# Manifolds of triangulations, braids on manifolds, groups $G_n^k$ and $\Gamma_n^k$ , and related groups

#### Vassily Olegovich Manturov

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Joint work with I.M.Nikonov On Groups  $G_n^k$  and  $\Gamma_n^k$ : A Study of Manifolds, Dynamics, and Invariants,arXiv: 1905.04916

Sixth Russian-Chinese conference on knot theory and related topics, Novosibirsk, June 2019.

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If dynamical systems describing a motion of n particles possess a nice codimension 1 property governed by k particles then they have topological invariants valued in groups  $G_n^k$ .

For two integers n > k, we define the group  $G_n^k$  as the group having the following  $\binom{n}{k}$  generators  $a_m$ , where m runs the set of all unordered k-tuples  $m_1, \ldots, m_k$ , whereas each  $m_i$  are pairwise distinct numbers from  $\{1, \ldots, n\}$ .

 $G_n^k = \langle a_m | (1), (2), (3) \rangle.$ 

For each (k + 1)-tuple U of indices  $u_1, \ldots, u_{k+1} \in \{1, \ldots, n\}$ , consider the k + 1 sets  $m^j = U \setminus \{u_j\}, j = 1, \ldots, k + 1$ . With U, we associate the relation

$$a_{m^1} \cdot a_{m^2} \cdots a_{m^{k+1}} = a_{m^{k+1}} \cdots a_{m^2} \cdot a_{m^1}; \qquad (1)$$

for two tuples U and  $\overline{U}$ , which differ by order reversal, we get the same relation.

Thus, we totally have  $\frac{(k+1)!\binom{n}{(k+1)}}{2}$  relations.

We shall call them the *tetrahedron relations*. Physists (tensor version): *Zamolodchikov relations*. For each (k + 1)-tuple U of indices  $u_1, \ldots, u_{k+1} \in \{1, \ldots, n\}$ , consider the k + 1 sets  $m^j = U \setminus \{u_j\}, j = 1, \ldots, k + 1$ . With U, we associate the relation

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# For k-tuples m, m' with $Card(m \cap m') < k - 1$ , consider the far commutativity relation:

$$a_m a_{m'} = a_{m'} a_m. \tag{2}$$

Note that the far commutativity relation can occur only if n > k + 1. Besides that, for all multiindices m, we write down the following relation:

$$a_m^2 = 1. (3)$$

Define  $G_n^k$  as the quotient group of the free group generated by all  $a_m$  for all multiindices m by relations (1), (2) and (3).

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# Dynamics of *n* points on the plane with respect to "three points are collinear" (k = 3) give rise to the map $PB_n \rightarrow G_n^3$ . Dynamics of *n* points on the plane with respect to "three points belong to the same circle/line" (k = 4) give rise to the map $PB_n \rightarrow G_n^4$ . The same works for projective spaces.

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## Projective duality: lines instead of points

Let us first pass from  $\mathbb{R}^{k-1}$  to  $\mathbb{R}P^{k-1}$ . We get:

#### Theorem

There are well defined homomorphisms  $\pi_1(C'_n(\mathbb{R}P^{k-1})) \to G^k_n$ .

Now, by using projective duality, we can consider cases of *moving projective hyperplanes* instead of *moving points*.

For example, for  $\mathbb{R}P^2$  we have moving lines and for "nice codimension one condition" it suffices to have that any two particles (lines) intersect transversally at one point and in the neighbourhood of a triple intersection everything is linear.

This allows one in fact to consider other spaces of curves.

Is it possible to detect any smooth structures on manifolds by using such spaces? Frankly speaking,  $C'_n(M_k)$  are **not** well defined unless we have geodesics, planes, by perplanes.

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Particles should not necessarily be points. When considering a collection of *hyperplanes* in  $\mathbb{R}P^{k-1}$ , where each k-1 are in general position, we get a configuration space whose fundamental group can be studied by methods similar to those described above.

We get a map from the fundamental group of each connected component of the above moduli space to  $G_n^{k+1}$ .

The set of subgroups of  $G_n^{k+1}$  serves as an invariant of the above moduli space.

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Consider a motion of n points in a closed 2-surface. If we want to consider the property "some three points are collinear", we need a Riemannian metric, but

- there is more than one geodesic passing through two points (this may yield the non-existing relation a<sub>123</sub>a<sub>124</sub> = a<sub>124</sub>a<sub>123</sub>);
- there are irrational cables; they destroy the whole situation: there is no chance to write a word since letters become everywhere dense.

A remedy for that is the possibility to consider only local situations when we only count situations where four nearest points belong to the same circle.

#### Definition

The group  $\Gamma_n^4$  is the group given by group presentation generated by  $\{d_{(ijkl)} \mid \{i, j, k, l\} \subset \overline{n}, |\{i, j, k\}| = 4\}$  subject to the following relations:

$$d^2_{(ijkl)} = 1 \text{ for } (i, j, k, l) \subset \overline{n},$$

- **2**  $d_{(ijkl)}d_{(stuv)} = d_{(stuv)}d_{(ijkl)}$ , for  $|\{i, j, k, l\} \cap \{s, t, u, v\}| < 3$ ,
- 3  $d_{(ijkl)}d_{(ijlm)}d_{(iklm)}d_{(iklm)}d_{(iklm)} = 1$  for distinct i, j, k, l, m.
- $d_{(ijkl)} = d_{(kjil)} = d_{(ilkj)} = d_{(klij)} = d_{(jkli)} = d_{(jilk)} = d_{(lkji)} = d_{(lijk)}$ , for distinct i, j, k, l, m.

### The pentagon relation



Figure: The pentagon relation

# A presentation of $\Gamma_n^4$ and braids on 2-surfaces

The main idea for the construction of this map is: *to associate a concrete map to a flip* and then take a composition over all moments when flips occur.



Figure: The flip,  $y = \frac{ac+bd}{x}$ 

## Solutions to the pentagon equation: the Ptolemy equation



Figure: The pentagon equation is satisfied

Figure 3 represents a generic braid isotopic to the identity and the map corresponding to it.

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# Mapping class group and $\Gamma_n^4$

There is a map from the mapping class group to  $\Gamma_n^4$  (joint work with S.Kim and J.Wang).



Figure: A map from Dehn twist to  $\Gamma_{n^{-1}}^4$ 

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Triangulations,  $G_n^k$ , and  $\Gamma_n^k$ 

# The group $\Gamma_n^5$ (I.M.Nikonov)



Figure: A Pachner move and configurations of codimension 2

#### Definition

The group  $\Gamma_n^5$  is the group with generators  $a_{ij,klm}$  and relations

- (index symmetry)  $a_{ij,klm} = a_{ji,klm} = a_{ij,lkm} = a_{ij,kml}$ ,
- (far commutativity)  $a_{ij,klm}a_{i'j',k'l'm'} = a_{i'j',k'l'm'}a_{ij,klm}$ , for  $|\{i,j,k,l,m\} \cap \{i',j',k',l',m'\}| < 4$ ,  $|\{i,j\} \cap \{i',j',k',l',m'\}| < 2$  and  $|\{i',j'\} \cap \{i,j,k,l,m\}| < 2$ ,
- (octahedron relation)  $a_{km,ijp}a_{ij,klm}^{-1}a_{lp,ikm}a_{km,jlp}^{-1}a_{ij,klp}a_{lp,ijm}^{-1} = 1$ ,
- (shifted octahedron relation)  $a_{lp,ijm}a_{kp,ijl}a_{km,jlp} = a_{km,ijl}a_{kp,ijm}a_{lp,ikm}$ .

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# Higher groups $\Gamma_n^k$ (I.M.Nikonov)

The group  $\Gamma_n^k$  is generated by

$$a_{P,Q}, \quad P,Q \in \{1,\ldots,n\}, P \cap Q = \emptyset, |P \cup Q| = k, |P|, |Q| \ge 2,$$

modulo:

- **1**  $a_{Q,P} = a_{P,Q}^{-1};$
- (far commutativity)  $a_{P,Q}a_{P',Q'} = a_{P',Q'}a_{P,Q}$  for each generators  $a_{P,Q}$ ,  $a_{P',Q'}$  such that

$$egin{aligned} |P \cap (P' \cup Q')| &< |P|, & |Q \cap (P' \cup Q')| &< |Q|, \ |P' \cap (P \cup Q)| &< |P'|, & |Q' \cap (P \cup Q)| &< |Q'|; \end{aligned}$$

③ ((k + 1)-gon relations) for any standard Gale diagram  $\bar{Y}$  of order k + 1 and any subset  $M = \{m_1, \ldots, m_{k+1}\} \subset \{1, \ldots, n\}$ 

$$\prod_{i=1}^{k+1} a_{M_R(\bar{Y},i),M_L(\bar{Y},i)} = 1$$

where  $M_R(\bar{Y}, i) = \{m_j\}_{j \in R_{\bar{Y}}(i)}, M_L(\bar{Y}, i) = \{m_j\}_{j \in L_{\bar{Y}}(i)}.$ 

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# The groups $\Gamma_n^k$ have non-trivial abelianisations which allow one to construct non-trivial invariants of paths in the configuration space.

One can pass to even stronger groups  $\tilde{\Gamma}_n^k$  by taking into account the orientation. Hence, for  $\tilde{\Gamma}_n^5$  there are twice as many generators as in  $\tilde{\Gamma}_n^4$ :  $a_{ijk}^{lm} = a_{ikj}^{ml} \neq a_{ikj}^{lm} = a_{ijk}^{ml}$ . The groups  $\Gamma_n^k$  have non-trivial abelianisations which allow one to construct non-trivial invariants of paths in the configuration space.

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- Vertices: triangulations;
- 2 Edges: Pachner moves;
- **③** 2-cells: Relations (far commutativity and inscribed polytopes with d + 3 vertices).

#### Definition

The *combinatorial n-strand braid groups* of the manifold M are the fundamental groups

$$B_c(M,n)_j = \pi_1((M_{comb}^{dn})_j), \quad j = 1, \ldots, q.$$

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## The manifold of triangulations: smooth approach

Fix a smooth manifold  $M^d$  with a Riemannian metric g. Let  $n \in \mathbb{N}, n \gg d$ . Consider the set of all Delaunay triangulations of M with n vertices  $x_1, \ldots, x_n$  (if exist). They are indexed by sets of vertices  $x_1, \ldots, x_n$ , hence, the set of such triangulation is a subset of C(M, n). It is clear that the above subset will be an open (not necessarily connected) manifold of dimension dn. The set  $x_1, \ldots, x_n$  if admissible if such a Delaunay triangulation exists. We get a non-compact (open) manifold  $M_{g}^{dn}$ . Here  $(x_1, \ldots, x_n)$  and  $(x'_1, \ldots, x'_n)$  are *adjacent* if there is a path  $(x_{1,t},\ldots,x_{n,t})$  for  $t \in [0,1]$ , s.t.  $(x_{1,0},\ldots,x_{n,0}) = (x_1,\cdots,x_n)$  and  $(x_{1,1}, \ldots, x_{n,1}) = (x'_1, \ldots, x'_n)$  and  $\exists ! t_0 \in [0, 1]$ , s.t. when passing through  $t = t_0$  the Delaunay triangulation  $(x_{1,t}, \ldots, x_{n,t})$  undergoes a flip (or Pachner move or bistellar move). Generally, the Pachner moves, not changing the number of points, may be not enough. In that case, we shall have many components and the invariant (valued  $\Gamma_n^k$ ) is be a set of groups. In dimension 3 the Pachner moves are enough. The flip corresponds to a position when some d + 2 points  $(x_{i_1,t_0}, \ldots, x_{i_{d+2},t_0})$  belong to the same sphere  $S^{d-1}$  such that no other point  $x_i$  lies inside the ball  $B, \partial B = S^{d-1}$ . Vassily Olegovich Manturov (BMSTU) Triangulations,  $G_n^k$ , and  $\Gamma_n^k$ VI Russian-Chinese Conference 23 / 30

Consider the manifold  $M^d$ . We consider all possible metrics  $g_\alpha$  as above. They naturally lead to  $M_{g_\alpha}^{dn}$  as described in the previous section. Those manifolds are naturally stratified. By a *generalised cell* of such a stratification we mean a connected component of the set of generic points of  $M_{g_\alpha}^{dn}$ .

We say that two generalised cells  $C_1$  and  $C_2$  are *adjacent* it there exist two points, say,  $x = (x_1, \ldots, x_n)$  in  $C_1$  and  $x' = (x'_1, \ldots, x'_n)$  in  $C_2$  and a path  $x_t = (x_1(t), \ldots, x_n(t))$ , such that  $x_i(0) = x_i$  and  $x_i(1) = x'(i)$  such that all points on this path are generic except for exactly one point, say, corresponding to  $t = t_0$ , which belongs to the stratum of codimension 1. We say that two generic strata of  $M_{g_{\alpha}}^{dn}, M_{g_{\beta}}^{dn}$  are *equivalent* if there is a homeomorphism of  $M_{g_{\alpha}}^{dn} \to M_{g_{\beta}}^{dn}$  taking one stratum to the other.

## The manifold of triangulations: topological approach

A 0-cell of the manifold of triangulations is an equivalence class of generic strata.

Analogously, we define 1-cells of the manifold of triangulations as equivalence classes of pairs of adjacent vertices for different metrics  $M_{g_{\alpha}}^{dn}$ ,  $M_{g_{\beta}}^{dn}$  to the pair of adjacent vertices equivalent to the initial ones. In a similar manner, we define 2-cells as equivalence classes of discs for metrics  $M_{g_{\alpha}}^{dn}$  such that:

- vertices of the disc are points in 0-strata;
- edges of the disc connect vertices from adjacent 0-strata; each edge intersects codimension 1 set exactly in one point;
- the cycle is spanned by a disc which intersects codimension 2 set exactly at one point;
- equivalence is defined by homeomorphism taking disc to disc, edge to an equivalent edge and vertex to an equivalent vertex and respects the stratification.

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#### Definition

The topological n-strand braid groups of the manifold M are the fundamental groups

$$B_t(M, n)_j = \pi_1((M_{top}^{dn})_j), \quad j = 1, \dots, q.$$

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In a similar way, one can define braid groups for any class of metrics for a given manifold: complex braid groups, simplicial braid groups, Kähler braid groups , etc.

To this end, one needs to consider all possible metrics from the given class and Delaunay triangulations with the given number of points.

#### Theorem

• The groups  $B_t(M, n)_i$  are invariants of the smooth type of  $M^n$ .

2 The groups  $B_c(M, n)_i$  are invariant of the PL-type of  $M^n$ .

### Associahedra

Let  $\mathcal{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ , n > d, be the vertex set of a polytope W in general position, namely, any d + 1 points of  $\mathcal{X}$  do not belong to one hyperplane. Let  $\mathcal{T}$  be the set of regular triangulations of W. We can consider  $\mathcal{T}$  as a graph whose vertices are regular triangulations and whose edges are Pachner moves. This graph is connected. Choose a regular triangulation  $T_0 \in \mathcal{T}$ . For any other triangulation  $\mathcal{T} \in \mathcal{T}$  there is a path  $\gamma = e_1 e_2 \ldots e_l$  from  $T_0$  to  $\mathcal{T}$  where each edge  $e_i$  is a Pachner move of type  $(P_i, Q_i)$ . We assign the word  $\varphi(\mathcal{T}) = \prod_{i=1}^l a_{P_i, Q_i} \in \Gamma_n^{d+2}$  to the triangulation  $\mathcal{T}$ . Let  $Cay(\Gamma_n^{d+2})$  be the Cayley graph of the standard presentation of  $\Gamma_n^{d+2}$ .

#### Theorem

The correspondence  $T \mapsto \varphi(T)$  defines an embedding of the graph  $\mathcal{T}$  into the graph  $Cay(\Gamma_n^{d+2})$ .

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## Some unsolved problems and work and progress:

- Solution to word problem for  $G_n^k$  except k = 2, n = k + 1, (n, k) = (5, 3) (the latter one — L.A.Bokut').
- **2** Realisation of groups  $\Gamma_n^k$  as fundamental groups.
- Solution of an invariant of knots valued in  $\Gamma_n^k$  by means of triangulations of space and Pachner moves (project with S.Kim and S.Yoon).
- Relations between groups  $G_n^k$  and  $\Gamma_n^k$ .
- Study the manifold of far triangulations. It is well known that Pentagon+Hexagon → braiding. In our case pentagon only yields braiding.
- **O** How to use Drinfeld's associator to get presentations of Γ?
- **②** Relation of the braid groups  $B_c(M, n)_j$ ,  $B_t(M, n)_j$  to Dijkgraaf-Witten invariants.
- One of the state of the stat

https://www.researchgate.net/project/Invariants-and-Pictures

Go to Kim Seongjeong's talk.

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Triangulations,  $G_n^k$ , and  $\Gamma_n^k$