Multi-switches, representations of braids and knot invariants

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A switch on a set X is a bijection $S : X \times X \to X \times X$, that satisfies the set theoretic Yang-Baxter equation

$$(S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S).$$

A switch S is called involutive if $S^2 = id$.

If X is not just a set but an algebraic system: quandle, group, module and so on, then the switch S on X is called a quandle switch, a group switch, a module switch and so on.

We will show that if we have a switch we can construct a representation of braid groups and an invariant of knots.

Example

Let X be an arbitrary set, and $T(a, b) = (b, a)$ for all $(a, b) \in X^2$. The map T is a switch which is called the twist. It is clear that T is involutive.

Example

Let X be a free left module over the ring $\mathbb{Z}[t^{\pm 1}]$, then the map $S_B(a, b) = (ta + (1 - t)b, a)$ for $(a, b) \in X^2$ is a switch. It is called the Burau switch.

Example

Let X be a free group. Then the map S_A defined by $S_A(a, b) = (aba^{-1}, a)$ for $(a, b) \in X^2$ is a switch which is called the Artin switch on a X. It is easy to check that $S_A^{-1}(a, b) = (b, b^{-1}ab)$.

Recall that a quandle Q is an algebraic system with one binary algebraic operation $(a, b) \mapsto a * b$ which satisfies the following axioms:

1. $a * a = a$ for all $a \in Q$,
2. the map $I_a : b \mapsto b * a$ is a bijection of Q for all $a \in Q$,
3. $(a * b) * c = (a * c) * (b * c)$ for all $a, b, c \in Q$.

A quandle Q is called trivial if $a * b = a$ for all $a, b \in Q$, the trivial quandle with n elements is denoted by T_n.

Quandles were introduced by S. V. Matveev and D. Joyce in 1982 as an invariant for links.

The following example gives a quandle switch.

Example

Let $(X, *)$ be a quandle. Then the map $S_Q(a, b) = (b * a, a)$ for $a, b \in X$ is a quandle switch. The inverse to S_Q is given by $S_Q^{-1}(a, b) = (b, a *^{-1} b)$, where $a *^{-1} b = I_b^{-1}(a)$.

Braid group B_n on $n \geq 2$ strands is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and is defined by relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| \geq 2. \end{aligned}$$

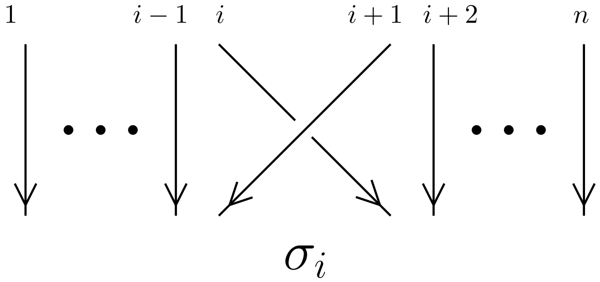


Figure: Geometric interpretation of σ_i

Pure braid group

There exists a homomorphism

$$\iota : B_n \to \Sigma_n$$

from the braid group B_n onto the symmetric group Σ_n on n symbols. This homomorphism maps the generator σ_i to the transposition $\tau_i = (i,\ i+1)$ for $i = 1, 2, \ldots, n-1$. The kernel of this homomorphism is called the pure braid group on n strands and is denoted by P_n.

Switches can be used in order to construct representations of braid groups.

Let $S \in \mathrm{Sym}(X^2)$ be a switch on X. For $i = 1, \ldots, n-1$ denote by

$$S_i = (id)^{i-1} \times S \times (id)^{n-i-1}.$$

From the relations of B_n and the Yang-Baxter equality follows that the map which maps σ_i to S_i for $i = 1, \ldots, n-1$ defines a representation of the braid group B_n into the symmetric group $\mathrm{Sym}(X^n)$. If S is involutive, then the map $\tau_i \mapsto S_i$ defines a representation of the symmetric group Σ_n into the group $\mathrm{Sym}(X^n)$.

Switches can also provide representations of braid groups by automorphisms of some algebraic system.

Let X be an algebraic system (for example, module, group, quandle and so on) generated by elements $x_1, \ldots, x_n$, and S is a switch on X with $S(a,b) = (L(a,b), R(a,b))$ for $a, b \in X$, then for $i = 1, \ldots, n-1$ denote by $S_i : \{x_1, \ldots, x_n\} \to X$ the map given by

$$S_i(x_k) = \begin{cases} L(x_k, x_{k+1}), & k = i, \\ R(x_k, x_{k+1}), & k = i+1, \\ x_k, & k \neq i, i+1. \end{cases}$$

If S_i induces an automorphism of X, then the map which maps σ_i to S_i for $i = 1, \ldots, n-1$ induces a representation

$$\varphi_S : B_n \to \mathrm{Aut}(X).$$

Example

If $X = M_n$ is the free left module of rank n over the ring $\mathbb{Z}[t^{\pm 1}]$, then the Burau switch S_B defines the Burau representation $\varphi_B : B_n \to \mathrm{Aut}(M_n) \cong \mathrm{GL_n}(\mathbb{Z}[t^{\pm 1}])$.

Example

If $X = F_n$ is the free group of rank n, then the Artin switch S_A defines the Artin representation $\varphi_A : B_n \to \mathrm{Aut}(F_n)$.

Example

If $X = FQ_n$ is the free quandle of rank n, then the quandle switch S_Q defines the representation $\varphi_Q : B_n \to \mathrm{Aut}(FQ_n)$.

By Alexander's theorem every link is a closure of some braid.

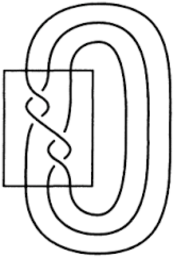


Figure: 3-strand braid β and its closure $\widehat{\beta}$

Suppose that L is the closure of some braid $\beta \in B_n$, then using representation $\varphi_S : B_n \to \mathrm{Aut}(X)$, constructed by a switch S, defined on the algebraic system $X = \langle x_1, x_2, \ldots, x_n \rangle$, we can define an algebraic system

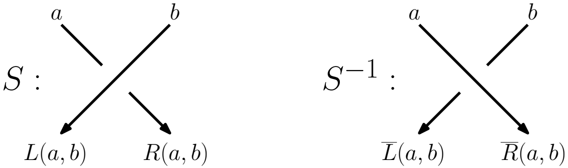
$$X_S(\beta) = \langle x_1, x_2, \ldots, x_n \mid\mid \varphi_S(\beta)(x_i) = x_i \; i = 1, 2, \ldots, n \rangle$$

that is a quotient of X.

Let X be an algebraic system and S be a switch on X such that

$$S(a,b) = (L(a,b), R(a,b)), \quad S^{-1}(a,b) = (\overline{L}(a,b), \overline{R}(a,b)).$$

For a diagram D_L link L we can construct an algebraic system $I_S(D_L)$ in the following way:
(1) the set of generators of $I_S(D_L)$ is the set of all arcs of D_L (strands going from one crossing to another crossing);
(2) the set of relations of $I_S(D_L)$ is the set of equalities which can be written from the crossings of D_L in the following way.



If $I_S(D_L)$ does not change under the Reidemeister moves R1-R3, then we get a link invariant. We will denote it by $I_S(L)$.

Theorem.

If $L = \widehat{\beta}$, then $X_S(\beta) \cong I_S(L)$.

Example

If X is the free left module then the Burau switch S_B defines the Alexander module $I_B(D_L)$ that is a quotient of X.

Example

If X is the free group, then the Artin switch S_A defines the group $I_A(D_L)$ of link L that is the fundamental group of compliment L in 3-sphere.

Example

If X is the free quandle, then the quandle switch S_Q defines the link quandle $I_Q(D_L) = Q(L)$.

Theorem [S. V. Matveev, D. Joyce, 1982].

If the knot quandles of two knots are isomorphic, then the (unoriented) knots are equivalent.

The virtual braid group VB_n is presented by L. Kauffman (1996).

VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the symmetric group $\Sigma_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. Generators $\rho_i, i = 1, \ldots, n-1$, satisfy the following relations:

$$\begin{aligned} \rho_i^2 &= 1 && \text{for } i = 1, 2, \ldots, n-1, \\ \rho_i \rho_j &= \rho_j \rho_i && \text{for } |i-j| \geq 2, \\ \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1} && \text{for } i = 1, 2 \ldots, n-2. \end{aligned}$$

Other defining relations of the group VB_n are mixed and they are as follows

$$\begin{aligned} \sigma_i \rho_j &= \rho_j \sigma_i && \text{for } |i-j| \geq 2, \\ \rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_{i+1} && \text{for } i = 1, 2, \ldots, n-2. \end{aligned}$$

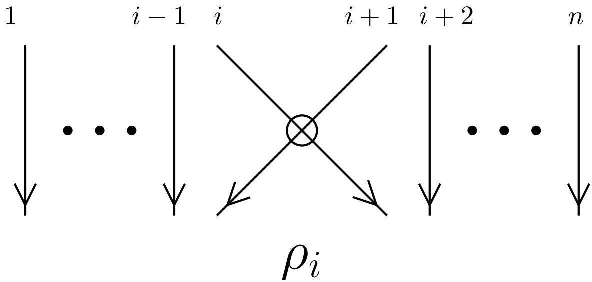


Figure: Geometric interpretation of ρ_i

Let $S, V \in \mathrm{Sym}(X^2)$ be a switch and an involutive switch on X, respectively. We say that the pair (S, V) is a virtual switch on X if the equality

$$(V \times id)(id \times V)(S \times id) = (id \times S)(V \times id)(id \times V)$$

holds in X^3.

Example

If X is a quandle, then (S_Q, T) is a virtual switch on X.

But if we define an invariant of virtual knots (L. Kauffman), using this virtual switch, then it is not strong invariants. It does not detect non-triviality of the virtual trefoil.

- Construct a faithful representation

$$\psi : VB_n \longrightarrow \mathrm{Aut}(H),$$

where H is a "good" algebraic system.

- Define a strong invariant for virtual links.

Usually, people consider virtual switches (S, V) on a set X, whee $V = T$ is the twist. So, invariants of virtual links are obtained from such virtual switches ignore the information in virtual crossings.

V. O. Manturov (2002) found a virtual switch (S, V), where V is not the twist. Let Q be a quandle, $T_1 = \{t\}$ a trivial quandle with one element, and $X = Q * T_1$ is the free product of Q and T_1. Take $S_Q : X^2 \to X^2$ the quandle switch $S_Q(a, b) = (b * a, a)$, and by $V : X^2 \to X^2$ the involutive switch with $V(a, b) = (b *^{-1} t, a * t)$ for $a, b \in X$. Then (S_Q, V) is a virtual switch on X. Using this virtual switch Manturov constructed a quandle invariant for virtual links which generalizes the quandle of Kauffman.

The Kishino knot is a non-trivial knot that is the connected sum of two trivial knots.

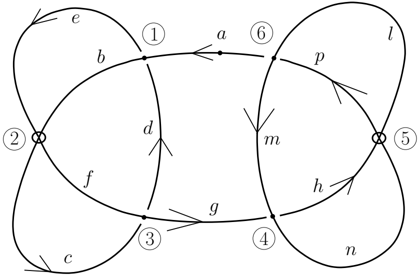


Figure: Kishino knot

R. Fenn, M. Jordan-Santana, L. Kauffman (2004) introduced biguandles as a tool for constructing invariants of virtual knots and links. Let S be a switch on X:

$$S(a,b) = (L(a,b), R(a,b)).$$

For $a, b \in X$ denote by $L(a,b) = b^a$, $R(a,b) = a_b$. The Yang-Baxter equation for S implies the following equalities

$$a^{bc} = a^{c_b b^c}, \qquad a_{bc} = a_{c^b b_c}, \qquad a_b^{c_{b^a}} = a^{c_b}{}_{c^a}$$

for all $a, b, c \in X$. A switch S is called the biquandle switch if the following conditions hold.

1. The maps $f^a, f_a : X \to X$ given by $f^a(x) = x^a$, $f_a(x) = x_a$ are bijective.
2. $a^{a^{-1}} = a_{a^{a^{-1}}}$ and $a_{a^{-1}} = a^{a_{a^{-1}}}$ for all $a \in X$.

Example

A quandle switch is a biquandle switch if we put $R(a,b) = a_b = a$.

For a virtual switch (S, V) on X and an integer $n \geq 2$ denote by

$$S_i = (id)^{i-1} \times S \times (id)^{n-i-1}, \qquad V_i = (id)^{i-1} \times V \times (id)^{n-i-1}$$

for $i = 1, \dots, n-1$. From the relations of VB_n and definition of virtual switch we see that the maps $\sigma_i \mapsto S_i$, $\rho_i \mapsto V_i$ induce a representation $VB_n \to \mathrm{Sym}(X^n)$.

If X is an algebraic system generated by elements $x_1, \dots, x_n$, and (S, V) is a virtual switch on X with $S(a, b) = (L(a, b), R(a, b))$, $V(a, b) = (U(a, b), W(a, b))$ for $a, b \in X$, then for $i = 1, \dots, n-1$ denote by $S_i, V_i : \{x_1, \dots, x_n\} \to X$ the maps given by

$$S_i(x_k) = \begin{cases} L(x_k, x_{k+1}), & k = i, \\ R(x_k, x_{k+1}), & k = i+1, \\ x_k, & k \neq i, i+1, \end{cases} \quad V_i(x_k) = \begin{cases} U(x_k, x_{k+1}), & k = i, \\ W(x_k, x_{k+1}), & k = i+1, \\ x_k, & k \neq i, i+1. \end{cases}$$

If S_i, V_i induce automorphisms of X, then the maps $\sigma_i \mapsto S_i$, $\rho_i \to V_i$ induce a representation

$$\varphi_{S,V} : VB_n \to \mathrm{Aut}(X).$$

Let $m \geq 1$ be an integer, $X_1, \dots, X_m$ non-empty sets, and $X = X_1 \times \cdots \times X_m$. Every switch S on X is called an m-switch, or a multi-switch (if m is not specified). We will think about X^2 as about $X_1^2 \times \cdots \times X_m^2$, so, for $A = (a_1, \dots, a_m), B = (b_1, \dots, b_m) \in X$ we write

$$S(A, B) = S(a_1, b_1; a_2, b_2; \dots; a_m, b_m).$$

If $X_1, \dots, X_m$ are not just sets but algebraic systems in some category: groups, modules and so on, then every multi-switch S on X is called a groups multi-switch, a module multi-switch and so on.

The notions of the involutive multi-switch and the virtual multi-switch are the same as for switches changing X by $X_1 \times \cdots \times X_m$.

Example

Every switch is a 1-switch.

Example (B.-Mikhalchishina-Neshchadim, 2017)

Let G be a group, X_2 be an abelian group, and $X_1 = G * X_2$ be the free product of G and X_2. For a fixed element $x_0 \in X_2$ denote by S the following map from $X_1^2 \times X_2^2$ to itself

$$S(a, b; x, y) = (ab^x a^{-x_0 y}, a^{x_0}; y, x), \qquad a, b \in X_1,\ x, y \in X_2.$$

The map S is a group 2-switch on $X = X_1 \times X_2$.

Example (S. Kamada, 2017)

Let G be a group, X_2, X_3 be abelian groups, and $X_1 = G * (X_2 \times X_3)$ be the free product of G and $X_2 \times X_3$. The map S given by

$$S(a, b; x, y; p, q) = (ab^x a^{-qy}, a^q; y, x; q, p)$$

for $a, b \in X_1$, $x, y \in X_2$, $p, q \in X_3$ is a group 3-switch on $X_1 \times X_2 \times X_3$.

We construct a virtual 2-switch on a biquandle.

Proposition [B.-Nasybullov, 2019].

Let X_1 be a biquandle, X_2 be a trivial subbiquandle of X_1, and $S, V : X_1^2 \times X_2^2 \to X_1^2 \times X_2^2$ be the maps defined by

$$S(a, b; x, y) = (b^a, a_b; y, x), \qquad V(a, b; x, y) = (b^{x^{-1}}, a^y; y, x)$$

for $a, b \in X_1$, $x, y \in X_2$. If for all $a, b \in X_1$, $x \in X_2$ the equalities

$$b^{ax} = b^{xa^x}, \qquad (a_b)^x = (a^x)_{b^x}$$

hold, then (S, V) is a virtual 2-switch on $X_1 \times X_2$.

As we noted above, (virtual) switches can be used to construct representations of (virtual) braid groups by automorphisms of algebraic systems.

However, there are representations $VB_n \to \mathrm{Aut}(G)$, where G is some group, which cannot be defined using any virtual switch on G. Take the representation, introduced by B.-Mikhalchishina-Neshchadim

$$\varphi_M : VB_n \to \mathrm{Aut}(F_{n,2n+1}),$$

where $F_{n,2n+1} = F_n * \mathbb{Z}^{2n+1}$ is a free product of the free group $F_n = \langle x_1, \dots, x_n \rangle$ and the free abelian group $\mathbb{Z}^{2n+1} = \langle u_1, \dots, u_n, v_0, \dots, v_n \rangle$.

This representation acts on the generators of $F_{n,2n+1}$ in the following way

$$\varphi_M(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1}^{u_i} x_i^{-v_0 u_{i+1}}, \\ x_{i+1} \mapsto x_i^{v_0}, \\ u_i \mapsto u_{i+1}, \\ u_{i+1} \mapsto u_i, \\ v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases} \qquad \varphi_M(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto x_i^{v_{i+1}}, \\ u_i \mapsto u_{i+1}, \\ u_{i+1} \mapsto u_i, \\ v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

cannot be defined using procedure described above for any virtual switch (S, V) on $F_{n,2n+1}$ (hereinafter we write only non-trivial actions on the generators, assuming that all other generators are fixed).

Describe a general construction of a representation $\varphi_{S,V} : VB_n \to \mathrm{Aut}(X)$ by a virtual switches (S, V).
Let X, $X_0, X_1, \dots, X_m$ be algebraic systems, $X_0, X_1, \dots, X_m \subseteq X$ such that

1. $X_i \cap X_j = \varnothing$ for $i \neq j$,
2. $X_0, X_1, \dots, X_m$ together generate X.

Let (S, V) be a virtual multi-switch on $X \times X_1 \times \dots \times X_m$, where
$S = (S_0, S_1, \dots, S_m)$, $V = (V_0, V_1, \dots, V_m)$ for

$$S_0 = (L_0, R_0), V = (U_0, W_0) : X^2 \times X_1^2 \times X_2^2 \times \dots \times X_m^2 \to X^2$$
$$S_i = (L_i, R_i), V_i = (U_i, W_i) : X_i^2 \to X_i^2, \text{ for } i = 1, 2, \dots, m.$$

Automorphic virtual multi-switch (AVMS)

For $a_0 \in X^2, a_i \in X_i^2$, $i = 1, 2, \ldots, m$, we write

$$S(a_0, a_1, \ldots, a_m) = (S_0(a_0, a_1, \ldots, a_m), S_1(a_1), \ldots, S_m(a_m)), \quad S_i = (L_i, R_i),$$

$$V(a_0, a_1, \ldots, a_m) = (V_0(a_0, a_1, \ldots, a_m), V_1(a_1), \ldots, V_m(a_m)), \quad V_i = (U_i, W_i).$$

Let $n \geq 2$ be an integer, and suppose that the system X_i is generated by elements $x_{i,1}, x_{i,2}, \ldots, x_{i,n}$ for $i = 0, \ldots, m$.
For $j = 1, \ldots, n-1$ denote by F_j, G_j the following maps from $X_0 \cup X_1 \cup \cdots \cup X_m$ to X

$$F_j : \begin{cases} x_{0,j} \mapsto L_0(x_{0,j}, x_{0,j+1}, x_{1,j}, x_{1,j+1}, \ldots, x_{m,j}, x_{m,j+1}), \\ x_{0,j+1} \mapsto R_0(x_{0,j}, x_{0,j+1}, x_{1,j}, x_{1,j+1}, \ldots, x_{m,j}, x_{m,j+1}), \\ x_{1,j} \mapsto L_1(x_{1,j}, x_{1,j+1}), \\ x_{1,j+1} \mapsto R_1(x_{1,j}, x_{1,j+1}), \\ \quad \vdots \\ x_{m,j} \mapsto L_m(x_{m,j}, x_{m,j+1}), \\ x_{m,j+1} \mapsto R_m(x_{m,j}, x_{m,j+1}), \end{cases}$$

$$G_j : \begin{cases} x_{0,j} \mapsto U_0(x_{0,j}, x_{0,j+1}, x_{1,j}, x_{1,j+1}, \ldots, x_{m,j}, x_{m,j+1}), \\ x_{0,j+1} \mapsto W_0(x_{0,j}, x_{0,j+1}, x_{1,j}, x_{1,j+1}, \ldots, x_{m,j}, x_{m,j+1}), \\ x_{1,j} \mapsto U_1(x_{1,j}, x_{1,j+1}), \\ x_{1,j+1} \mapsto W_1(x_{1,j}, x_{1,j+1}), \\ \quad \vdots \\ x_{m,j} \mapsto U_m(x_{m,j}, x_{m,j+1}), \\ x_{m,j+1} \mapsto W_m(x_{m,j}, x_{m,j+1}). \end{cases}$$

If the maps F_j, G_j induce automorphisms of X, then we say that (S, V) is an automorphic virtual multi-switch (shortly, AVMS) on $X \times X_1 \times \cdots \times X_m$ with respect to the set of generators $\{x_{i,j} \mid i = 0, \ldots, m, j = 1, \ldots, n\}$.
A virtual multi-switch can be AVMS with respect to one generating set of X, but not AVMS with respect to another generating set of X.

Proposition [B.-Nasybullov, 2019].

Let (S, V) be an AVMS on $X \times X_1 \times \cdots \times X_m$ with respect to the set of generators $\{x_{i,j} \mid i = 0, \dots, m, j = 1, \dots, n\}$. Then the map

$$\varphi_{S,V} : VB_n \to \mathrm{Aut}(X)$$

which is defined on the generators of VB_n as

$$\varphi_{S,V}(\sigma_j) = F_j, \qquad \varphi_{S,V}(\rho_j) = G_j, \qquad for\ j = 1, 2, \dots, n-1,$$

is a representation of VB_n.

As corollary of the general construction is the following

Theorem [B.-Nasybullov, 2019].

Let FQ_n be a free quandle on the set of generators $\{x_1, \dots, x_n\}$ and $T_n = \{y_1, \dots, y_n\}$ be a trivial quandle. Then the map φ given by

$$\varphi(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1} * x_i, \\ x_{i+1} \mapsto x_i, \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i, \end{cases} \qquad \varphi(\rho_i) : \begin{cases} x_i \mapsto x_{i+1} *^{-1} y_i, \\ x_{i+1} \mapsto x_i * y_{i+1}, \\ y_i \mapsto y_{i+1}, \\ y_{i+1} \mapsto y_i, \end{cases}$$

induces a homomorphism $VB_n \to \mathrm{Aut}(FQ_n * T_n)$.

Using a virtual multi-switch (S, V) on X one can construct an algebraic system which is an invariant of virtual links.
To do it we construct an algebraic system $X_{S,V}(\beta)$ for $\beta \in VB_n$.

Let X be an algebraic system, and $X^{(2)} < X^{(3)} < \dots$ be an ascending series of subsystems of X such that $X = \bigcup_n X^{(n)}$. For $n \geq 2$ let $X_0^{(n)}, X_1^{(n)}, \dots, X_m^{(n)}$ be algebraic systems in $X^{(n)}$ such that

1. $X_i^{(n)} \cap X_j^{(n)} = \varnothing$ for $i \neq j$,
2. $X_0^{(n)}, X_1^{(n)}, \dots, X_m^{(n)}$ together generate $X^{(n)}$,
3. $X_i^{(n)}$ is generated by elements $x_{i,1}, \dots, x_{i,n}$ for $i = 0, \dots, m$.

Let (S, V) be a virtual $(m+1)$-switch on $X \times X_1 \times \cdots \times X_m$, such that the maps S, V fix $X^{(n)} \times X_1^{(n)} \times \cdots \times X_m^{(n)}$ for all n, and the restriction $\left(S^{(n)}, V^{(n)}\right)$ is an automorphic virtual $(m+1)$-switch on $X^{(n)} \times X_1^{(n)} \times \cdots \times X_m^{(n)}$ with respect to the set of generators $\{x_{i,j} \mid i = 0, \ldots, m, j = 1, \ldots, n\}$. By Proposition for $n = 2, 3, \ldots$ we have representations

$$\varphi_{S^{(n)},V^{(n)}} : VB_n \to \mathrm{Aut}\left(X^{(n)}\right).$$

Denote by $VB_\infty = \bigcup_n VB_n$, and by $\varphi_{S,V} : VB_\infty \to \mathrm{Aut}(X)$ the homomorphism which is equal to $\varphi_{S^{(n)},V^{(n)}}$ on VB_n (this homomorphism is well defined since $\varphi_{S^{(n)},V^{(n)}}$ agree with each other). Now we can write $\varphi_{S,V} : VB_n \to \mathrm{Aut}(X^{(n)})$ meaning the restriction of $\varphi_{S,V}$ to VB_n.

For $\beta \in VB_n$ put

$$X_{S,V}(\beta) = \langle x_{i,j}, \quad i = 0, \ldots, m, j = 1, \ldots, n \mid \varphi_{S,V}(\beta)(x_{ij}) = x_{ij},$$

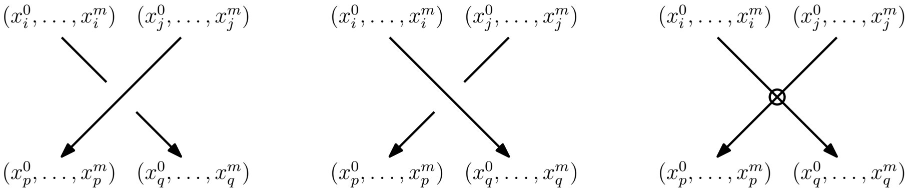
$$i = 0, \ldots, m, j = 1, \ldots, n \rangle.$$

We will find conditions on (S, V) under which $X_{S,V}(\beta)$ is an invariant of $L = \widehat{\beta}$. To do it we give another definition of $X_{S,V}(\beta)$, using a diagram D_L of L.

Let L has a diagram D_L. We call an edge of D_L any arc from one crossing of D_L (virtual or classical) to another crossing of D_L (virtual or classical). Suppose that D_L has n edges. Label edges of D_L by $(m+1)$-tuples $(x_j^0, x_j^1, \dots, x_j^m)$ for $j = 1, \dots, n$, where $x_j^0, x_j^1, \dots, x_j^m$ are generators of X. By the definition, a subsystem of X generated by all x_j^i from labels on edges of D_L is $X^{(n)}$. Denote by $I_{S,V}(D_L)$ the quotient of $X^{(n)}$ by the relations which can be written from the crossings of D_L in the following way:

$$
\begin{aligned}
S(x_i^0, x_j^0; x_i^1, x_j^1; \dots; x_i^m, x_j^m) &= (x_p^0, x_q^0; x_p^1, x_q^1; \dots; x_p^m, x_q^m), \quad \text{positive crossing},\\
S^{-1}(x_i^0, x_j^0; x_i^1, x_j^1; \dots; x_i^m, x_j^m) &= (x_p^0, x_q^0; x_p^1, x_q^1; \dots; x_p^m, x_q^m), \quad \text{negative crossing},\\
V(x_i^0, x_j^0; x_i^1, x_j^1; \dots; x_i^m, x_j^m) &= (x_p^0, x_q^0; x_p^1, x_q^1; \dots; x_p^m, x_q^m), \quad \text{virtual crossing},
\end{aligned}
$$

The labels $(x_i^0, \ldots, x_i^m)$, $(x_j^0, \ldots, x_j^m)$, $(x_p^0, \ldots, x_p^m)$, $(x_q^0, \ldots, x_q^m)$ are as on the following picture.



Note that each crossings gives $2m+2$ relations, so, $I_{S,V}(D_L)$ is obtained from $X^{(n)}$ adding $2n(m+1)$ relations.

Theorem [B.-Nasybullov, 2019].

Let X be an algebraic system, and $X_0, \dots, X_m$ be subsystems of X which satisfy conditions (1)-(3). If (S,V) is an automorphic virtual $(m+1)$-switch on X such that S, V are biquandles switches on $X \times X_1 \times \dots \times X_m$, then $I_{S,V}(D_L)$ is an invariant of L.

Using the Artin switch S_A we can construct the Artin representation of the braid group B_n by automorphisms of the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$:

$$\varphi_A : B_n \to \mathrm{Aut}(F_n).$$

Take the Fox derivatives and using the abelianization map $F_n \to F_n^{ab} = \langle t_1, t_2, \ldots, t_n \rangle$, we can define a map

$$B_n \to GL_n(\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}])$$

by the rule

$$\sigma_i \to A_i = I_{i-1} \oplus \begin{pmatrix} 1 - t_{i+1} & t_i \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1} \quad i = 1, 2, \ldots, n-1.$$

To get a representation of B_n it is need to have relations

$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}.$$

It is easy to check that this relation is true if and only if $t_1 = t_2 = \ldots = t_n$ and we get the Burau representation of $\varphi_B : B_n \to \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$.

Question.

Is it possible to construct some representation of B_n which depends on n variables?

Consider the free left module M with free basis $e_1, e_2, \ldots, e_n$ over $K = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ and take the additive group of this module. Put $X = M$, $X_1 = K$, then the map

$$S(a, b; t, \tau) = ((1-\tau)a + tb, a, \tau, t)$$

is a 2-switch on M.

Define a map

$$\varphi_G : B_n \to \mathrm{Aut}(M)$$

by action on the generators:

$$\varphi_G(\sigma_i) : \begin{cases} e_i \longmapsto (1 - t_{i+1})e_i + t_i e_{i+1}, \\ e_{i+1} \longmapsto e_i, \\ t_i \longmapsto t_{i+1}, \\ t_{i+1} \longmapsto t_i, \end{cases} \quad , \quad i = 1, 2, \ldots, n-1.$$

Hence, any element $\beta \in B_n$ acts on element of $m \in M$ by the rule

$$\left(\sum_{k=1}^{n} \alpha_k e_k \right)^{\beta} = \sum_{k=1}^{n} \varphi_G(\beta)(\alpha_k)\, \varphi_G(\beta)(e_k).$$

Note that we consider $\mathrm{Aut}(M)$ as the automorphisms of additive group.

Proposition [B.-Nasybullov, 2019].

The map $\varphi_G : B_n \to \mathrm{Aut}(M)$ is a representation of the braid group B_n. Its restriction to the pure braid group P_n is the Gassner linear representation $P_n \to GL_n(K)$.

In the paper of S. Barden and R. Fenn (2004) was proved that the Kishino knot is non-trivial using the virtual switch $(S(t), T)$ on a left free module over algebra of quaternions $\mathbb{H}$, where

$$S(t) = \begin{pmatrix} 1+i & -tj \\ t^{-1}j & 1+i \end{pmatrix}, T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

L. Kauffman and V. O. Manturov (2005) also proved that the Kishino knot is non-trivial using the virtual switch (S, V), where

$$S = \begin{pmatrix} 1+i & -j \\ j & 1+i \end{pmatrix}, V = \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix}.$$

We can formulate

Problem.

Let a virtual switch (X, S, V) defined an invariant of virtual links which distinguishes two virtual links L and L'. Is it true that there is a switch $(X, \widetilde{S})$ such that the invariant that is defined by virtual switch $(X, \widetilde{S}, T)$ distinguishes L and L'?

Thank you!