# 3-submanifolds of $S^3$ which admit complete surface systems (CSS)

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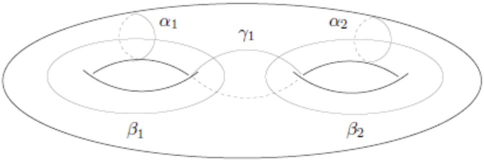
# Content

### 1.1 Some Definitions and fundamental facts on CCS and CSS

**Def:** Let $F = F_n$ be a closed connected orientable surface of genus $n \geq 1$. A complete curve system (CCS, for simplicity) on $F$ is a collection $\mathcal{J} = \{J_1, \cdots, J_n\}$ of $n$ pairwise disjoint simple closed curves on $F$ such that the surface obtained by cutting $F$ open along $\mathcal{J}$ is a $2n$-punctured sphere.

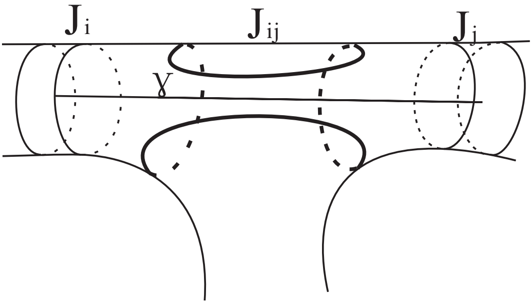



On the surface $F$ genus 2 as above, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \beta_2\}$, $\{\beta_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, $\{\alpha_1, \gamma_1\}$, and $\{\alpha_2, \gamma_1\}$ are examples of CCS for $F$.

# Band sum move

**Def:** Let $\mathcal{J} = \{J_1, \cdots, J_n\}$ be a CCS on $F$. For $1 \leq i \neq j \leq n$, let $\gamma$ be a simple arc on $S$ such that $\gamma \cap J_i$ is an end point of $\gamma$, $\gamma \cap J_j$ is another end point of $\gamma$, and the interior of $\gamma$ is disjoint from $\bigcup_{1 \leq i \leq n} J_i$. Let $P = N(J_i \cup \gamma \cup J_j)$ be a small compact regular neighborhood of $J_i \cup \gamma \cup J_j$ on $S$. Denote by $J_{ij} = J_i \#_\gamma J_j$ the boundary component of $P$ which is not isotopic to $J_i$ or $J_j$ on $P$, and call it the band sum of $J_i$ and $J_j$ along $\gamma$. We may assume that $J_{ij}$ is disjoint from the curves in $\mathcal{J}$. Replace $J_i$ or $J_j$ by $J_{ij}$ in $\mathcal{J}$ to get a new CCS $\mathcal{J}'$ on $S$. We call $\mathcal{J}'$ a band sum move of $\mathcal{J}$.

# Band sum move




It is clear that if $\mathcal{J}'$ is a band sum move of $\mathcal{J}$, then $\mathcal{J}$ is also a band sum move of $\mathcal{J}'$.

**Def:** Two CCS $C_1$ and $C_2$ on a closed surface $S$ of genus $n > 0$ are called equivalent if one can be obtained from another by a finite number of band sum moves and isotopies.

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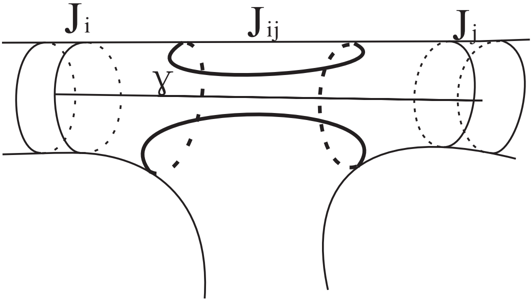


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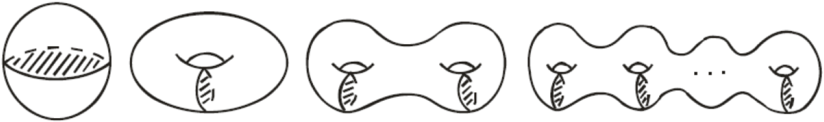
**Def:** Two CCS $\mathcal{C}_1$ and $\mathcal{C}_2$ on a closed surface $S$ of genus $n > 0$ are called equivalent if one can be obtained from another by a finite number of band sum moves and isotopies.

**Def:** Let $M$ be a compact 3-manifold with a single boundary component $F$ of genus $g(F) = n \geq 1$. Let $\mathcal{J} = \{J_1, \cdots, J_n\}$ be a CCS on $F$. If there exists a collection of pairwise disjoint compact connected orientable surfaces $S_1, \cdots, S_n$ properly embedded in $M$ such that $\partial S_i = J_i$ for each $1 \leq i \leq n$, we call $\mathcal{S} = \{S_1, \cdots, S_n\}$ a complete surface system (CSS) in $M$, and call $\mathcal{J}$ a complete spanning curve system (CSCS) for $M$ on $F$. Sometimes we say that $M$ admits a CSS or CSCS.

**Example 1:** A handlebody $H$ of genus $n$ is a 3-manifold which admits a complete disk system $\mathcal{D} = \{D_1, \cdots, D_n\}$ such that the manifold obtained by cutting $H$ open along $\mathcal{D}$ is a 3-ball.

Clearly, $\mathcal{D}$ is a CSS for handlebody $H$, usually called a complete disk system for $H$, and $\partial\mathcal{D} = \{\partial D_1, \cdots, \partial D_n\}$ is a CSCS for $H$.



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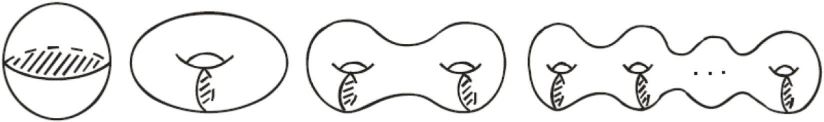

**Example 2:** Let $K$ be a knot in $S^3$, $N(K)$ a regular neighbourhood of $K$ in $S^3$, and $M_K = \overline{M \setminus N(K)}$ the complement of $K$. Let $S'$ be a Seifert surface of $K$ in $S^3$ with $S' \cap N(K)$ an annulus. Let $S = S' \cap M_K$, and $J = \partial S$. Then $J$ is a CSCS for $M_K$, and $S$ is a CSS for $M_K$.

**Example 3:** Let $L = \{l_1, \cdots, l_n\}$ be a boundary link in $S^3$. $L$ bounds a disjoint union of $n$ Seifert surfaces $S_1 \cdots, S_n$ in $S^3$ such that $l_i$ bounds $S_i$ for $i = 1, \cdots, n$. Choose a point $P$ in $S^3$ so that $P$ is not contained in any $S_i$, $1 \leq i \leq n$. For each $i$, $1 \leq i \leq n$, choose a simple arc $\alpha_i$ in $S^3$ connecting $P$ and a point $P_i \in l_i$, such that $\alpha_i \cap S_i = \alpha_i \cap l_i = P_i$, and for $i \neq j$, $\alpha_i \cap \alpha_j = \{P\}$. Set $\Gamma = \bigcup_{i=1}^{n} \alpha_i \cup l_i$. Then $\Gamma$ is a connected graph with $\chi(\Gamma) = -n$. Let $H$ be a regular neighborhood of $\Gamma$ in $S^3$. $H$ is a handlebody of genus $n$. Clearly, $M = \overline{S^3 \setminus H}$ admits a CSCS on $\partial M$.

# Some other examples of CSCS

**Example 2:** Let $K$ be a knot in $S^3$, $N(K)$ a regular neighbourhood of $K$ in $S^3$, and $M_K = \overline{M \setminus N(K)}$ the complement of $K$. Let $S'$ be a Seifert surface of $K$ in $S^3$ with $S' \cap N(K)$ an annulus. Let $S = S' \cap M_K$, and $J = \partial S$. Then $J$ is a CSCS for $M_K$, and $S$ is a CSS for $M_K$.

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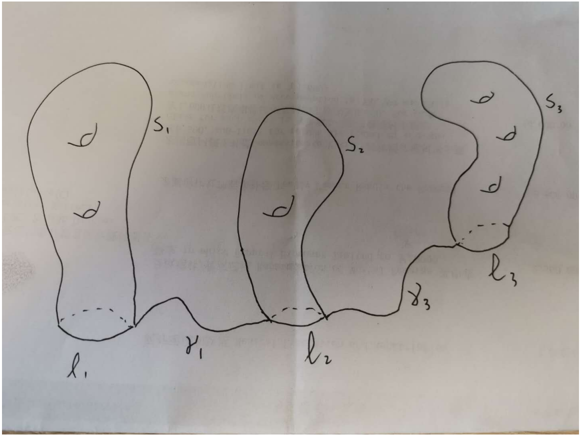


**Question:** For a 3-submanifold $M$ in $S^3$ which admits a CSS, can $M$ be obtained from a boundary link in a way as above?

# Equivalence of CSS

**Def:** Let $\mathcal{S} = \{S_1, \cdots, S_n\}$, $\mathcal{S}' = \{S'_1, \cdots, S'_n\}$ be two CSS for 3-manifold $M$, and $\mathcal{J}$, $\mathcal{J}'$, the corresponding CSCSs on $F = \partial M$. We say that $\mathcal{S}$ and $\mathcal{S}'$ are equivalent if $\mathcal{J}$ and $\mathcal{J}'$ are equivalent on $F$.

**Remark:**

1. The equivalence of CSS for $M$ only depends on the equivalence of their corresponding boundaries.

2. For a CSCS $\mathcal{J}$ for a 3-manifold $M$, the spanned surfaces of $\mathcal{J}$ in $M$ may not unique. For example, the knot complements. We emphasize the existence of a CSS, not the individual of the CSS. That's the reason why we use CSCS to denote CSS.

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# Fundamental properties

The following facts about handlebodies are well known:

**Proposition**

Let $H$ be a handlebody of genus $n \geq 1$.

(1) The only complete surface system in $H$ is the complete disk system.

(2) Let $\mathcal{D} = \{D_1, \cdots, D_n\}$ be a complete disk system for $H$, and $\mathcal{J} = \partial\mathcal{D} = \{\partial D_1, \cdots, \partial D_n\}$. Then any CCS $\mathcal{K}$ on $\partial H$ which is equivalent to $\mathcal{J}$ is also a CSS for $H$, therefore the boundary of a complete disk system for $H$.

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The previous proposition can be generalized directly as follows:

**Proposition**

Let $M$ be a compact 3-manifold with a single boundary component $F$ of genus $g(F) = n \geq 1$. Let $\mathcal{K} \subset F$ be a CSCS for $M$. Then any CCS $\mathcal{J}$ on $F$ which is equivalent to $\mathcal{K}$ is also a CSCS for $M$. Moreover, for any GCCS $\mathcal{J}' = \{J_1, \cdots, J_k\}$ on $F$ which contains $\mathcal{J}$ as a subset, there exists a collection of pairwise disjoint compact orientable surfaces $S_1, \cdots, S_k$ properly embedded in $M$, such that $\partial S_i = J_i$ for each $1 \leq i \leq k$.

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Let $S$ be a connected closed orientable surface of genus $n$ in a closed orientable 3-manifold $M$. If $S$ cuts $M$ into two handlebodies $H$ and $H'$, we say that $S$ is a Heegaard surface of $M$, and $H \cup_S H'$ is a Heegaard splitting (HS) for $M$. $n$ is called the genus of the Heegaard splitting.

We use $g(M)$ to denote the Heegaard genus of $M$, which is the minimal genus of all Heegaard splittings of $M$.

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For a Heegaard splitting $H \cup_S H'$ for $M$, let $\mathcal{J} = \{J_1, \cdots, J_n\}$ ($\mathcal{J}' = \{J'_1, \cdots, J'_n\}$) be a CSCS for $H$ ($H'$, resp.). We call $(H; \mathcal{J}')$ (or $(H; \mathcal{J}')$, or $(S; \mathcal{J}, \mathcal{J}')$) a Heegaard diagram associated the Heegaard splitting $H \cup_S H'$ of $M$.

Let $(H; \mathcal{J}')$ be a Heegaard diagram for $M$. One can obtain $M$ by adding 2-handles along each curve in $\mathcal{J}'$, then capping of the resulting manifold by a 3-ball.

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A Heegaard splitting $H \cup_S H'$ is stabilized if $\exists$ essential disks $D \subset H$ and $D' \subset H'$ s.t. $|\partial D \cap \partial D'| = 1$. Otherwise, $H \cup_S H'$ is unstabilized.

A stabilized HS $H \cup_S H'$ can be viewed as a connected sum of a HS $V \cup_F V'$ (with genus $g(S) - 1$) and a genus 1 HS of $S^3$. $H \cup_S H'$ is called an elementary stabilization of $V \cup_F V'$. $H \cup_S H'$ is called an stabilization of $V \cup_F V'$, if $H \cup_S H'$ can be obtained from $V \cup_F V'$ by a finite number of elementary stabilization.

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# Uniqueness theorem of the HS for $S^3$

**Theorem (Waldhausen, 1968)**

Let $V \cup_S W$ be a Heegaard splitting of genus $n \geq 1$ for $S^3$. Then $V \cup_S W$ is a stabilization of the Heegaard splitting of genus 0 for $S^3$, i.e., for each genus, the Heegaard splitting for $S^3$ is unique.

As a direct consequence, we have

**Corollary**

Let $V \cup_S W$ be a Heegaard splitting of genus $n \geq 1$ for $S^3$. Then there exists a Heegaard diagram $(S; \{\alpha_1, \cdots, \alpha_n\}, \{\beta_1, \cdots, \beta_n\})$ for $S^3$ associated to the splitting such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$.

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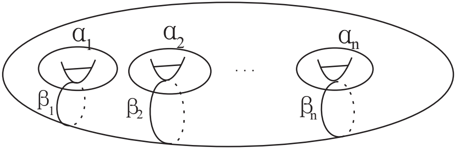
# Canonical Heegaard diagram for $S^3$

We call the Heegaard diagram $(S; \{\alpha_1, \cdots, \alpha_n\}, \{\beta_1, \cdots, \beta_n\})$ for $S^3$ in the previous Corollary the canonical Heegaard diagram for $S^3$. See Figure below,




where $\{\alpha_1, \cdots, \alpha_n\}$ bound a complete disk system in one handlebody $V$, and $\{\beta_1, \cdots, \beta_n\}$ bound a complete disk system in another handlebody $W$.

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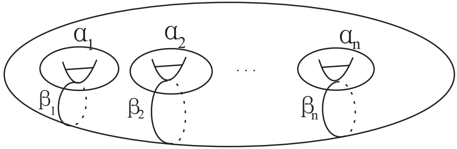


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# A characterization of the 3-sphere

There is a very elegant characterization of the 3-sphere in terms of any corresponding Heegaard diagram.

**Theorem**

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(1) The theorem was attributed to Moise and some others for the homotopy 3-spheres, and first stated in Haken's paper. By Perelman's work on Thurston's Geometrization Conjecture (which implies Poincaré Conjecture), the homotopy 3-sphere is the 3-sphere.

(2) The embedding $i : V \hookrightarrow S^3$ might be complicated, even for a Heegaard diagram $(V; J)$ associated to the genus 1 Heegaard splitting $V \cup_S W$ of $S^3$, where $i(J)$ could be any knot in $S^3$.

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The following is a classical re-embedding theorem of Fox for a compact connected 3-submanifold of $S^3$:

**Theorem**

Let $X$ be a compact connected 3-submanifold of $S^3$. Then $X$ can be re-embedded in $S^3$ so that the complement of the image of $X$ is a union of handlebodies.

In the following, we will always assume that $M$ is a compact 3-submanifold of $S^3$ with one boundary component $F$ which admits CSCS. We say that there always exists such 3-submanifold in $S^3$, for example, let $M_K$ be the complement of a non-trivial knot $K$ in $S^3$ , then the preferred longitude for $K$ is a CSCS for $M_K$ on the $\partial M_K$ which can be spanned to a Seifert surface for $K$.

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By Fox's re-embedding theorem, the previous characterization of the 3-sphere in term of HS can be stated in a strong version as follows.

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We call the 3-manifold $W'$ in the above theorem a quasi-handlebody.

**Question:** Classify quasi-handlebodies.

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# 3-submanifolds in $S^3$ which admit CSCS

The following theorem shows that the equivalent classes of CSCS for such 3-submanifolds of $S^3$ are unique.

**Theorem (Zhao-Lei-Li)**

Let $\mathcal{J}$, $\mathcal{K}$ be two CSCS for $M$. Then $\mathcal{J}$ and $\mathcal{K}$ are equivalent.

**A natural question:** Is there a 3-manifold $M$ which admits two non-equivalent CSCS?

**Theorem (Zhao-Lei)**

Suppose that $M$ is a 3-submanifold of $S^3$ which admits a CSCS $\mathcal{J}$. Assume that $\partial M$ is compressible in $M$. Let $\mathcal{D}$ be a maximal collection of pairwise disjoint compression disks for $\partial M$ in $M$. Then there exists CSCS $\mathcal{J}'$ for $M$ (which is equivalent to $\mathcal{J}$) such that $\mathcal{J}'$ is disjoint from $\partial\mathcal{D}$.

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**Theorem (Zhao-Lei-Li)**

Suppose that $M$ is a 3-submanifold of $S^3$ which admits a CSCS $\mathcal{J}$, and the complement $H = \overline{S^3 \setminus M}$ is a handlebody. Then there exists a CSCS $\{\alpha_1, \cdots, \alpha_n\}$ for $M$ on $\partial M$ which is equivalent to $\mathcal{J}$ and a CSCS $\{\beta_1, \cdots, \beta_n\}$ for $H$ (which bounds a complete disk system for $H$) such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$.

**Corollary**

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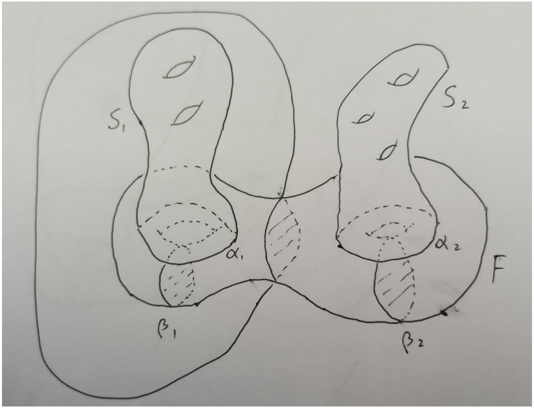
# A figure to show




This is just the picture we have seen in the boundary link case, therefore the theorem gives a positive answer to the question mentioned earlier up to eqivalence.

In general, we have

**Theorem (Zhao-Lei-Li)**

Let $F$ be a closed surface of genus $n \geq 1$ in $S^3$ which splits $S^3$ into two 3-manifolds $M_1$ and $M_2$, each admits a CSCS. Then there exists a CSCS $\{\alpha_1, \cdots, \alpha_n\}$ for $M_1$ and a CSCS $\{\beta_1, \cdots, \beta_n\}$ for $M_2$, such that $|\alpha_i \cap \beta_i| = 1$ for $1 \leq i \leq n$, and $|\alpha_i \cap \beta_j| = 0$ for $1 \leq i \neq j \leq n$.

**Remark:** The proofs of these theorems are essentially dependent on the uniqueness theorem for Heegaard splittings of $S^3$. It is clear that the above theorem implies Poincaré Conjecture. Thus, it is equivalent to Poincaré Conjecture.

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**THANKS FOR YOUR ATTENTION!**