disconnected union of several such pairs. We denote the set of all vertices by (Γ). Vertices $V, W \in (\Gamma)$ are joined by an oriented edge $\overrightarrow{VW}$ if and only if the set of knots (W). is obtained from the set of knots (V) by applying a non trivial annular reduction to some pair $(F \times I, K) \in V$.

Definition 3

- We say that a vertex W of the graph Γ is *a root* of a vertex V if
- (i) there exists a coherently oriented path along edges of the graph from V to W;
- (ii) W is a sink, i.e., there are no outgoing edges.

- It follows at once from Definition 1 that a nontrivial knot $K \subset F \times I$ decomposes into prime summands if and only if the corresponding vertex has a root. Moreover, the summands of such a decomposition are determined uniquely if and only if the root is unique.

- For this reason, the question of the existence and uniqueness of a root for any vertex of the graph Γ is very important.

- We need two useful conditions, (CF) (complexity function) and (EE) (edge equivalence).
- (CF) There exists a function $c(V) \to Z_+ = \{1,2, \dots\}$ such that, for any edge $\overrightarrow{VW}$, the inequality $c(V) > c(W)$ holds.
- (EE) Any two edges outgoing from any vertex V are equivalent in the sense of the relation generated by elementary equivalences of the form $\overrightarrow{VU} \sim \overrightarrow{VW}$, where the edges $\overrightarrow{VU}$ and $\overrightarrow{VW}$ are chosen so that the vertices U and W have a common root.

Theorem (C. Hog-Angeloni, S.Matveev, Roots of 3-manifolds)

- *Let be an oriented graph possessing properties* (CF) *and* (EE). *Then any vertex has a unique root.*

- This simple Theorem asserts that conditions (CF) and (EE) are sufficient for both the existence and the uniqueness of a root for any vertex of an abstract oriented graph.

Property (CF).

- Let us show that the graph Γ constructed above has property (CF). We apply annular reduction to all knots contained in a vertex V as long as possible. This process is finite, because the total number of successive non trivial reductions which can be applied to a given pair $(F \times I, K)$ is bounded by a constant depending only on this pair. Therefore, the required function $c(V) \to \mathbb{Z}_+$ can be defined as follows: the number $c(V)$ equals the length of the longest chain of nontrivial reductions that can be applied to the union of knots forming the vertex V.

Property (EE).

- Let us show that Γ has property (EE). Suppose that its edges $\overrightarrow{VW_1}$ and $\overrightarrow{VW_2}$ correspond to reductions along annuli A_1 and A_2, respectively.

Case 1.

- Suppose that $A_1 \cap A_2 = \emptyset$. Then, the annulus A_1 survives under the reduction along the annulus A_2. Thus, to one of the knots forming the vertex W_2 we can apply the reduction along the annulus A_1 (if it is nontrivial in W_2). The same result is obtained by applying the reduction along the annulus A_2 to W_1. Therefore, W_1 and W_2 have a common root. If the annulus A_1 is trivial in W_2, then A_1 and A_2 are parallel in V, and the vertices W_1 and W_2 simply coincide.

- Now, we apply decreasing induction on the number $n = \#(A_1 \cap A_2)$ of components in the intersection of the annuli A_1 and A_2. The base of induction is the case $n = 0$ considered above. To perform the inductive step, it suffices to construct an admissible mediator annulus $A_3 \subset F \times I$ for the given annuli $A_1, A_2 \subset F \times I$ such that the numbers $\#(A_1 \cap A_3)$ and $\#(A_2 \cap A_3)$ are strictly less than n. The cases in which there is no such a mediator annulus are very special, and the existence of a common root for the vertices W_1 and W_2 in these cases can be proved directly.

Case 2.

- Suppose that $A_1 \cap A_2$ contains a trivial circle or a trivial arc. Then a mediator annulus is obtained by using the standard procedure of intersection elimination by means of a surgery of one of the annuli along the innermost circle or the outermost arc of the other annulus.
- If the intersection $A_1 \cap A_2$ contains no trivial circles and arcs, then it consists of either nontrivial circles or radial arcs (i.e., arcs joining different circle on the boundaries of the annuli). In the case of nontrivial circles, a mediator annulus is constructed in the same way as above by means of a surgery of one of the annuli A_1 and A_2 along two circles in their intersection which are neighboring with respect to the second annulus.

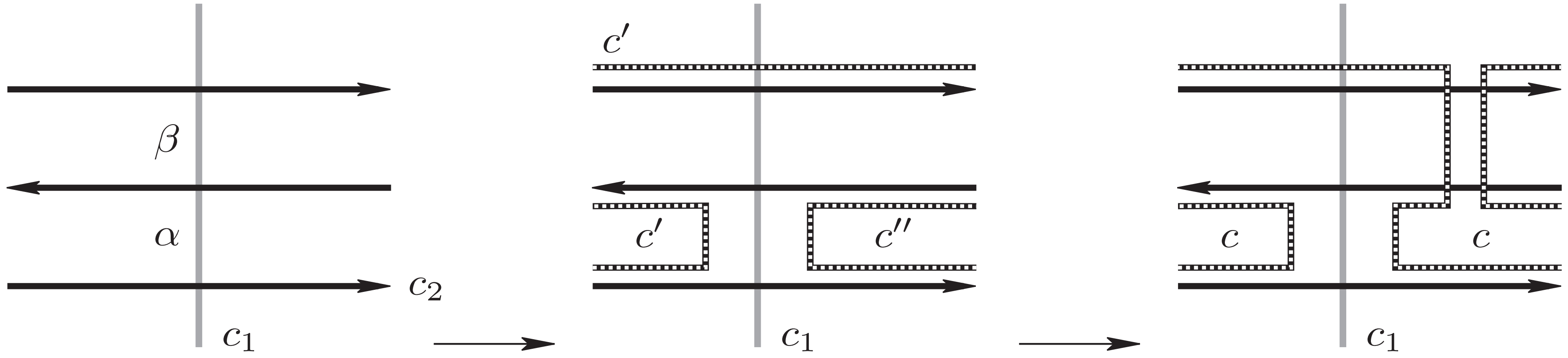
- In order to consider the case of radial arcs, we introduce a surgery of the given one dimensional manifold $C \subset F$ along an arc $\alpha \subset F$ adjacent to C at the endpoints. The surgery consists in cutting the manifold C along $\partial\alpha$ and joining the four endpoints of the obtained arcs by two parallel copies of the arc α.
- Suppose that the intersection $A_1 \cap A_2$ consists of radial arcs. Then the annuli have the form $A_i = C_i \times I$ for $i = 1,2,$ where C_i are separating simple closed curves in F. Therefore, we can forget for a while about the annuli and deal with curves in F; instead of the knot K, we consider its projection K in F. Since the annuli are admissible, it follows that K intersects each curve in precisely two points.

Case 3.

- Suppose that $C_1 \cap C_2$ consists of $k \geq 5$ points. These points divide each circle into k arcs. Since $k \geq 5$, each of the circles (e.g., C_1) contains a pair of neighboring arcs α and β not intersecting the projection of the knot. The surgery of C_2 along the arc α yields the union of two disjoint circles, which we denote by C_2' , and C_2''. The surgery of $C_2' \cup C_2''$ along a parallel copy of the arc β that joins C_2' with C_2'' gives a nontrivial separating circle $C_3 \subset F$ such that $C_3 \cap \overline{K}$ consists of two points, $\#(C_3 \cap C_1) = k - 2$, and $\#(C_3 \cap C_2) = 4$. It follows that the annulus $C_3 \times I$ is a mediator.

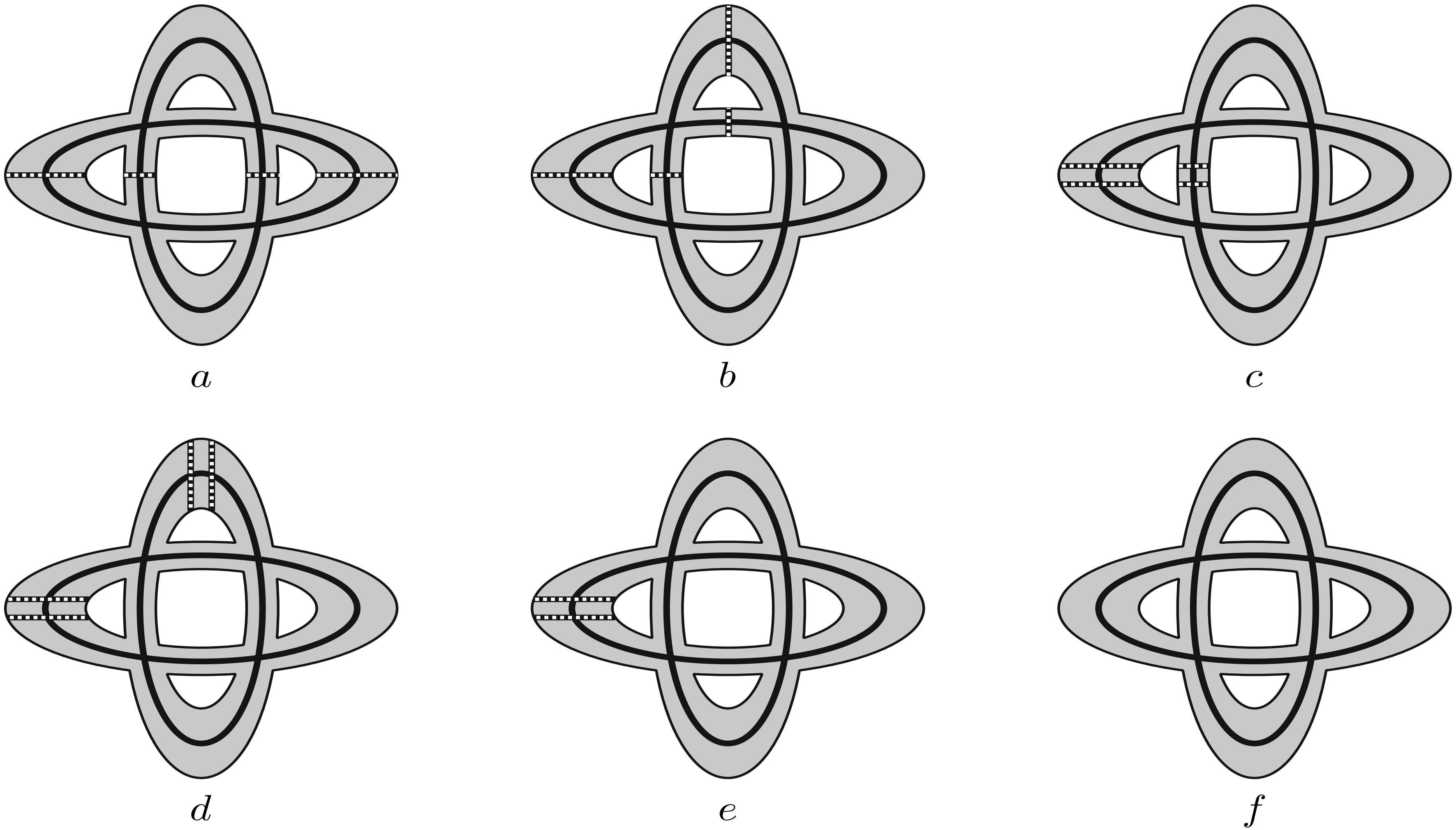
Case 4.

- In the only remaining case, the intersection $C_1 \cap C_2$ consists of two or four points. There are two possibilities:
- (a) $\overline{K}$ intersects precisely one arc of each circle, and each such intersection consists of two points;
- (b) $\overline{K}$ intersects two arcs of every circle, each at one point.

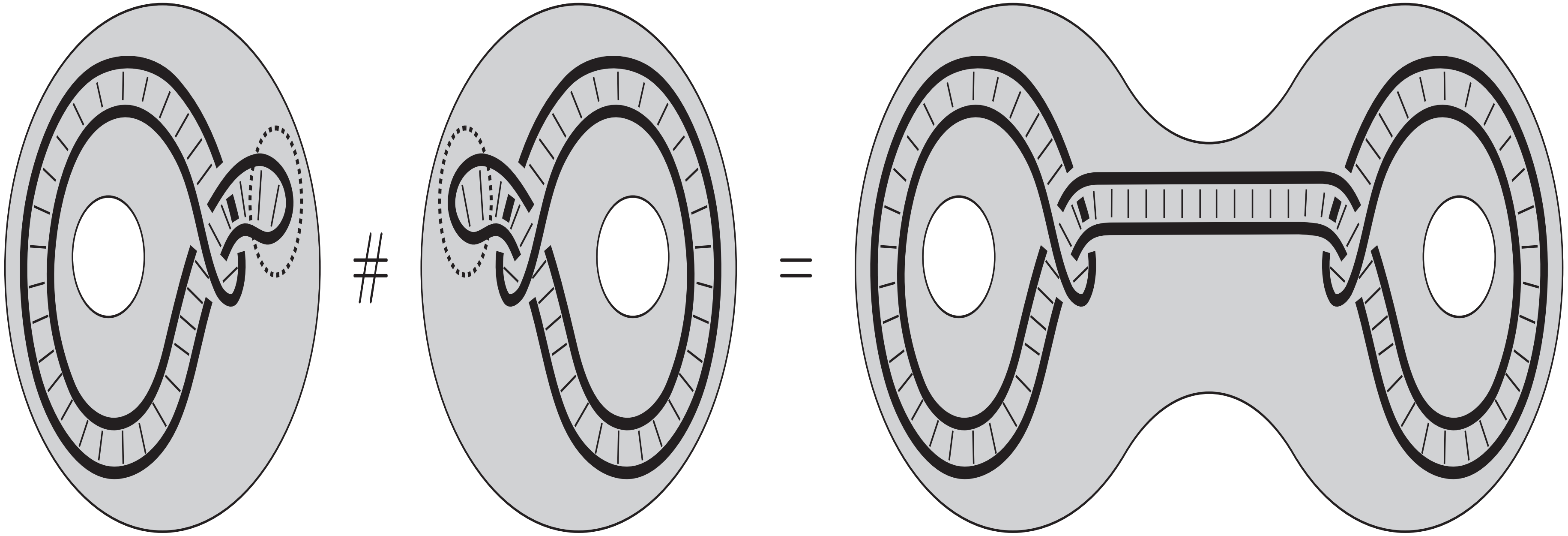


- In any of these cases, we can either find a mediator annulus or (thanks to the simplicity of the situation) directly construct a common root of the vertices W_1 and W_2. Under all other mutual arrangements of the curves C_1 and C_2 and the projection $\overline{K}$, there is a biangle intersecting $\overline{K}$ in an odd number of points. This contradicts the assumption that the knot K is homologically trivial (this assumption is used only at this place).

Possible intersections of a homologically trivial knot with N.

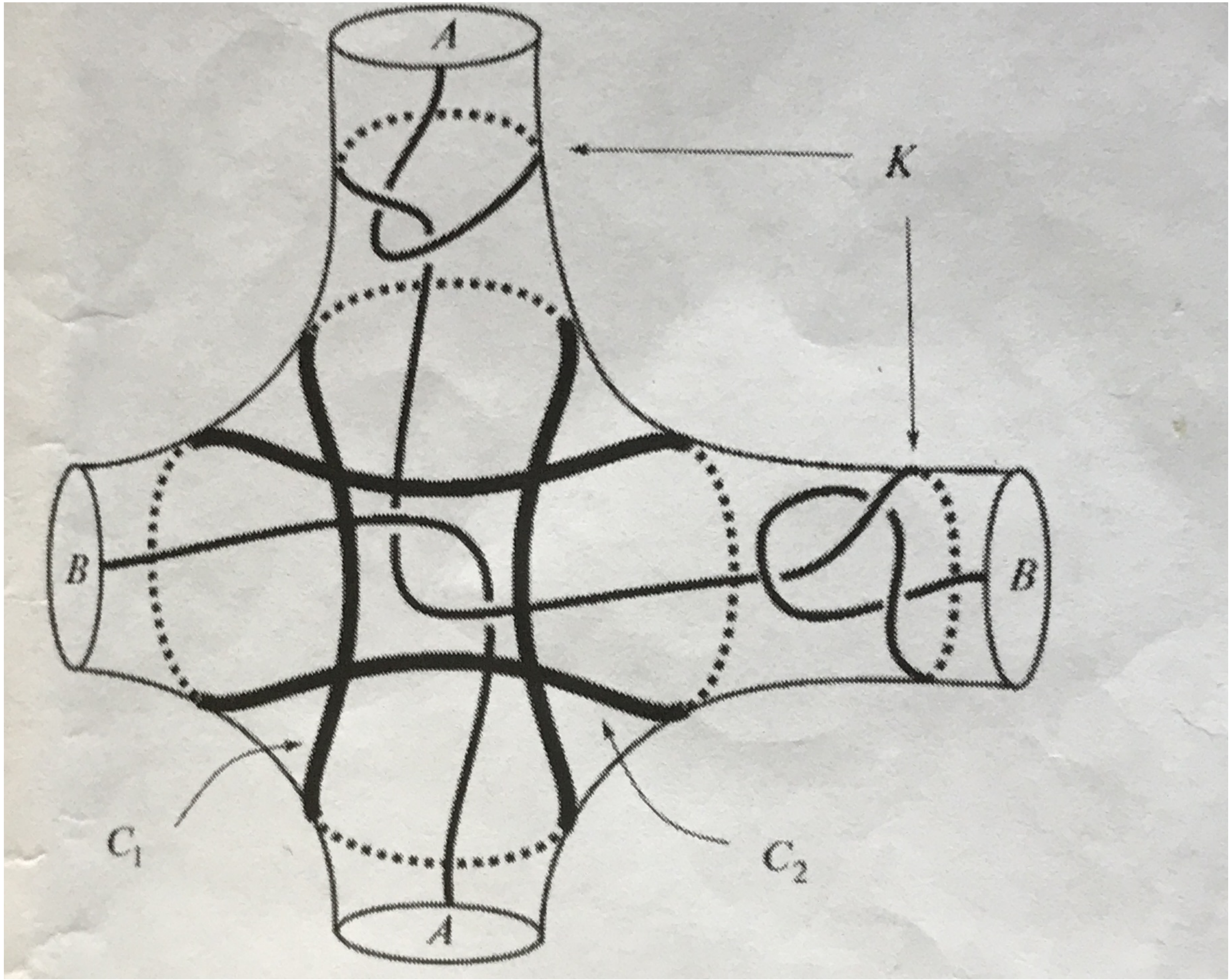


Summation of ribbons in $T \times I$ can produce infinitely many different knots in a thickened surface of genus 2 by means of variation.



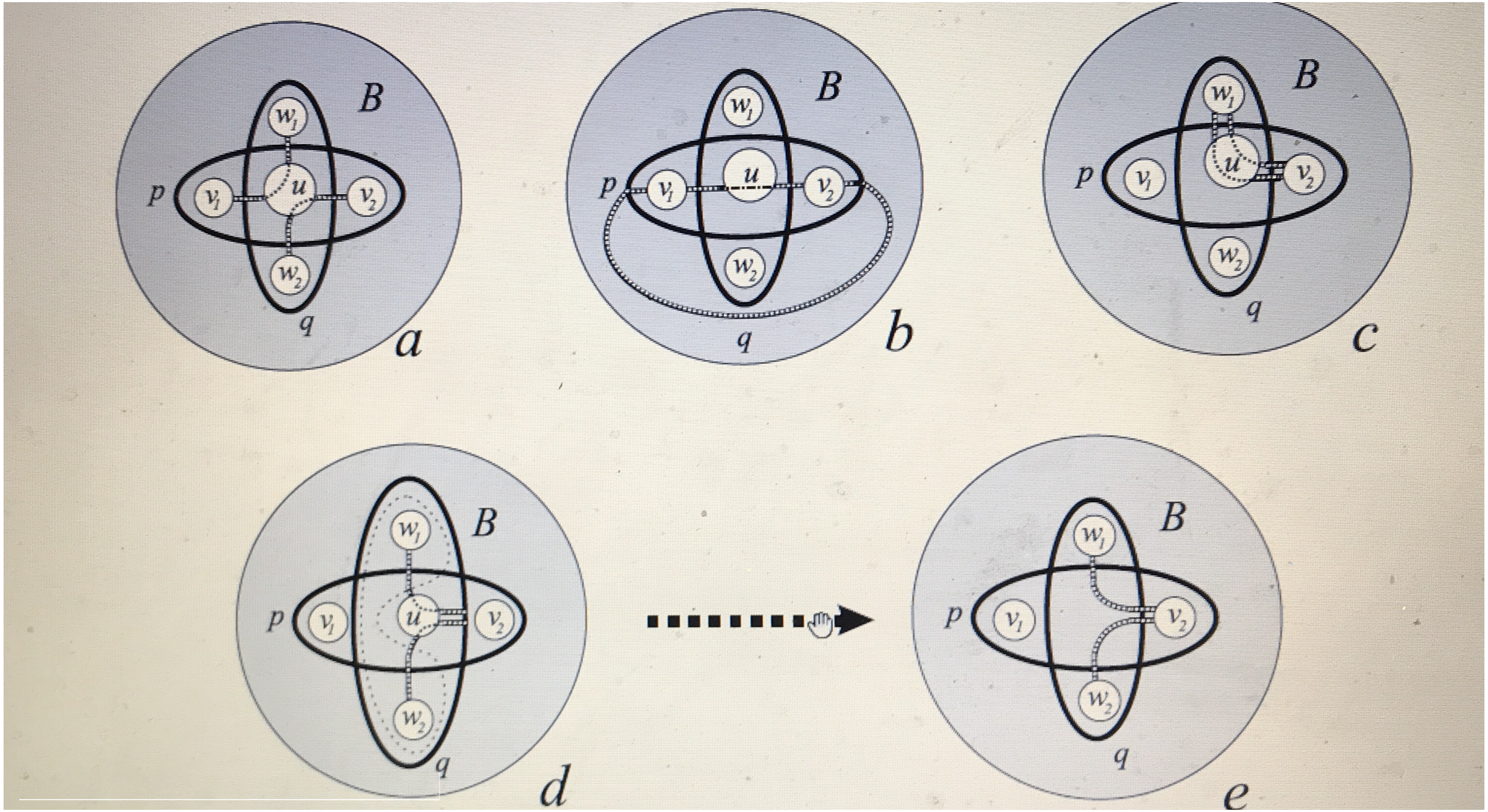
Remark 1.

- The condition of homologous triviality of a node is essential. In next figure shows a diagram of a knot in a thickened surface of genus 2, which $C_1 \times I$ allows two expansions with different terms. Reduction along the ring gives the union of two primary knots in two copies of the standard thickened torus $T^2 \times I$. Reduction along the ring $C_2 \times I$ also gives a pair of primary knots in $T^2 \times I$, but completely different.


A
K
B
B
C
C_1
C_2
A

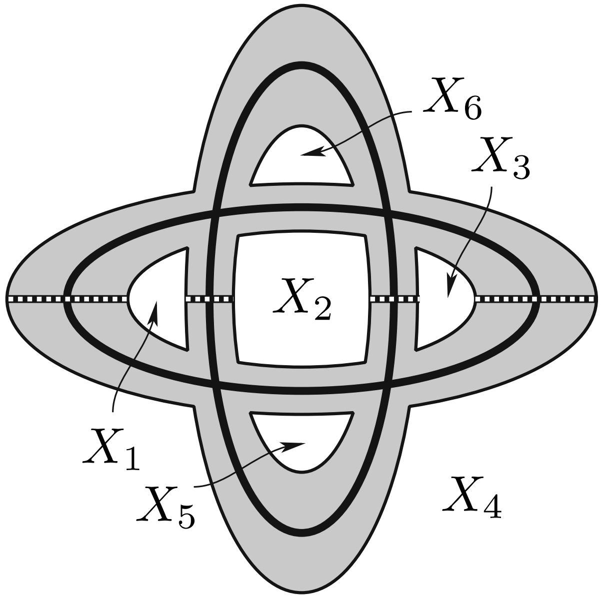
Remark 2.

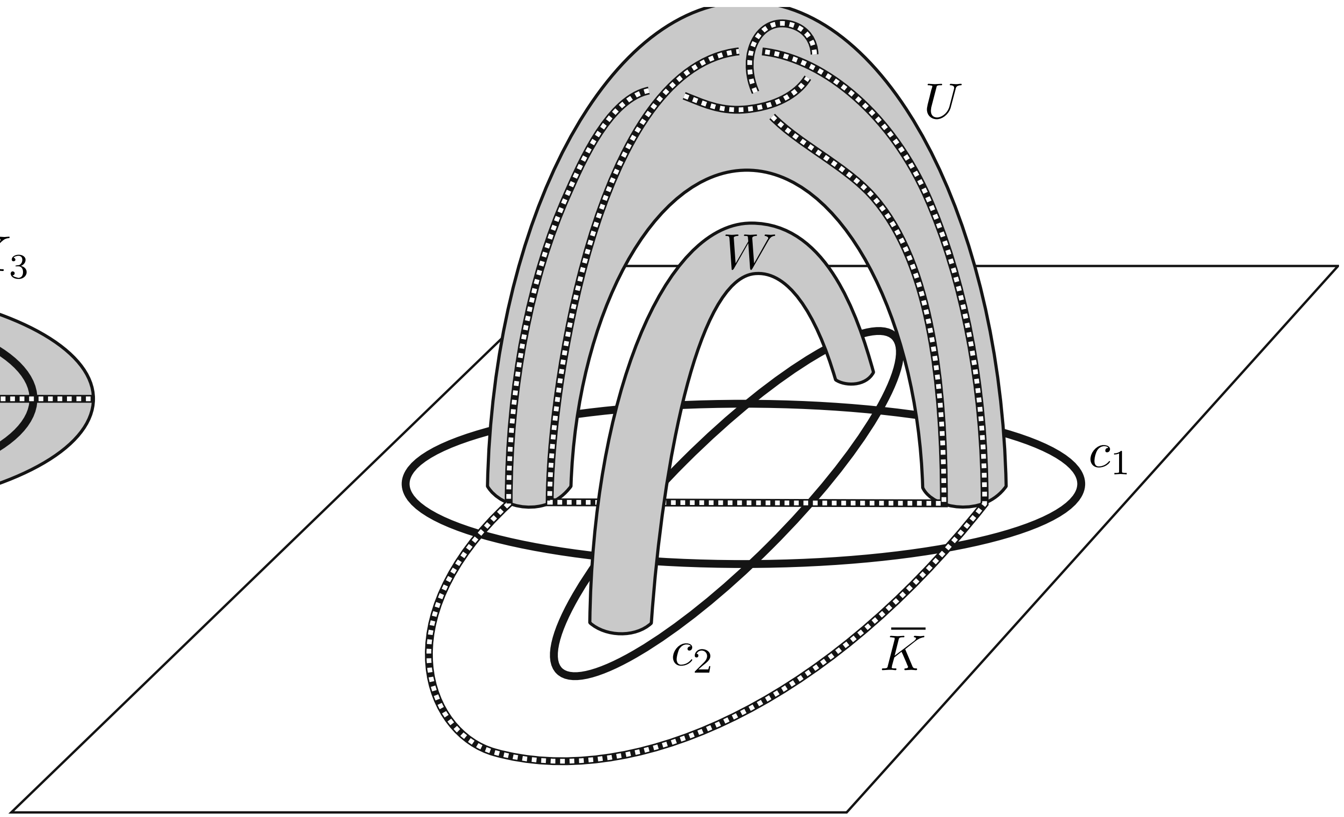
- As in the classical case, the theory of knots in $F \times I$ has several different interpretations. And knots, and surfaces, and ambient manifolds (i.e., direct products of surfaces into segments) can be considered as with orientations, or without.
- In all cases, the theorem on the existence and uniqueness of decomposition of a node into a connected sum of primary terms remains valid (with a corresponding change in the concept of a connected sum).



B
p
q
w1
w2
v1
v2
u
a
b
c
d
e

Reductions along the circles c_1 and c_2 give the same result, regardless of how the knot passes through U.





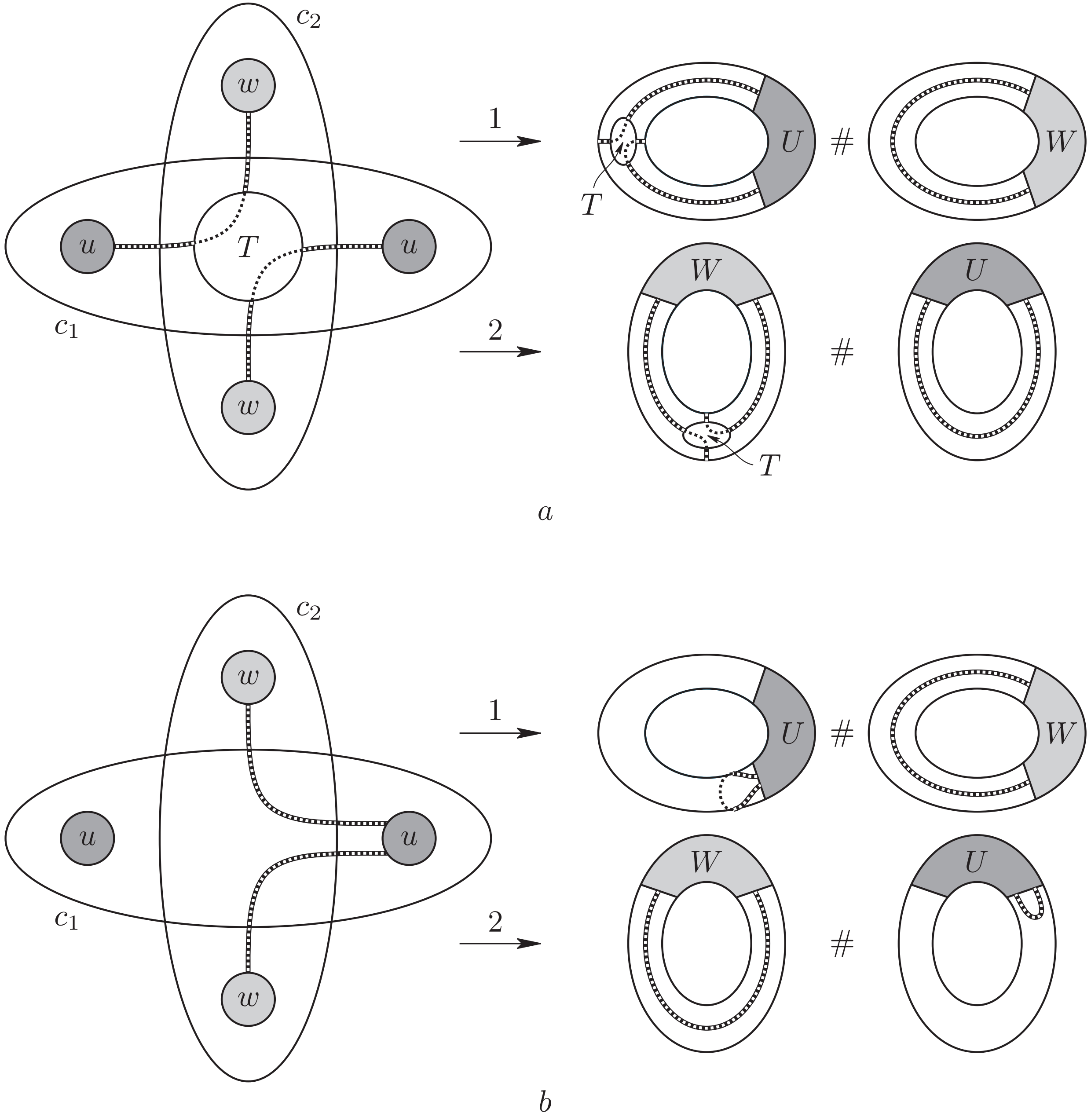
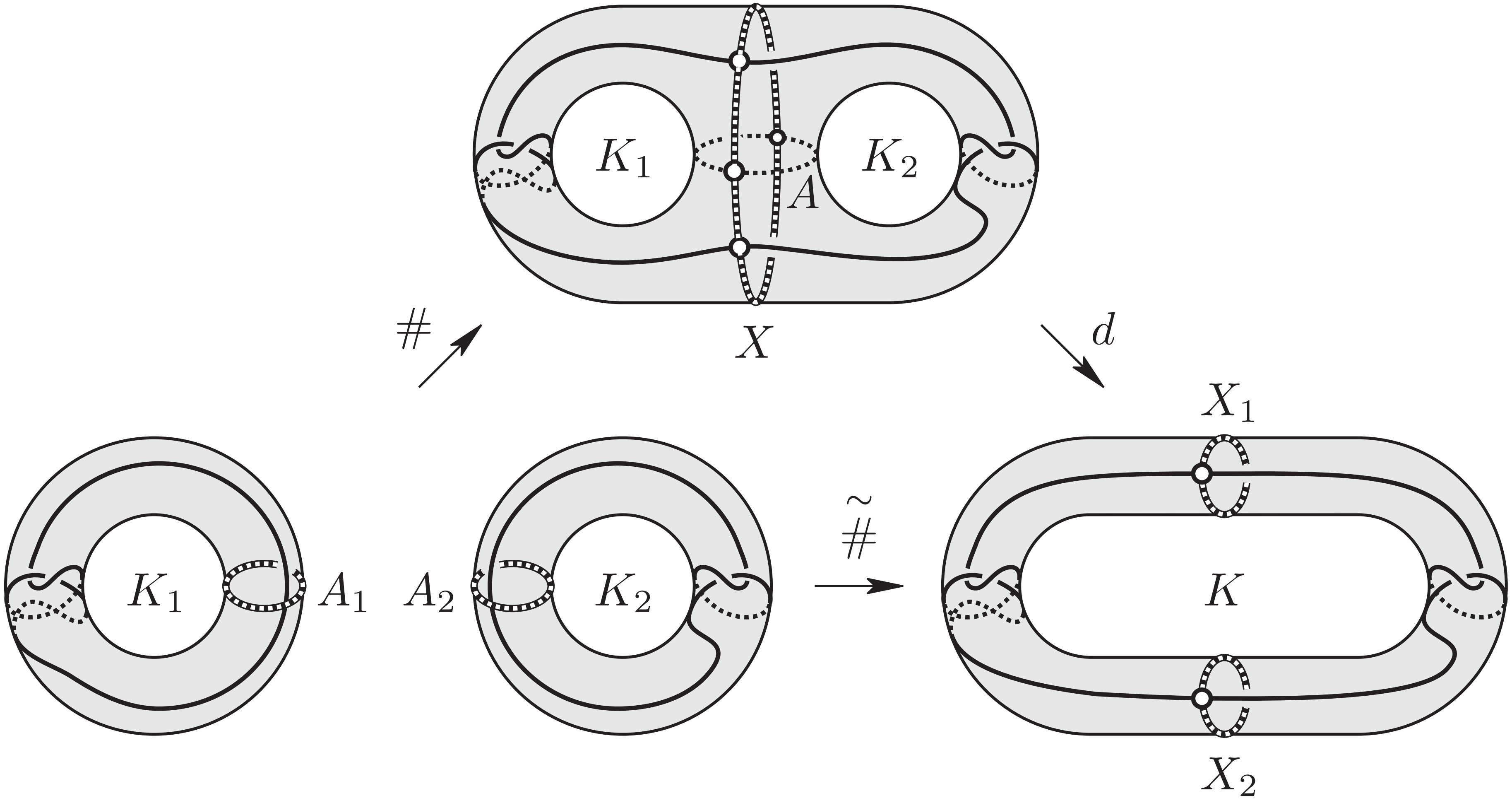


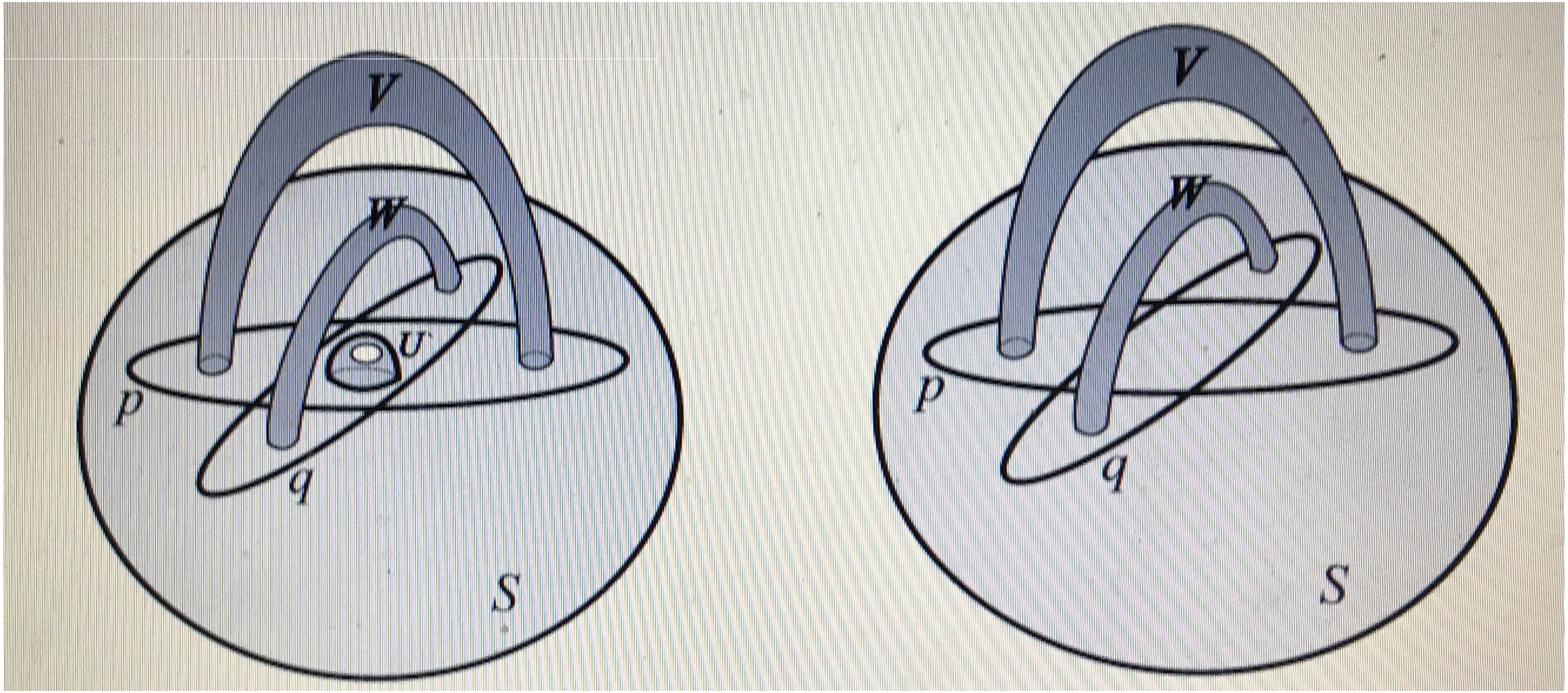
Figure 17. Two types of counterexamples.

END

- Thank you for your attention

A special connected sum # is a superposition of an annular connected sum # and a destabilization d along A.




V
W
U
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q
S
V
W
p
q
S