VI Russian-Chinese Conference on Knot Theory and Related Topics

Ribbon Surface-Link and Stable-Ribbon Surface-Link

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This talk is an explanation of the following papers in ongoing project:

[K1] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link.

[K2] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, II. General case.

[K3] A. Kawauchi, Triviality of a surface-link with meridian-based free fundamental group.

http://www.sci.osaka-cu.ac.jp/kawauchi/

Plan of this talk:

1. **A ribbon surface-link**
2. **A stable surface-link and handles**
3. **Uniqueness of an orthogonal 2-handle pair**
4. **Main result: Cancelling the stableness**

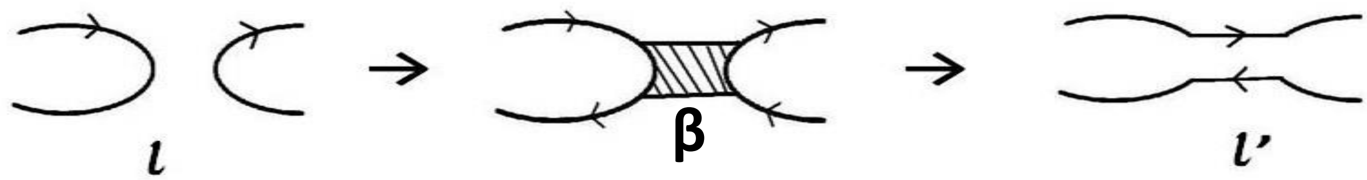
1. A ribbon surface-link

[KSSI1982] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space, I : Normal forms, Math. Sem. Notes, Kobe Univ. 10 (1982),75-125.

A ***surface-link*** is a closed oriented (possibly disconnected) surface F smoothly embedded in the 4-space $R^4 = \{(x,t) \mid x \in R^3, t \in R\}$.

A surface-knot F is ***equivalent*** to a surface-knot F', which is denoted by $F \cong F'$, if ∃ an equivalence (i.e., an orientation-preserving diffeomorphism $f: R^4 \to R^4$ sending F to F' orientation-preservingly).

A band surgery on a link : $l \to l'$ in R^3



For a subset $A \subset R^3$ and an interval [a,b], use the notation $A[a,b]=\{(x,t)\,|\, x \in A,\ t \in [a,b]\}$.

The realizing surface of a band surgery $l \to l'$ by a system β of bands $\beta_1,\ldots,\beta_m$ is a surface F_a^b in $R^3[a,b]$ defined by:

$$F_a^b \cap R^3[t] = \begin{cases} l'[t], & (a+b)/2 < t \leqq b \\ (l \cup \beta)[t], & t=(a+b)/2 \\ l[t], & a \leqq t < (a+b)/2. \end{cases}$$

Let $F_a^b = F_{a_0}^{a_1} \cup F_{a_1}^{a_2} \cup \ldots \cup F_{a_{m-1}}^{a_m}$, $a = a_0 < a_1 < \ldots < a_m = b$, be the realizing surface of a band surgery sequence $l_0 \to l_1 \to \ldots \to l_{m-1} \to l_m$ in R^3.

Assume $l_0 = o_0$ and $l_m = o_m$ are trivial links with d' and d any bounding disk systems, respectively. The ***closed realizing surface*** in $R^3[a,b]$ of $o_0 \to l_1 \to \ldots \to l_{m-1} \to o_m$ in R^3 is:

$$\bar{F}_a^b = d'[a] \cup F_a^b \cup d\,[b].$$

Lemma 1.1 ([KSSI1982]). ∀ surface-link F is equivalent to the closed realizing surface $\bar{F}^b_a$ of a band surgery sequence o' → o. Further, the equivalence class of $\bar{F}^b_a$ is independent on choices of the disks d' and d.

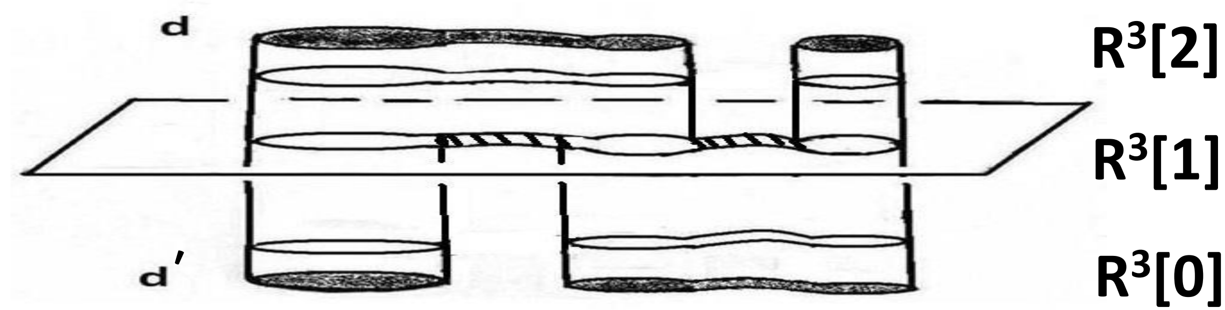


Figure: $\bar{F}^2_0$

A surface-link F in R^4 is ***ribbon*** if $F = \bar{F}_a^b$ for a band surgery sequence o → l → o with o a trivial link and the band surgery l → o is the inverse of o → l.

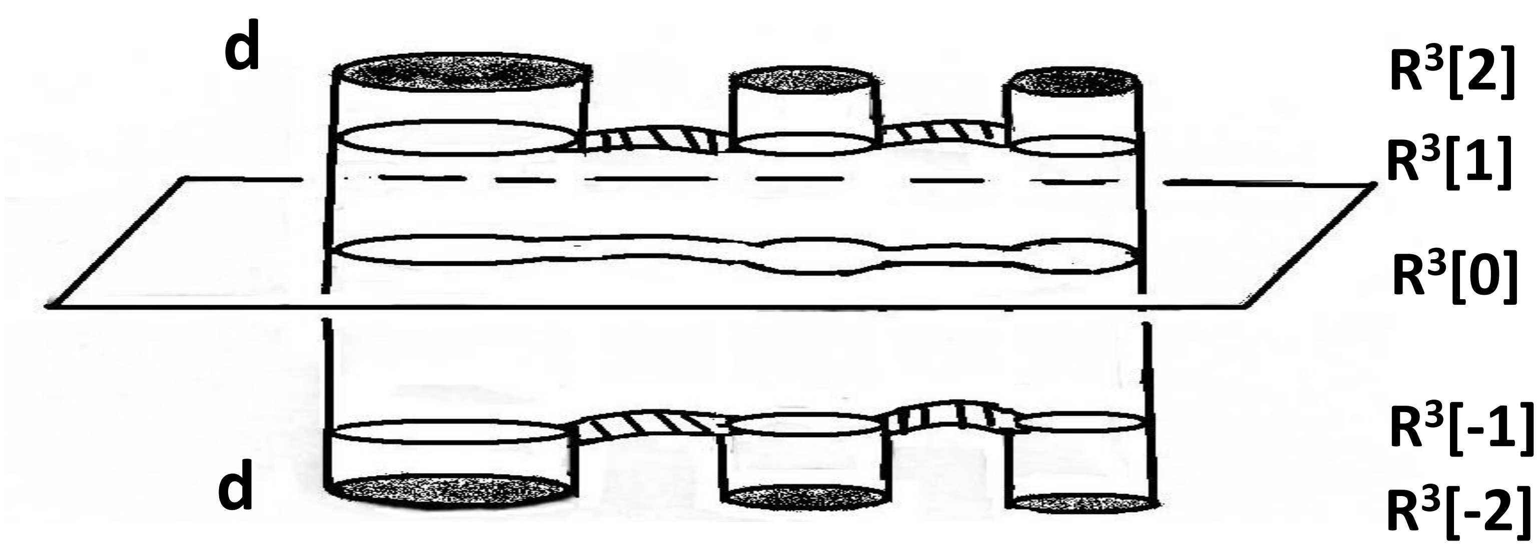


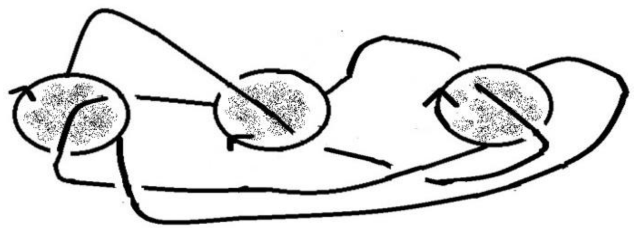
Figure: $\bar{F}_{-2}^{2}$

Observation. The ribbon surface-link $\bar{F}_a^b$ is given by the surgery of the trivial S^2-link $O=\partial(d[a,b])$ in R^4 along embedded 1-handles $N(\alpha)=\beta[a',b']$ with $a<a'<b'<b$, where d is a disk system with $\partial d=o$ and α is the centerline system of the band system β.

By [HK1979] , the equivalence class of $\bar{F}_a^b$ is determine by the core arcs α of the 1-handles $N(\alpha)$ and independent of framings of α.

[HK1979] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-space, Osaka J. Math. 16 (1979), 233-248.

<u>Theorem</u> ([K2015, K2017, K2018]): **Every ribbon surface-link is identified with a chord diagram consisting of a based loop system and a chord system modulo the moves M0, M1, M2.**



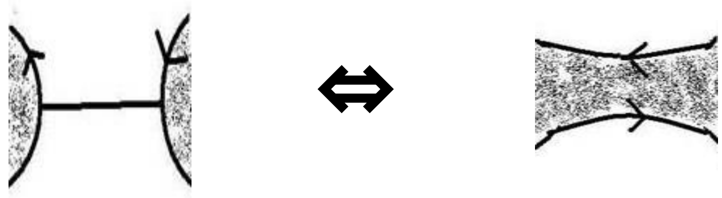
[K2015] A. Kawauchi, A chord diagram of a ribbon surface-link, JKTR 24 (2015), 1540002 (24pp.).

[K2017] A. Kawauchi, Supplement to a chord diagram of a ribbon surface-link, JKTR 26 (2017), 1750033 (5pp.).

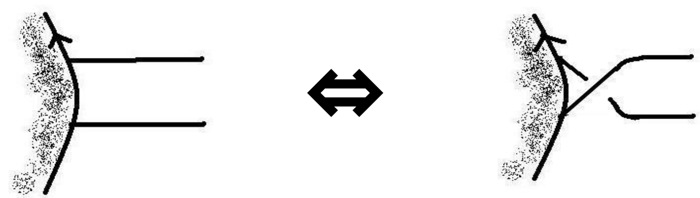
[K2018] A. Kawauchi, Faithful equivalence of equivalent ribbon surface-links, JKTR 27 (2018), 1843003 (23pp.).

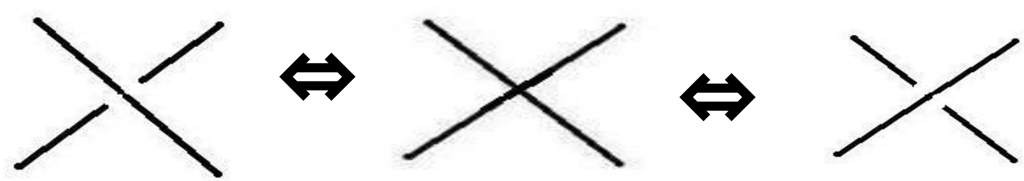
The move M0: Reidemeister-moves.

The move M1: Fusion-fission move.



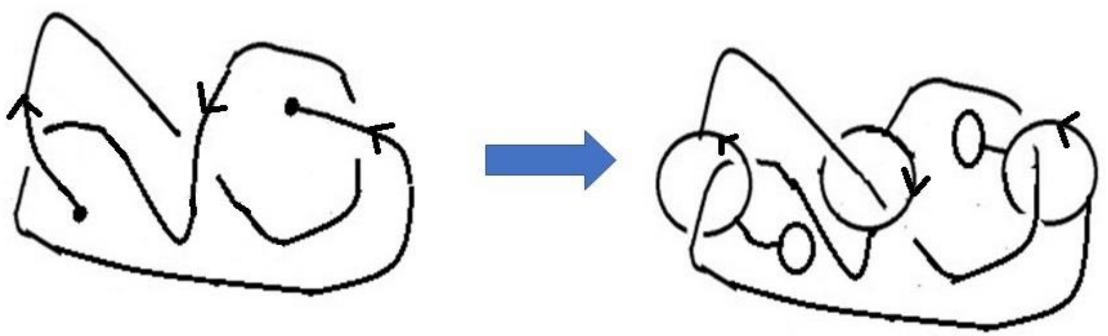
The move M2: Chord moves.





<u>Note:</u> **∃ canonical maps inducing the same groups:**

(a virtual knot) → (a chord diagram)/(M0,M1,M2),
(a knotoid) → (a chord diagram))/(M0,M1,M2).



2. A stable surface-link and handles

A surface-link F is *trivial* (or *unknotted and unlinked*) if ∃ mutually disjoint handlebodies V in R^4 such that ∂V=F.

A *stabilization* of a surface-link F is a connected sum $\overline{F} = F \# T_1 \# T_2 \#...\# T_m$ for trivial torus-knots T_i (i=1,2,…,m) for some m.

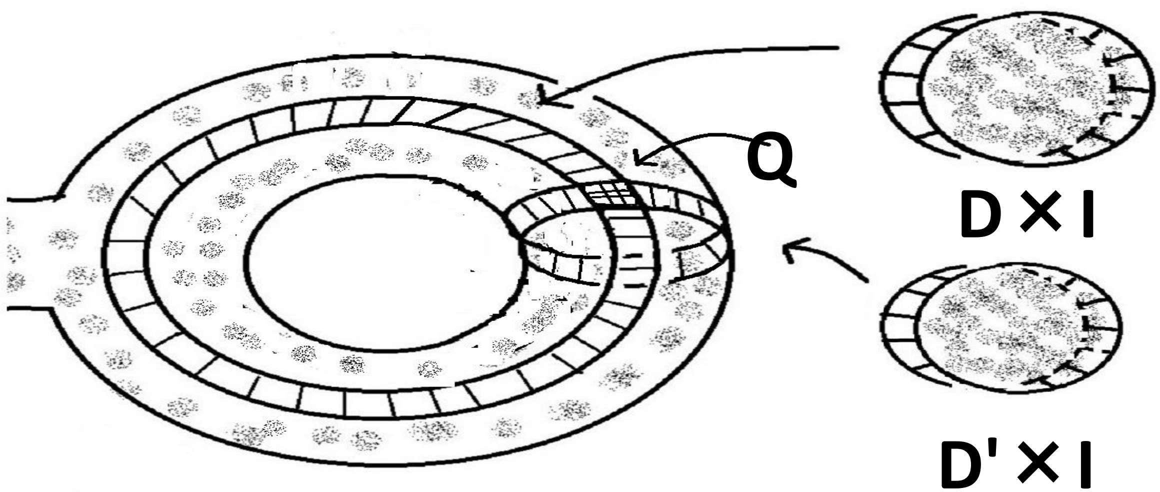
A *handle-irreducible summand* of a surface-link F is a surface-link F* of minimal total genus such that a stabilization $\overline{F}^*$ is equivalent to F.

A 1-*handle* on a surface-link F in R^4 is an embedded 1-handle $I \times D$ on F with $I \times D \cap F = (\partial I) \times D$.
A 2-*handle* on a surface-link F in R^4 is an embedded 2-handle $D \times I$ on F with $D \times I \cap F = (\partial D) \times I$.

Note ([HK1979]). ∀surface-link F is obtained from a trivial surface-knot by the surgery along finitely many disjoint 2-handles.
(∵) A handlebody is obtained from ∀connected Seifert hypersurface V for F by the surgery along mutually disjoint 1-handles in V.

An ***orthogonal 2-handle pair (:= O2-handle pair)*** is a 2-handle pair $(D\times I, D'\times I)$ on a surface-link F such that the core disks D and D' meet transversely at just one point p in F with

$$D\times I\cap D'\times I = (\partial D)\times I\cap (\partial D')\times I = p\times I\times I =: Q.$$



An orthogonal 2-handle pair (:= O2-handle pair)

Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F.

Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively.

Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the once-punctured surface

$$F^c = cl(F-((\partial D) \times I \cup (\partial D') \times I))$$

and the plumbed disk $\delta = D \times (\partial I) \cup Q \cup D' \times (\partial I)$.

Lemma 2.1 ([K1]). For any O2-handle pair $(D \times I, D' \times I)$ on $\forall$ surface-link F,

$$F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I).$$

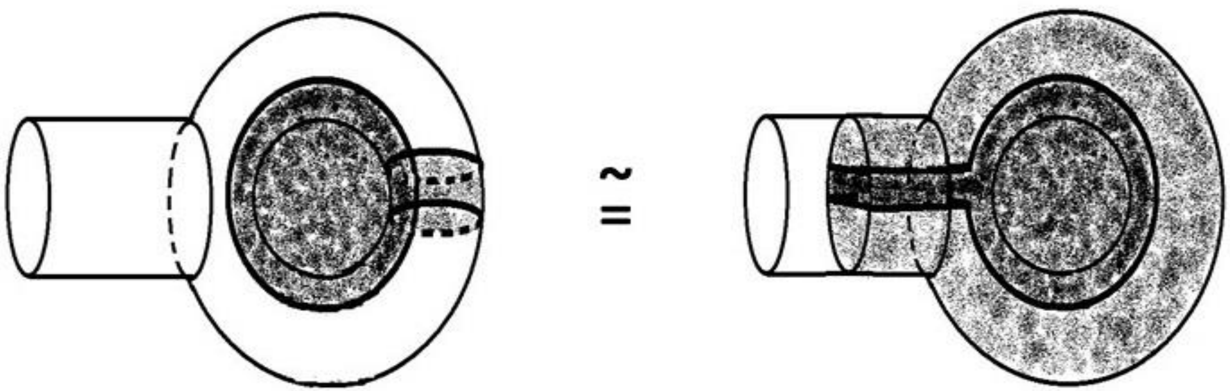
A compact once-punctured torus T^0 in a 3-ball B is *trivial* if T^0 is smoothly and properly embedded in B and $\exists$ a solid torus V in B with $\partial V = T^0 \cup \delta$ for a disk δ in ∂B.

A *bump* B of a surface-link F is a 3-ball in R^4 such that $F \cap B = T^0$ in B.

Let F(B) be a surface-link $cl(F - T^0) \cup \delta$ for a disk δ in ∂B with $\partial\delta = \partial T^0$.

Lemma 2.2 ([K1]). (1) A bump B is obtained uniquely from $\forall$ given O2-handle pair $(D\times I, D'\times I)$ on a surface-link F with $F(B) = F(D\times I, D'\times I)$.

(2) An O2-handle unordered pair $(D\times I, D'\times I)$ is uniquely obtained from $\forall$ given bump B on a surface-knot F in R^4 with $F(D\times I, D'\times I) = F(B)$.



$$\Delta = D \times I \cup D' \times I$$

Lemma 2.3 ([K2]). **For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F and a trivial torus-knot T with a spin loop basis (e, e'), $\exists$ an equivalence $f: R^4 \to R^4$ from F to $F(D \times I, D' \times I) \# T$ such that**

$$f(\partial D, \partial D') = (e, e').$$

Lemma 2.4 ([K1]). Let ($D \times I$, $D' \times I$) be an O2-handle pair on a surface-link F such that D is an immersed disk. Then there is an embedded 2-handle $D^* \times I$ on F with $(\partial D^*) \times I = (\partial D) \times I$ such that ($D^* \times I$, $D' \times I$) is an O2-handle pair on F.

3. Uniqueness of an orthogonal 2-handle pair

A surface-link F has ***only unique O2-handle pair in the rigid sense*** if for ∀ O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F with $(\partial D) \times I = (\partial E) \times I$ and $(\partial D') \times I = (\partial E') \times I$, ∃ an equivalence $f: R^4 \to R^4$ keeping F^c fixed such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$.

A surface-link F has ***only unique O2-handle pair in the soft sense*** if for ∀ O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F attached to the same component, ∃ an equivalence $f: R^4 \to R^4$ from $F(D \times I, D' \times I)$ to $F(E \times I, E' \times I)$.

<u>Theorem 3.1 ([K1])</u>.

For ∀ O2-handle pairs ($D\times I$, $D'\times I$) and ($E\times I$, $E'\times I$) on F with $(\partial D)\times I = (\partial E)\times I$ and $(\partial D')\times I = (\partial E')\times I$, ∃ an ambient isotopy $f_t : R^4 \to R^4$ ($t\in[0,1]$) keeping F^c fixed such that $f_0 = 1$ and

$$f_1(D\times I)= E\times I \text{ and } f_1(D'\times I)= E'\times I.$$

Thus, ∀ surface-link has only unique O2-handle pair in the rigid sense.

<u>Theorem 3.3 ([K2])</u>. **∀surface-link has only unique O2-handle pair in the soft sense.**

4. Main result: Cancelling the stableness

The following is a characterization of a ribbon surface-link:

Lemma 4.1 ([K2]). A surface-link F is ribbon if and only if ∃ a punctured handlebody system V in R^4 such that $\partial V = F \cup O$ for a trivial S^2-link O with $F \cap O = \emptyset$.

Call V a *semi-unknotted punctured handlebody system (=: a SUPH system)* for F.

Lemma 4.2([K2]). **The following (1) and (2) hold.**

(1) ∀stable-ribbon surface-link is ribbon.

(2) If F is a ribbon surface-link with an O2-handle pair $(D\times I, D'\times I)$ on F, then $F(D\times I, D'\times I)$ is a ribbon surface-link.

Theorem 3.3 (on uniqueness of the O2-handle pair in the soft sense) and Lemma 4.2 imply:

Theorem 4.3 ([K2]). A handle-irreducible summand F* of every surface-link F is a surface-link which is determined uniquely from F up to equivalences. Further, if F is stable-ribbon, then F* is ribbon.

Corollary 4.4 ([K2]). **If a connected sum F#F' of surface-links F and F' is a ribbon surface-link, then F and F' are ribbon surface-links.**

Note. **Every 1-knot K is a connected summand of a ribbon 1-knot.**

Stabilized Triviality.

(1)([HK1979]) If a surface-knot F has $\pi_1(R^4\text{-}F) \cong Z$, then $\exists$ a stabilization $\overline{F}$ of F is a trivial surface-knot.

(2)([K3]) If a surface-link F has a meridian-based free $\pi_1(R^4\text{-}F)$, then $\exists$ a stabilization $\overline{F}$ of F is a trivial surface-link.

By Stabilized Triviality and Theorem 4.3, we have:

Corollary 4.5.

(1)([K1]) If a surface-knot F has $\pi_1(R^4\text{-}F) \cong Z$, then F is a trivial surface-knot.

(2)([K3]) If a surface-link F has a meridian-based free $\pi_1(R^4\text{-}F)$, then F is a trivial surface-link.