Some remarks on the chord index

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A brief review on virtual knot theory

Classical knot theory: Knot types={knot diagrams}/{Reidemeister moves}

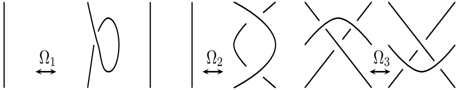

Figure 1: Reidemeister moves

Virtual knot theory: Besides over crossing and under crossing, we add another structure to a crossing point: virtual crossing

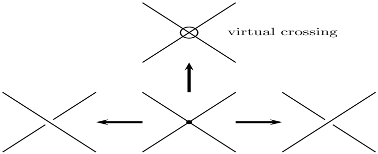

Figure 2: virtual crossing

A brief review on virtual knot theory

Virtual knot types= {all virtual knot diagrams}/{generalized Reidemeister moves}

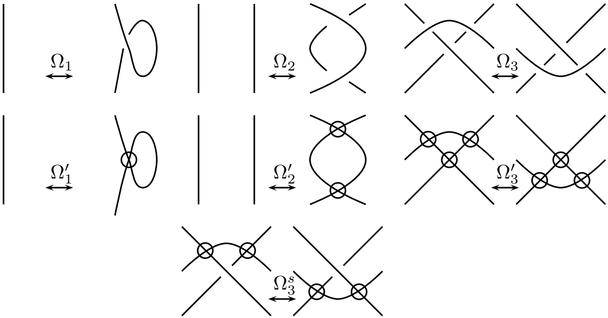

Figure 3: generalized Reidemeister moves

A brief review on virtual knot theory

Flat virtual knots (or virtual strings by Turaev) can be regarded as virtual knots without over/undercrossing information. More precisely, a flat virtual knot diagram can be obtained from a virtual knot diagram by replacing all real crossing points with flat crossing points. By replacing all real crossing points with flat crossing points in Figure 3 one obtains the flat generalized Reidemeister moves. Then

Flat virtual knot types= {all flat virtual knot diagrams}/{flat generalized Reidemeister moves}

A brief review on virtual knot theory

Virtual knot theory was introduced by L. Kauffman. Roughly speaking, there are two motivations to extend the classical knot theory to virtual knot theory.

From the topological viewpoint, a classical knot is an embedding of S^1 into R^3 up to isotopies. It is equivalent to replace the ambient space R^3 with $S^2 \times [0, 1]$. A virtual knot is an embedded circle in the thickened closed orientable surface $\Sigma_g \times [0, 1]$ up to isotopies and (de)stabilizations.

A brief review on virtual knot theory

Another motivation comes from realizing an arbitrary Gauss diagram.
For a given classical knot diagram, there exists a unique Gauss diagram corresponding to it. However, there exist some Gauss diagrams which cannot be realized as a classical knot diagram. Therefore one has to add some virtual crossing points.

A brief review on virtual knot theory

Given a virtual knot diagram, a Gauss diagram is:

- a circle together with some chords, each chord connecting the preimages of a crossing point
- an orientation from the preimage of the overcrossing to the preimage of the undercrossing
- the writhe of the each crossing point

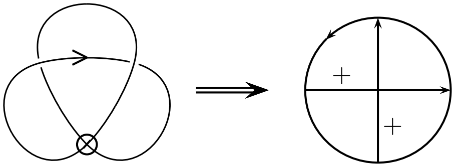

Figure 4: virtual trefoil and its Gauss diagram

The importance of realizing all Gauss diagrams also comes from Goussarov, Polyak and Viro's work on finite type invariants[1]。They show that that any integer-valued finite type invariant of degree n of a (long) knot D can be expressed in terms of Gauss diagrams. Note that not all Gauss diagrams can be realized by classical knots, it seems more reasonable to study the finite type invariants in the world of virtual knots.

[1] **M. Goussarov**, **M. Polyak**, **O. Viro**, **Finite-type invariants of classical and virtual knots**, **Topology (2000)**, **1045-1068**.

Two virtual knot invariants derived from the chord index

In 2016, Cheng proposed the chord index axioms[2], which is a generalization of the parity axioms of Manturov[3].

Assume for each real crossing point c of a diagram, we can assign an index to it(e.g. an integer, a polynomial, a group etc.). We say this index satisfies the chord index axioms if it satisfies the following five conditions:

[2] **Z. Cheng**, **The chord index**, **its definitions**, **applications and generalizations**, **arXiv:1606.01446**.

[3] **V. Manturov**, **Parity in knot theory**, **Sbornik: Mathematics 201 (2010)**, **no. 5**, **693-733**.

Two virtual knot invariants derived from the chord index

1. The real crossing point involved in Ω_1 has a fixed index;
2. The two crossing points involved in Ω_2 have the same indices;
3. The indices of the three crossing points involved in Ω_3 are preserved under Ω_3 respectively;
4. The index of the real crossing point involved in Ω_3^s is preserved under Ω_3^s;
5. The index of any real crossing point not involved in a generalized Reidemeister move is preserved under this move.

Two virtual knot invariants derived from the chord index

Our aim is trying to construct some chord index type of invariants. More precisely, Using the flat virtual knot/link-valued chord index we introduce two invariants, which take values in the following module of $\mathcal{M}_1^u$ and $\mathcal{M}_2^o$.

$\mathcal{M}_1^u =$ the free $\mathbf{Z}$-module generated by the set of all unoriented flat virtual knots.

$\mathcal{M}_2^o =$ the free $\mathbf{Z}$-module generated by the set of all oriented 2-component flat virtual links.

Two virtual knot invariants derived from the chord index

One can compare these invariants with some other flat virtual knots/graphs-valued virtual knot invariants. For example, the polynomial $\nabla(K)$ of Turaev which takes values in the polynomial algebra generated by nontrivial flat virtual knots[4], or the $sl(3)$ invariant introduced by Kauffman and Manturov, which is valued in a module generated by graphs[5].

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[4] **V. Turaev**, **Virtual strings**, **Annales de linstitut Fourier (2004)**, **no. 7**, **2455-2525**.

[5] **L. H. Kauffman**, **V. Manturov**,
A graphical construction of the sl(3) invariant for virtual knots,
Quantum Topology (2014), **no. 4**, **523-539**.

Two virtual knot invariants derived from the chord index

An $\mathcal{M}_1^u$-valued virtual knot invariant

Let K be a virtual knot diagram and c a real crossing point. Associate it a unoriented flat virtual knot $\widetilde{K}_c$: There are two kinds of resolution on c. Use *0-smoothing* to denote the one which preserves the number of components and *1-smoothing* to denote the other.

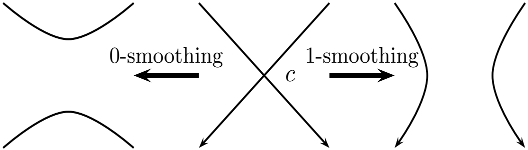

Figure 5: two kinds of resolution

Two virtual knot invariants derived from the chord index

Applying 0-smoothing at c we get another virtual knot, which is unoriented even if K is oriented. Use K_c to denote this unoriented virtual knot and $\widetilde{K}_c$ to denote the corresponding flat virtual knot. Then we have

Theorem 2.1 $\widetilde{K}_c$ satisfies the chord index axioms.

Proof of Theorem 2.1:

1. If c is a crossing point appearing in Ω_1, after performing 0-smoothing at c, it is easy to observe that $\widetilde{K}_c = \widetilde{K}$ without considering the orientation, which is a fixed element in $\mathcal{M}_1^u$.

Two virtual knot invariants derived from the chord index

Proof of Theorem 2.1(continue):

2. Consider the two crossing points in Ω_2, say c_1 and c_2. As illustrated in the next figure, in both cases the chord indices $\widetilde{K}_{c_1}$ and $\widetilde{K}_{c_2}$ are equivalent as flat virtual knots.

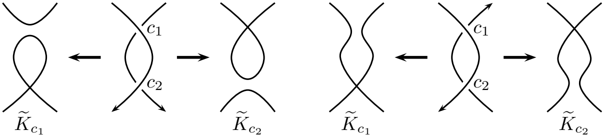

Figure 6: Resolutions of crossing points in Ω_2

Two virtual knot invariants derived from the chord index

Proof of Theorem 2.1(continue):

3. Assume K and K' are related by an Ω_3 move, use c_1, c_2, c_3 to denote the three crossing points in K, and c_1', c_2', c_3' to denote the corresponding crossing points in K'. It is easy to see that $\widetilde{K}_{c_1} = \widetilde{K'}_{c_1'}, \widetilde{K}_{c_2} = \widetilde{K'}_{c_2'}$ and $\widetilde{K}_{c_3} = \widetilde{K'}_{c_3'}$ from the figure below.

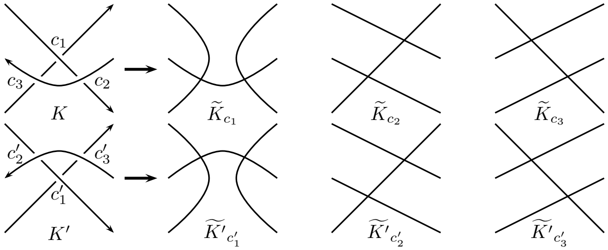

Figure 7: Resolutions of crossing points in Ω_3

Proof of Theorem 2.1(continue):

4. For Ω_3^s, there exist two possibilities: in one case one chord index can be obtained from the other one by two flat Ω_2'-moves, in the other case two chord indices are the same.

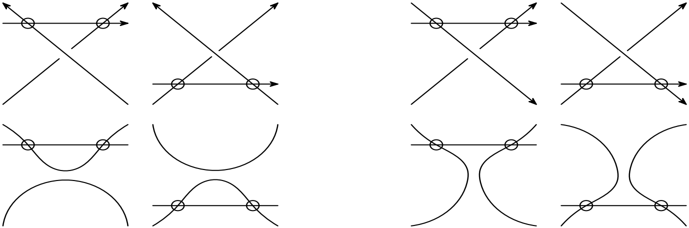

Figure 8: resolution of crossing point in Ω_3^s

Proof of Theorem 2.1(continue):

5. If a crossing point c is not involved in the move, then the two chord indices before and after the move are related by a flat version of this move.

Two virtual knot invariants derived from the chord index

Theorem 2.2: Let K be a virtual knot diagram, then $\mathcal{F}(K) = \sum_{c} w(c)\widetilde{K}_c - w(K)\widetilde{K} \in \mathcal{M}_1^u$ is a virtual knot invariant. Here $\widetilde{K}$ should be understood as the corresponding flat virtual knot of K without the orientation, the sum runs over all the real crossing points of K, and $w(c), w(K)$ denote the writhe of c and K respectively.

Two virtual knot invariants derived from the chord index

Proof of Theorem 2.2:

Notice that $\widetilde{K}_c = \widetilde{K}$ if c is the crossing point involved in Ω_1 and the writhes of the two crossing points involved in Ω_2 are distinct. The result follows directly from the chord index axioms.

Two virtual knot invariants derived from the chord index

Corolarry 2.1 If K is a classical knot, then $\mathcal{F}(K) = 0$.

Actually, since any flat virtual knot diagram with one or two flat crossing points represent the unknot. It follows that if the number of real crossing points of K is less than or equal to 2, then $\mathcal{F}(K) = 0$.

Example 2.1 Let K be the virtual knot described in the figure below. Direct calculation shows that $\mathcal{F}(K) = \widetilde{\text{Kishino}} + 4U - 5\widetilde{K}$, where $\widetilde{\text{Kishino}}$ denotes the Kishino flat virtual knot and U the unknot. Since $\widetilde{\text{Kishino}}$ is nontrivial, we conclude that $\mathcal{F}(K) \neq 0$.

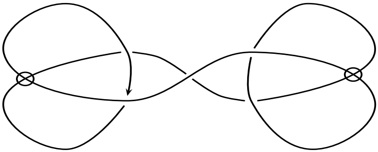

Figure 9: A virtual knot K with nontrivial $\mathcal{F}$

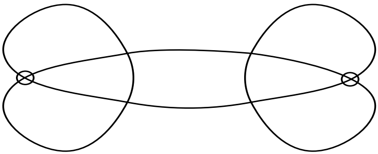

Figure 10: Kishino flat virtual knot

Remark Actually, in Example 2.1, the flat knots $\widetilde{\text{Kishino}}$ and $\widetilde{K}$ are also different. They can be distinguished by a flat invariant $\overline{B}_K(t,s)$ defined by Dr. M. Xu[6].

[6] **M. Xu**, **Writhe polynomial for virtual links**, **arXiv:1812.05234.**.

Two virtual knot invariants derived from the chord index

Remark The invariant $\mathcal{F}(K)$ can be easily extended to an invariant of n-component virtual links ($n \geq 2$). In this case we need change $\mathcal{M}_1^u$ to a free **Z**-module $\mathcal{M}^u$ generated by all unoriented flat virtual links. Now the the chord index of a self-crossing point is a unoriented n-component flat virtual link, and the chord index of a mixed-crossing point is a unoriented $(n-1)$-component flat virtual link.

Two virtual knot invariants derived from the chord index

An $\mathcal{M}_2^o$-valued virtual knot invariant

Let K be a virtual knot diagram and c a real crossing point of K. Now performing 1-smoothing at c transforms K into an oriented 2-component virtual link L_c. Denote $\widetilde{L}_c$ the corresponding flat virtual link, which lies in $\mathcal{M}_2^o$. Then we have:

Theorem 2.3 $\widetilde{L}_c$ satisfies the chord index axioms.

Two virtual knot invariants derived from the chord index

Similar to Theorem 2.2, this chord index also provides us a virtual knot invariant.

Theorem 2.4 Let K be a virtual knot, then $\mathcal{L}(K)=\sum\limits_{c} w(c)\widetilde{L}_c - w(K)(\widetilde{K}\cup U)\in\mathcal{M}_2$ defines a virtual knot invariant.

Here each $\widetilde{L}_c$ should be understood as a unordered 2-component flat virtual link.

Two virtual knot invariants derived from the chord index

The invariants $\mathcal{F}(K)$ and $\mathcal{L}(K)$ contain some information about "mirror images" of virtual knots.

Proposition 2.1 Let K be an oriented virtual knot diagram, if we use $r(K)$ to denote the diagram obtained from K by reversing the orientation, and $m(K)$ denotes the diagram obtained from K by switching all real crossing points, then we have

$\mathcal{F}(r(K)) = \mathcal{F}(K)$ and $\mathcal{F}(m(K)) = -\mathcal{F}(K)$.

$\mathcal{L}(r(K)) = r(\mathcal{L}(K))$ and $\mathcal{L}(m(K)) = -\mathcal{L}(K)$.

Two virtual knot invariants derived from the chord index

It is evident that $\mathcal{L}(K) = 0$ if K is a classical knot. But unlike $\mathcal{F}(K)$, which vanishes on virtual knots with two real crossing points, $\mathcal{L}(K)$ is able to distinguish the virtual trefoil knot from the unknot.

Example 2.2 Consider the virtual trefoil knot K. We have $\mathcal{L}(K) = 2\widetilde{HL} - 2(U \cup U)$, here $\widetilde{HL}$ denotes the flat virtual Hopf link which can be obtained from the classical Hopf link diagram by replacing the two real crossing points with one virtual crossing point and one flat crossing point.

To prove $\mathcal{L}(K) \neq 0$, it suffices to show that $\widetilde{HL}$ is nontrivial.

Two virtual knot invariants derived from the chord index

Define a flat linking number of a unordered 2-component flat virtual link as follows:

For a unordered 2-component flat virtual link $\widetilde{L} = \widetilde{K}_1 \cup \widetilde{K}_2$. Denote C_{12} to be the set of all flat crossing points between $\widetilde{K}_1$ and $\widetilde{K}_2$. Replacing each flat crossing point in C_{12} with a real crossing point such that the over-strand belongs to $\widetilde{K}_1$, then define the flat linking number by $lk(\widetilde{L}) = |\sum_{c \in C_{12}} w(c)|$, where $w(c)$ means the writhe of c.

This definition does not depend on the order of $\widetilde{K}_1$ and $\widetilde{K}_2$, and it is invariant under all flat generalized Reidemeister moves. Clearly, $lk(\widetilde{HL}) = 1$ and it follows that $\mathcal{L}(K) \neq 0$.

Two virtual knot invariants derived from the chord index

Theorem 2.2 and Theorem 2.4 can be regarded as a general method to construct virtual invariants. Combining them with some concrete flat invariants we may obtain concrete and sometimes much more usable virtual knot invariants. Here we will show that several known invariants can be recovered from our invariants.

Two virtual knot invariants derived from the chord index

Example 2.3 The writhe polynomial $W_K(t)$ is defined independently by Cheng-Gao, Dye, Kauffman, Im-Kim-Lee and Satoh-Taniguchi. The key point to define $W_K(t)$ is the chord index, which assigns an integer $\mathrm{Ind}(c)$ to each real crossing point c of a virtual knot diagram. Here we mainly follow the approach of Folwaczny and Kauffman[7].

[7] **L. Folwaczny**, **L. Kauffman**, **A linking number definition of the affine index polynomial and applications**, **JKTR 22 (2013)**, **no. 12**, **1341004**.

Two virtual knot invariants derived from the chord index

Assuming K is a virtual knot diagram and c a real crossing point. Applying 1-smoothing to c will get a 2-component virtual link $L_c = K_1 \cup K_2$.

The order of these two components is arranged as follows: refer to Figure 5, if c is positive we call the component on the left side K_1 and those on the right side K_2; Conversely, if c is negative we use K_1 to denote the right side component and use K_2 to denote the left side component.

All the real crossing points between K_1 and K_2 can be divided into two sets C_{12} and C_{21}. Where C_{12} denotes the set of real crossing points where the over-strands belong to K_1 and C_{21} those the over-strands belong to K_2.

Two virtual knot invariants derived from the chord index

Now we can define the *index* of c as

$$\mathrm{Ind}(c) = \sum_{c \in C_{12}} w(c) - \sum_{c \in C_{21}} w(c)$$

and the *writhe polynomial* as

$$W_K(t) = \sum_{c} w(c) t^{\mathrm{Ind}(c)} - w(K).$$

According to the definition of the index, $\mathrm{Ind}(c)$ is invariant under the crossing change of any real crossing point of L_c. Hence it is an invariant of $\widetilde{L}_c$. So $W_K(t)$ is a special case of $\mathcal{L}(K)$.

Example 2.4 The next example concerns the sequence of 2-variable polynomial invariants $L^n_K(t, l)$ recently introduced by Kaur-Prabhakar-Vesnin[8]. We will show that this sequence of polynomial invariants combines some information of $\mathcal{F}(K)$ and $\mathcal{L}(K)$.

[8] **K. Kaur**, **M. Prabhakar**, **A. Vesnin**, **Two-variable polynomial invariants of virtual knots arising from flat virtual knot invariants**, **arXiv:1803.05191**.

Two virtual knot invariants derived from the chord index

Let K be a virtual knot diagram and c a real crossing point. Assume the writhe polynomial $W_K(t) = \sum_n a_n t^n$, it is known that $W_K(t) - W_K(t^{-1})$ is a flat virtual knot invariant. $W_K(t) - W_K(t^{-1})$ has the form of $\sum_n (a_n - a_{-n}) t^n$, and the coefficient of t^n equals $a_n - a_{-n}$, which is also a flat virtual knot invariant. It is called the *n-th dwrithe* and denote by $\nabla J_n(K)$ by Kaur-Prabhakar-Vesnin.

Now the polynomials $L_K^n(t, l)$ defined by Kaur-Prabhakar-Vesnin are

$$L_K^n(t, l) = \sum_c w(c) t^{\mathrm{Ind}(c)} l^{|\nabla J_n(K_c)|} - w(K) l^{|\nabla J_n(K)|}.$$

Two virtual knot invariants derived from the chord index

It is easy to check that $t^{\mathrm{Ind}(c)} l^{|\nabla J_n(K_c)|}$ satisfies the chord index axioms, and if a crossing point is involved in Ω_1 then the index equals $l^{|\nabla J_n(K)|}$.

As we mentioned above, $\mathrm{Ind}(c)$ is a flat virtual knot invariant of L_c and $|\nabla J_n(K_c)|$ is a flat virtual knot invariant of K_c, hence $L^n_K(t, l)$ can be regarded as a mixture of some information coming from $\mathcal{F}(K)$ and $\mathcal{L}(K)$.

From the viewpoint of finite type invariant

Recall that a finite type invariant (or Vassiliev invariant) of degree n is a (virtual) knot invariant valued in an Abelian group which vanishes on all singular knots with k singularities provided that $k \geq n + 1$.

More precisely, if $f : K \to A$ is a virtual knot invariant which associates each virtual knot with an element in an Abelian group A. Then we can extend f from virtual knots to singular virtual knots via the following recursive relation

$$f^{(n)}(K) = f^{(n-1)}(K_+) - f^{(n-1)}(K_-),$$

here $K_+(K_-)$ is obtained from K, a singular virtual knot with n singularities, by replacing a singular point with a positive (negative) crossing point. We set $f^{(0)} = f$ as the initial condition.

From the viewpoint of finite type invariant

Now we say f is a finite type invariant of degree n if $f^{(n+1)}(K) = 0$ for any singular virtual knot K with $n+1$ singularities, and there exists a singular virtual knot K with n singularities which satisfies $f^{(n)}(K) \neq 0$.

For classical knots, this definition coincides with the definition given by Birman and Lin[9]

[9] **J. Birman**, **X. Lin**, **Knot polynomials and Vassiliev's invariants**, **Invent. Math.**, **1993**, **225-270**.

From the viewpoint of finite type invariant

Theorem 3.1 Both $\mathcal{F}(K)$ and $\mathcal{L}(K)$ are finite type invariants of degree one.

Proof of Theorem 3.1:

We only prove that $\mathcal{F}(K)$ is a finite type invariant of degree one, the other proof is similar.

We need to show that $\mathcal{F}^{(2)}$ vanishes on any singular virtual knot with two singularities and there is a singular virtual knot with one singularity which has nontrivial $\mathcal{F}^{(1)}$.

From the viewpoint of finite type invariant

Proof of Theorem 3.1(continue):

Let K be a virtual knot diagram and c_1, c_2 be two real crossing points. Without loss of generality, we assume that $w(c_1) = w(c_2) = +1$. We use $K_{-+}(K_{+-})$ to denote the virtual knot diagram obtained from K by switching $c_1(c_2)$, and use K_{--} to denote the diagram obtained from K by switching both c_1 and c_2, and $K_{++} = K$.

The recursive relation $f^{(n)}(K) = f^{(n-1)}(K_+) - f^{(n-1)}(K_-)$ and $f^{(2)}(K) = 0$ is now equivalent to

$\mathcal{F}(K_{++}) - \mathcal{F}(K_{+-}) - \mathcal{F}(K_{-+}) + \mathcal{F}(K_{--}) = 0.$

From the viewpoint of finite type invariant

Proof of Theorem 3.1(continue):

One computes

$$
\begin{aligned}
&\mathcal{F}(K_{++}) - \mathcal{F}(K_{+-}) - \mathcal{F}(K_{-+}) + \mathcal{F}(K_{--})\\
&= (\sum_c w(c)\widetilde{K_{++c}} - w(K_{++})\widetilde{K_{++}}) - (\sum_c w(c)\widetilde{K_{+-c}} - w(K_{+-})\widetilde{K_{+-}})\\
&-(\sum_c w(c)\widetilde{K_{-+c}} - w(K_{-+})\widetilde{K_{-+}}) + (\sum_c w(c)\widetilde{K_{--c}} - w(K_{--})\widetilde{K_{--}})\\
&= \sum_c w(c)\widetilde{K_{++c}} - \sum_c w(c)\widetilde{K_{+-c}} - \sum_c w(c)\widetilde{K_{-+c}} + \sum_c w(c)\widetilde{K_{--c}}\\
&= (\widetilde{K_{++c_1}} + \widetilde{K_{++c_2}}) - (\widetilde{K_{+-c_1}} - \widetilde{K_{+-c_2}}) - (-\widetilde{K_{-+c_1}} + \widetilde{K_{-+c_2}}) + (-\widetilde{K_{--c_1}} - \widetilde{K_{--c_2}})\\
&= (\widetilde{K_{++c_1}} - \widetilde{K_{--c_1}}) + (\widetilde{K_{++c_2}} - \widetilde{K_{--c_2}}) + (\widetilde{K_{-+c_1}} - \widetilde{K_{+-c_1}}) + (\widetilde{K_{+-c_2}} - \widetilde{K_{-+c_2}})\\
&= 0 + 0 + 0 + 0\\
&= 0
\end{aligned}
$$

From the viewpoint of finite type invariant

Proof of Theorem 3.1(continue):

On the other hand, consider the positive Kishino virtual knot K below

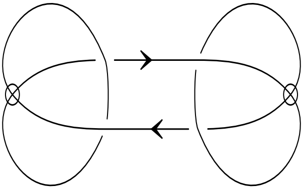

Figure 11: positive Kishino knot K

From the viewpoint of finite type invariant

Proof of Theorem 3.1(continue):

Denote the positive crossing point on the top left of K by c. We still use K_+ to denote K and use K_- to denote the diagram after switching c. Then we have

$$\begin{aligned}
&\mathcal{F}(K_+) - \mathcal{F}(K_-) \\
&= (\sum_c w(c)\widetilde{K_{+}}_c - w(K_+)\widetilde{K_+}) - (\sum_c w(c)\widetilde{K_{-}}_c - w(K_-)\widetilde{K_-}) \\
&= 2U - 2\widetilde{\text{Kishino}} \\
&\neq 0
\end{aligned}$$

Flat virtual knot invariants

Recall that we have introduced two virtual knot invariants $\mathcal{F}(K)$ and $\mathcal{L}(K)$, which take values in $\mathcal{M}_1^u$ and $\mathcal{M}_2^o$ respectively. However, in general it is still not easy to distinguish two elements in $\mathcal{M}_1^u$ or $\mathcal{M}_2^o$. To this end, we extend the main idea of $\mathcal{F}(K)$ and $\mathcal{L}(K)$ from virtual knots to flat virtual knots, and we can define two invariants for flat virtual knots, which take values in $\mathcal{M}_1^o$ and $\mathcal{M}_2^o$ respectively. Here $\mathcal{M}_1^o$ denotes the free $\mathbf{Z}$-module generated by all oriented flat virtual knots, and as before, $\mathcal{M}_2^o$ is referred to the free $\mathbf{Z}$-module generated by all oriented 2-component flat virtual links.

Flat virtual knot invariants

Given an oriented flat virtual knot diagram $\widetilde{K}$, we can define its Gauss diagram $G(\widetilde{K})$. At first glance, since there is no over/undercrossing information, we can just connect the two preimages of each flat crossing with a chord without direction and sign. However, we can still assign a direction to each chord as follows:

Replacing each flat crossing point with a positive crossing point and then add an arrow to each chord in $G(\widetilde{K})$, directed from the preimage of the overcrossing to the preimage of the undercrossing. Now we obtain a Gauss diagram $G(\widetilde{K})$ of which each chord has a direction but no signs.

Flat virtual knot invariants

The following result is well-known, see for example[10].

Lemma 4.1 Each $G(\widetilde{K})$ corresponds to a unique flat virtual knot $\widetilde{K}$.

[10] **V. Turaev**, **Virtual strings**, **Annales de linstitut Fourier (2004)**, **no. 7**, **2455-2525**.

Flat virtual knot invariants

Next, we give a sign to each chord or flat crossing. For a given flat virtual knot diagram $\widetilde{K}$, let us consider the corresponding positive virtual knot diagram, say K^+. Now we can use the index of Example 2.3 to assign an integer $\mathrm{Ind}(c)$ to each positive crossing point of K^+.

For the corresponding flat crossing point c in $\widetilde{K}$, we define its writhe as follows

$$w(c) = \mathrm{sgn}(\mathrm{Ind}(c)) = \begin{cases} \frac{|\mathrm{Ind}(c)|}{\mathrm{Ind}(c)} & \text{if } \mathrm{Ind}(c) \neq 0; \\ 0 & \text{if } \mathrm{Ind}(c) = 0. \end{cases}$$

Flat virtual knot invariants

Define the writhe of a flat virtual knot as $w(\widetilde{K}) = \sum_{c} w(c)$, where the sum runs over all flat crossing points.

It is easy to check that

Lemma 4.2 $w(\widetilde{K})$ is a flat virtual knot invariant.

Flat virtual knot invariants

Let c be a flat crossing point of $\widetilde{K}$. Similar as in section 2, we can perform 0-smoothing or 1-smoothing to resolve this flat crossing point to get a unoriented flat virtual knot $\widetilde{K}_c$ or an oriented 2-component flat virtual link $\widetilde{L}_c$. Now for $\widetilde{K}_c$ we fix an orientation according to the Figure below.

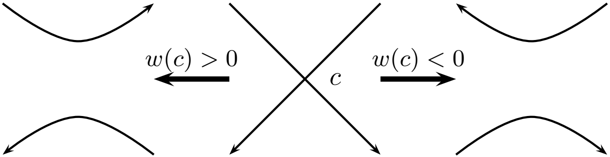

Figure 12: The orientation of $\widetilde{K}_c$

Flat virtual knot invariants

Define $\widetilde{\mathcal{F}}(\widetilde{K}) = \sum\limits_{c} w(c)\widetilde{K}_c \in \mathcal{M}_1^o$

$\widetilde{\mathcal{L}}(\widetilde{K}) = \sum\limits_{c} w(c)\widetilde{L}_c \in \mathcal{M}_2^o$.

Then we have

Theorem 4.1 $\widetilde{\mathcal{F}}(\widetilde{K})$ and $\widetilde{\mathcal{L}}(\widetilde{K})$ are both flat virtual knot invariants.

Proposition 4.1 Suppose $\widetilde{\mathcal{F}}(\widetilde{K}) = \sum_{i=1}^{n} a_i \widetilde{K}_i$ ($a_i \neq 0$), then $\widetilde{K}_i \neq \widetilde{K}$ for any $1 \leq i \leq n$.

Proof If $\widetilde{K}$ is a unknot, then we have $\widetilde{\mathcal{F}}(\widetilde{K}) = 0 \neq \widetilde{K}$. If $\widetilde{K}$ is nontrivial, we choose a minimal diagram of it, i.e. a diagram which realizes the minimal flat crossing number. According to the definition of $\widetilde{\mathcal{F}}(\widetilde{K})$, each $\widetilde{K}_i$ has strictly fewer flat crossing points, hence it cannot be equivalent to $\widetilde{K}$.

Remark This proposition says that the map $\widetilde{\mathcal{F}}$ turns an oriented flat virtual knot K into a linear combination of "strictly simpler" flat virtual knots. Therefore sometimes one can use known nontrivial flat virtual knots to detect the nontrivially of some more complicated flat virtual knots.

Thank you!